



Subject: Calculus

Chapter: 1,2,3,4 (Unit 1 & 2)

Category: Assignment 1 Solution

Answer 1.

There is some simplification that we can do.

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{2(-3+h)^2 - 18}{h} &= \lim_{h \rightarrow 0} \frac{2(9 - 6h + h^2) - 18}{h} \\ &= \lim_{h \rightarrow 0} \frac{18 - 12h + 2h^2 - 18}{h} \\ &= \lim_{h \rightarrow 0} \frac{-12h + 2h^2}{h}\end{aligned}$$

So, upon multiplying out the first term we get a little cancellation and now notice that we can factor an h out of both terms in the numerator which will cancel against the h in the denominator and the division by zero problem goes away and we can then evaluate the limit.

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{2(-3+h)^2 - 18}{h} &= \lim_{h \rightarrow 0} \frac{-12h + 2h^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{h(-12 + 2h)}{h} \\ &= \lim_{h \rightarrow 0} -12 + 2h = -12\end{aligned}$$

Answer 2:

a) For this part we can notice that because there are values of x on both sides of $x = 4$ in the range $x < 6$ we won't need to worry about one-sided limits here.

Here is the work for this part

$$\lim_{x \rightarrow 4} g(x) = \lim_{x \rightarrow 4} (2x) = 2 \lim_{x \rightarrow 4} x = 2(4) = g(4)$$

So, we can see that;

$$\lim_{x \rightarrow 4} g(x) = g(4)$$

and so the function is continuous at $x=4$.

b) we'll need to take a look at the two one sided limits to compute the overall limit and again because we are being asked to determine if the function is continuous at this point we'll need to resort to basic limit properties to compute the one-sided limits and not just plug in the point (which assumes continuity again...).

Here is the work for this part.

$$\begin{aligned}\lim_{x \rightarrow 6^-} g(x) &= \lim_{x \rightarrow 6^-} (2x) = 2 \lim_{x \rightarrow 6^-} x = 2(6) = 12 \\ \lim_{x \rightarrow 6^+} g(x) &= \lim_{x \rightarrow 6^+} (x - 1) = \lim_{x \rightarrow 6^+} x - \lim_{x \rightarrow 6^+} 1 = 6 - 1 = 5\end{aligned}$$

So we see that;

$\lim_{x \rightarrow 6^-} g(x) \neq \lim_{x \rightarrow 6^+} g(x)$ and so $\lim_{x \rightarrow 6} g(x)$ does not exist.

Answer 3

You start by multiplying the numerator and denominator by the conjugate of the denominator,

$$3 + \sqrt{x}.$$

$$= \lim_{x \rightarrow 9} \frac{(x-9)(3+\sqrt{x})}{(3-\sqrt{x})(3+\sqrt{x})}$$

Now multiply out the part of the fraction containing the conjugate pair (the denominator in this problem).

$$= \lim_{x \rightarrow 9} \frac{(x-9)(3+\sqrt{x})}{(9-x)}$$

Cancel.

$$= \lim_{x \rightarrow 9} (-1(3+\sqrt{x}))$$

Remember that any fraction of the form

$$\frac{a-b}{b-a}$$

always equals -1.

Now plug in.

$$= -1(3+\sqrt{9})$$

$$= -6$$

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Answer 4

We can't just plug in the point to evaluate the limit, we make certain simplifications and then work.

a) $x = -1$

So, here we go.

$$\lim_{x \rightarrow -1} f(x) = \lim_{x \rightarrow -1} \frac{4x+5}{9-3x} = \frac{\lim_{x \rightarrow -1} (4x+5)}{\lim_{x \rightarrow -1} (9-3x)} = \frac{4 \lim_{x \rightarrow -1} x + \lim_{x \rightarrow -1} 5}{\lim_{x \rightarrow -1} 9 - 3 \lim_{x \rightarrow -1} x} = \frac{4(-1) + 5}{9 - 3(-1)} = f(-1)$$

So, we can see that

$$\lim_{x \rightarrow -1} f(x) = f(-1)$$

and so the function is continuous at $x = -1$.

b) $x = 0$

Here is the work for this part.

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{4x + 5}{9 - 3x} = \frac{\lim_{x \rightarrow 0} (4x + 5)}{\lim_{x \rightarrow 0} (9 - 3x)} = \frac{4 \lim_{x \rightarrow 0} x + \lim_{x \rightarrow 0} 5}{\lim_{x \rightarrow 0} 9 - 3 \lim_{x \rightarrow 0} x} = \frac{4(0) + 5}{9 - 3(0)} = f(0)$$

So, we can see that

$$\lim_{x \rightarrow 0} f(x) = f(0)$$

and so the function is continuous at $x = 0$.

c) $x = 3$.

For the value of $x = 3$, we see that the denominator becomes zero. i.e. $9 - 3(3) = 0$. Therefore, we can see that the function is not continuous at $x = 3$.

$$\lim_{x \rightarrow 3} f(x) \text{ does not exist.}$$

Answer 5

We look at the denominator and determine the exponential function with the "largest" exponent which we will then factor out from both numerator and denominator. In the case since we are looking at a limit at plus infinity we only look at exponentials with positive exponents.

So, we'll factor an e^{4x} out of both then numerator and denominator. Once that is done we can cancel it.

Here is the work for this limit,

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{6e^{4x} - e^{-2x}}{8e^{4x} - e^{2x} + 3e^{-x}} &= \lim_{x \rightarrow \infty} \frac{e^{4x} (6 - e^{-6x})}{e^{4x} (8 - e^{-2x} + 3e^{-5x})} \\ &= \lim_{x \rightarrow \infty} \frac{6 - e^{-6x}}{8 - e^{-2x} + 3e^{-5x}} \\ &= \frac{6 - 0}{8 - 0 + 0} \\ &= \frac{3}{4} \end{aligned}$$

Answer 6.

a) In this case $y=6$ is completely inside the second interval for the function and so there are values of y on both sides of $y=6$ that are also inside this interval. This means that we can just use the fact to evaluate this limit.

$$\begin{aligned}\lim_{y \rightarrow 6} g(y) &= \lim_{y \rightarrow 6} (1 - 3y) \\ &= -17\end{aligned}$$

b)

In this case the point that we want to take the limit for is the cutoff point for the two intervals. So, let's do the two one-sided limits and see what we get.

$$\begin{aligned}\lim_{y \rightarrow -2^-} g(y) &= \lim_{y \rightarrow -2^-} (y^2 + 5) \quad \text{since } y \rightarrow -2^- \text{ implies } y < -2 \\ &= 9\end{aligned}$$

$$\begin{aligned}\lim_{y \rightarrow -2^+} g(y) &= \lim_{y \rightarrow -2^+} (1 - 3y) \quad \text{since } y \rightarrow -2^+ \text{ implies } y > -2 \\ &= 7\end{aligned}$$

So, in this case we can see that,

$$\lim_{y \rightarrow -2^-} g(y) = 9 \neq 7 = \lim_{y \rightarrow -2^+} g(y)$$

and so since the two one sided limits aren't the same

$$\lim_{y \rightarrow -2} g(y)$$

doesn't exist.

Answer 7

The derivative using the product rule.

$$f'(t) = (8t - 1)(t^3 - 8t^2 + 12) + (4t^2 - t)(3t^2 - 16t) = 20t^4 - 132t^3 + 24t^2 + 96t - 12$$

Answer 8.

The derivative using the quotient rule.

$$R'(w) = \frac{(3 + 4w^3)(2w^2 + 1) - (3w + w^4)(4w)}{(2w^2 + 1)^2} = \boxed{\frac{4w^5 + 4w^3 - 6w^2 + 3}{(2w^2 + 1)^2}}$$

Answer 9

Using the rules for derivatives of exponentials, we get

$$\boxed{f'(x) = 2e^x - 8^x \ln(8)}$$

Answer 10

Using the rules for derivatives:

$$\boxed{y' = 5z^4 - e^z \ln(z) - \frac{e^z}{z}}$$

Answer 11

The derivative is

a)

$$f'(x) = 4(6x^2 + 7x)^3(12x + 7) = \boxed{4(12x + 7)(6x^2 + 7x)^3}$$

b) Note that we only need to use the Chain Rule on the second term as we can differentiate the first term without the Chain Rule. The derivative is then,

$$\boxed{f'(t) = (4 + 7t^6) e^{4t+t^7}}$$

Answer 12

a) The first derivative is

$$\frac{dz}{dx} = \frac{-3x^2}{7 - x^3}$$

The second derivative is then

$$\frac{d^2z}{dx^2} = \frac{-6x(7-x^3) - (-3x^2)(-3x^2)}{(7-x^3)^2} = \boxed{\frac{-42x - 3x^4}{(7-x^3)^2}}$$

b) The first derivative is

$$Q(v) = 2(6+2v-v^2)^{-4}$$

$$Q'(v) = -8(2-2v)(6+2v-v^2)^{-5}$$

The second derivative is then

$$\boxed{Q''(v) = 16(6+2v-v^2)^{-5} + 40(2-2v)^2(6+2v-v^2)^{-6}}$$

c) The first derivative is

$$f'(t) = \frac{2t}{1+t^2}$$

The second derivative is then

$$\begin{aligned} f''(t) &= \frac{2(1+t^2) - (2t)(2t)}{(1+t^2)^2} \\ &= \frac{2-2t^2}{(1+t^2)^2} \end{aligned}$$

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Answer 13

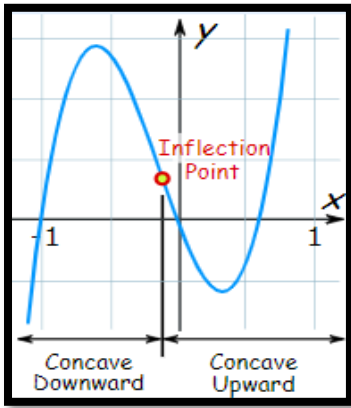
Let's work out the second derivative:

- The derivative is $f'(x) = 15x^2 + 4x - 3$
- The second derivative is $f''(x) = 30x + 4$

And $30x + 4$ is negative up to $x = -4/30 = -2/15$, and positive from there onwards. So:

$f(x)$ is concave downward up to $x = -2/15$

$f(x)$ is concave upward from $x = -2/15$ on



Answer 14

$$\text{Let } y = f(x) = x^3 - 3x^2 - 9x + 12$$

$$f'(x) = 3x^2 - 3(2x) - 9(1) + 0$$

$$f'(x) = 3x^2 - 6x - 9$$

$$f'(x) = 0$$

$$3x^2 - 6x - 9 = 0$$

$$\div \text{ by } 3 \Rightarrow x^2 - 2x - 3 = 0$$

$$x + 1 = 0 \text{ Therefore, } x = -1$$

$$x - 3 = 0 \text{ Therefore, } x = 3$$

$$f'(x) = 3x^2 - 6x - 9$$

$$f''(x) = 3(2x) - 6(1) - 0$$

$$f''(x) = 6x - 6$$

$$\text{Put } x = -1$$

$$f''(-1) = 6(-1) - 6$$

$$= -6 - 6$$

$$f''(-1) = -12 < 0 \text{ Maximum}$$

To find the maximum value let us apply $x = -1$ in the given function

$$f(x) = x^3 - 3x^2 - 9x + 12$$

$$f(-1) = (-1)^3 - 3(-1)^2 - 9(-1) + 12$$

$$= -1 - 3(1) + 9 + 12$$

$$= -1 - 3 + 9 + 12$$

$$= -4 + 21$$

$$= 17$$

$$\text{Put } x = 3$$

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$$f''(3) = 6(3) - 6$$

$$= 18 - 6$$

$$f''(3) = 12 > 0 \text{ Minimum}$$

To find the minimum value let us apply $x = 3$ in the given function

$$f(x) = x^3 - 3x^2 - 9x + 12$$

$$f(3) = 3^3 - 3(3)^2 - 9(3) + 12$$

$$= 27 - 3(9) - 27 + 12$$

$$= 27 - 27 - 27 + 12$$

$$= -27 + 12$$

$$= -15$$

Therefore the maximum value = 17 and

The minimum value = -15

Answer 15.

$$\text{Let } y = f(x) = 4x^3 - 18x^2 + 24x - 7$$

$$f'(x) = 4(3x^2) - 18(2x) + 24(1) - 0$$

$$f'(x) = 12x^2 - 36x + 24$$

$$f'(x) = 0$$

$$12x^2 - 36x + 24 = 0$$

$$\div \text{ by } 12 \Rightarrow x^2 - 3x + 2 = 0$$

$$x - 1 = 0$$

$$x = 1$$

$$x - 2 = 0$$

$$x = 2$$

$$f'(x) = 12x^2 - 36x + 24$$

$$f''(x) = 12(2x) - 36(1) + 0$$

$$f''(x) = 24x - 36$$

$$\text{Put } x = 1$$

$$f''(1) = 24(1) - 36$$

$$= 24 - 36$$

$$f''(1) = -12 < 0 \text{ Maximum}$$

To find the maximum value let us apply $x = 1$ in the given function.

$$f(x) = 4x^3 - 18x^2 + 24x - 7$$

$$f(1) = 4(1)^3 - 18(1)^2 + 24(1) - 7$$

$$= 4(1) - 18(1) + 24 - 7$$

$$= 4 - 18 + 24 - 7$$

$$= 28 - 25$$

$$= 3$$

Put $x = 2$

$$f'(2) = 24(2) - 36$$

$$= 48 - 36$$

$$f'(2) = 12 > 0 \text{ Minimum}$$

To find the minimum value let us apply $x = 2$ in the given function

$$f(x) = 4x^3 - 18x^2 + 24x - 7$$

$$f(2) = 4(2)^3 - 18(2)^2 + 24(2) - 7$$

$$= 4(8) - 18(4) + 48 - 7$$

$$= 32 - 72 + 48 - 7$$

$$= 80 - 79$$

$$= 1$$

Therefore the maximum value is 3 and the minimum value is 1.

Answer 16.

The function is defined for all $x \in \mathbb{R}$. It has the following x -intercepts:

$$f(x)=0, \Rightarrow x^2(x+3) = 0, \Rightarrow x_1 = 0, x_2 = -3.$$

The y -intercept is equal to $f(0)=0$.

The function is positive on the intervals $(-3,0)$ and $(0,+\infty)$ and negative on $(-\infty,-3)$.

The function is neither even nor odd, and it has no asymptotes.

Take the derivative:

$$f'(x) = (x^2(x+3))' = (x^3 + 3x^2)' = 3x^2 + 6x.$$

Find the critical points:

$$f'(x) = 0, \Rightarrow 3x^2 + 6x = 0, \Rightarrow 3x(x+2) = 0, \Rightarrow x_1 = 0, x_2 = -2.$$

We can see from the sign chart that $x=-2$ is a point of maximum, and $x=0$ is a point of minimum.

The y -values of these points are

$$f(-2) = (-2)^2(-2+3) = 4;$$

$$f(0) = 0.$$

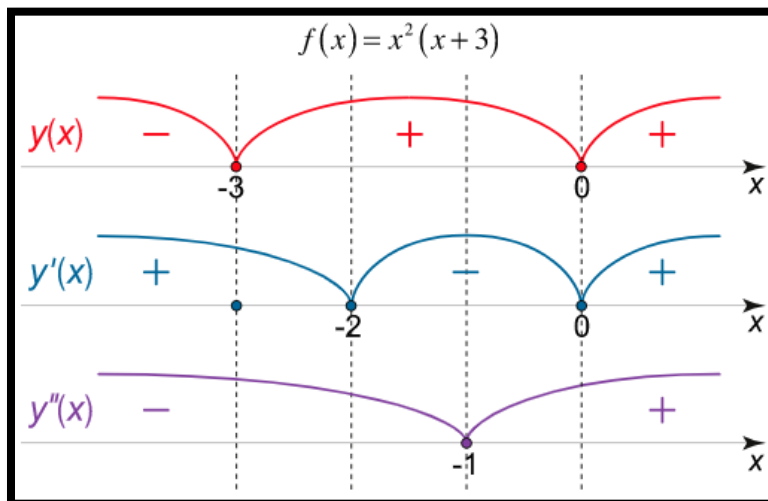
We differentiate once more to get the second derivative:

$$f''(x) = (3x^2 + 6x)' = 6x + 6.$$

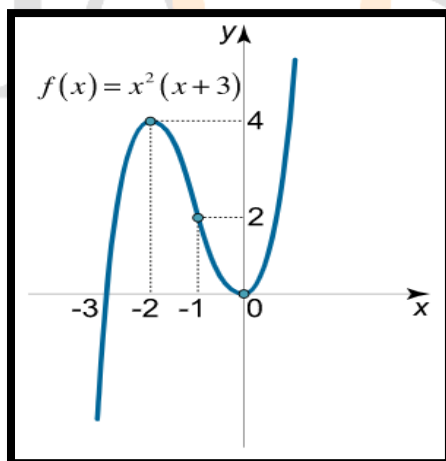
$$f''(x) = 0, \Rightarrow 6x + 6 = 0, \Rightarrow x = -1.$$

The graph of the function is concave downward on $(-\infty, -1)$ and concave upward on $(-1, +\infty)$. Therefore, $x = -1$ is a point of inflection. The y -coordinate of this point is $f(-1) = (-1)^2(-1 + 3) = 2$.

Given these results, we can draw a schematic graph of the function



Detailed Graph for the function:-



Answer 17.

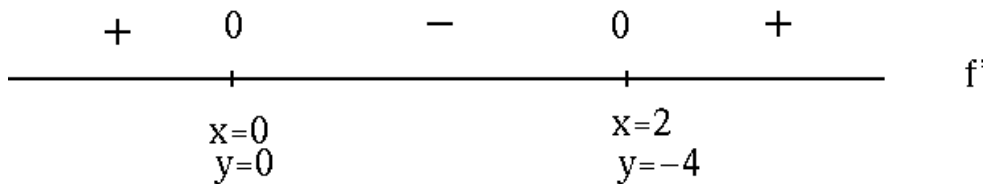
The domain of f is all x -values. Now determine a sign chart for the first derivative, f' :

$$f(x) = 3x^2 - 6x$$

$$= 3x(x - 2)$$

$$= 0$$

for $x = 0$ and $x = 2$. See the adjoining sign chart for the first derivative, f' .



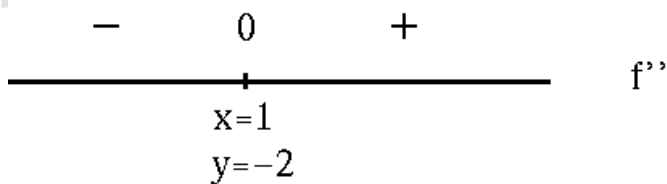
Now determine a sign chart for the second derivative, f'' :

$$f''(x) = 6x - 6$$

$$= 6(x - 1)$$

$$= 0$$

or $x = 1$. See the adjoining sign chart for the second derivative, f'' .



FROM f' :

f is () for $x < 0$ and $x > 2$;

f is () for $0 < x < 2$;

f has a relative maximum at $x=0, y=0$;

f has a relative minimum at $x=2, y=-4$.

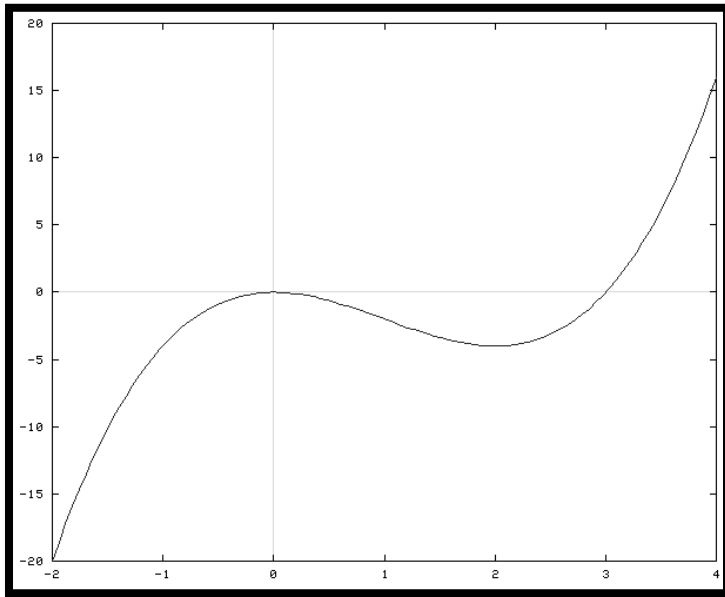
FROM f'' :

f is () for $x > 1$;

f is () for $x < 1$;

f has an inflection point at $x=1, y=-2$

If $x=0$, then $y=0$ so that $y=0$ is the y -intercept. If $y=0$, then $x^3-3x^2=x^2(x-3)=0$ so that $x=0$ and $x=3$ are the x -intercepts. There are no vertical or horizontal asymptotes since f is a polynomial. See the adjoining detailed graph of f .



Answer 18.

$$\text{When } x^2 = 1, f(x^2 - 1) = f(1 - 1) = f(0) = (1)^2 - 7(1) + k_1$$

$$f(0) = -6 + k_1 \dots\dots\dots(1)$$

Essentially, we have replaced all x^2 with 1.

$$\text{When } x^3 = 2, f(x^3 - 2) = f(2 - 2) = f(0) = (2)^2 - 9(2) + k_2$$

$$f(0) = -14 + k_2 \dots\dots\dots(2)$$

Essentially, we have replaced all x^3 with 2.

Equating $f(0)$ in equations (1) and (2)

$$(-6 + k_1) = (-14 + k_2)$$

$$\text{or } k_2 - k_1 = 8$$

Answer 19.

Correct Answer - (4)

Solution:

An odd function is a function whose value reverses in sign for a reversal in sign of its argument, i.e. $f(x) = -f(-x)$.

Except $f(x) = |x|^3$ all other functions mentioned in the choices change values.

Answer 20.

$$f(x) = x^2 - 3x + 2.$$

To find $f(f(x))$

$$f(f(x)) = f(x)^2 - 3f(x) + 2.$$

$$= (x^2 - 3x + 2)^2 - 3(x^2 - 3x + 2) + 2$$

By using the formula $(a-b+c)^2 = a^2 + b^2 + c^2 - 2ab + 2ac - 2bc$, we get

$$= (x^2)^2 + (3x)^2 + 2^2 - 2x^2(3x) + 2x^2(2) - 2(3x)(2) - 3(x^2 - 3x + 2) + 2$$

Now, substitute the values

$$= x^4 + 9x^2 + 4 - 6x^3 - 12x + 4x^2 - 3x^2 + 9x - 6 + 2$$

$$= x^4 - 6x^3 + 9x^2 + 4x^2 - 3x^2 - 12x + 9x - 6 + 2 + 4$$

Simplify the expression, we get,

$$f(f(x)) = x^4 - 6x^3 + 10x^2 - 3x$$

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