



Subject: Calculus

Chapter: 5,6,7 (Unit 3 & 4)

Category: Assignment 2 Solution

1. Here it is,

$$\begin{aligned} u = \frac{2}{w} \quad du = -\frac{2}{w^2} dw &\Rightarrow \frac{1}{w^2} dw = -\frac{1}{2} du \\ w = 2 \quad \Rightarrow \quad u = 1 \quad w = \frac{1}{50} &\Rightarrow \quad u = 100 \end{aligned}$$

Here is the integral.

$$\begin{aligned} \int_{\frac{1}{50}}^2 \frac{e^{\frac{2}{w}}}{w^2} dw &= -\frac{1}{2} \int_{100}^1 e^u du \\ &= -\frac{1}{2} e^u \Big|_{100}^1 \\ &= -\frac{1}{2} (e^1 - e^{100}) \end{aligned}$$

2. Clearly the derivative of the denominator, ignoring the exponent, differs from the numerator only by a multiplicative constant and so the substitution is,

$$u = t^4 + 2t \quad du = (4t^3 + 2) dt = 2(2t^3 + 1) dt \quad \Rightarrow \quad (2t^3 + 1) dt = \frac{1}{2} du$$

After a little manipulation of the differential we get the following integral.

$$\begin{aligned} \int \frac{2t^3 + 1}{(t^4 + 2t)^3} dt &= \frac{1}{2} \int \frac{1}{u^3} du \\ &= \frac{1}{2} \int u^{-3} du \\ &= \frac{1}{2} \left(-\frac{1}{2} \right) u^{-2} + c \\ &= -\frac{1}{4} (t^4 + 2t)^{-2} + c \end{aligned}$$

3. The solution is;

$$f(x) = \int f'(x) dx = \int x^4 + 3x - 9 dx = \frac{1}{5} x^5 + \frac{3}{2} x^2 - 9x + c$$

4. In this case, $x=1$ is between the limits of integration. Thus we can write the integral as follows,

$$\int_{-2}^3 f(x) dx = \int_{-2}^1 f(x) dx + \int_1^3 f(x) dx$$

The mathematics is then,

$$\begin{aligned} \int_{-2}^3 f(x) dx &= \int_{-2}^1 f(x) dx + \int_1^3 f(x) dx \\ &= \int_{-2}^1 3x^2 dx + \int_1^3 6 dx \\ &= x^3 \Big|_{-2}^1 + 6x \Big|_1^3 \\ &= 1 - (-8) + (18 - 6) \\ &= 21 \end{aligned}$$

5. It looks like the substitution should be,

$$u = 4y^2 - y \quad du = (8y - 1) dy$$

The integral is then,

$$\begin{aligned} \int 3(8y - 1) e^{4y^2 - y} dy &= 3 \int e^u du \\ &= 3e^u + c \\ &= 3e^{4y^2 - y} + c \end{aligned}$$

6. Let's first get the most general possible first derivative by integrating the second derivative.

$$\begin{aligned} f'(x) &= \int f''(x) dx \\ &= \int 15x^{\frac{1}{2}} + 5x^3 + 6 dx \\ &= 15 \left(\frac{2}{3} \right) x^{\frac{3}{2}} + \frac{5}{4} x^4 + 6x + c \\ &= 10x^{\frac{3}{2}} + \frac{5}{4} x^4 + 6x + c \end{aligned}$$

We can now find the most general possible function by integrating the first derivative which we found above.

$$f(x) = \int 10x^{\frac{3}{2}} + \frac{5}{4}x^4 + 6x + c \, dx$$

$$= 4x^{\frac{5}{2}} + \frac{1}{4}x^5 + 3x^2 + cx + d$$

Now, plug in the two values of the function that we've got.

$$-\frac{5}{4} = f(1) = 4 + \frac{1}{4} + 3 + c + d = \frac{29}{4} + c + d$$

$$404 = f(4) = 4(32) + \frac{1}{4}(1024) + 3(16) + c(4) + d = 432 + 4c + d$$

This gives us a system of two equations in two unknowns that we can solve.

$$\begin{aligned} -\frac{5}{4} &= \frac{29}{4} + c + d & \Rightarrow & \quad c = -\frac{13}{2} \\ 404 &= 432 + 4c + d & & \quad d = -2 \end{aligned}$$

The function is then,

$$f(x) = 4x^{\frac{5}{2}} + \frac{1}{4}x^5 + 3x^2 - \frac{13}{2}x - 2$$

7. Given $z = f(x + iy) + g(x - iy)$

Diff. partially w.r.t. x and y we get

$$\frac{\partial z}{\partial x} = p = f'(x + iy) + g'(x - iy)$$

$$\frac{\partial z}{\partial y} = q = if'(x + iy) - ig'(x - iy)$$

Again differentiating w.r.t x and y we get

$$\frac{\partial^2 z}{\partial x^2} = f''(x + iy) + g''(x - iy)$$

$$\frac{\partial^2 z}{\partial y^2} = -f''(x + iy) - g''(x - iy)$$

Adding the above equations we get

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

8. Get things separated out and then integrate.

$$\begin{aligned}\frac{1}{r^2} dr &= \frac{1}{\theta} d\theta \\ \int \frac{1}{r^2} dr &= \int \frac{1}{\theta} d\theta \\ -\frac{1}{r} &= \ln|\theta| + c\end{aligned}$$

Now, apply the initial condition to find c.

$$-\frac{1}{2} = \ln(1) + c \quad c = -\frac{1}{2}$$

So, the implicit solution is then,

$$-\frac{1}{r} = \ln|\theta| - \frac{1}{2}$$

Solving for r gets us our explicit solution.

$$r = \frac{1}{\frac{1}{2} - \ln|\theta|}$$

INSTITUTE OF ACTUARIAL
& QUANTITATIVE STUDIES

9. First, we need to get the differential equation in the correct form.

$$\frac{dv}{dt} + 0.196v = 9.8$$

From this we can see that $p(t)=0.196$ and so $\mu(t)$ is then.

$$\mu(t) = e^{\int 0.196 dt} = e^{0.196t}$$

Now multiply all the terms in the differential equation by the integrating factor and do some simplification.

$$\begin{aligned}e^{0.196t} \frac{dv}{dt} + 0.196e^{0.196t}v &= 9.8e^{0.196t} \\ (e^{0.196t}v)' &= 9.8e^{0.196t}\end{aligned}$$

Integrate both sides and don't forget the constants of integration that will arise from both integrals.

$$\int (e^{0.196t}v)' dt = \int 9.8e^{0.196t} dt$$

$$e^{0.196t}v + k = 50e^{0.196t} + c$$

We can subtract k from both sides to get.

$$e^{0.196t}v = 50e^{0.196t} + c - k$$

Both c and k are unknown constants and so the difference is also an unknown constant. We will therefore write the difference as c. So, we now have

$$e^{0.196t}v = 50e^{0.196t} + c$$

The final step in the solution process is then to divide both sides by $e^{-0.196t}$ or to multiply both sides by $e^{-0.196t}$. Either will work, but we usually prefer the multiplication route. Doing this gives the general solution to the differential equation.

$$v(t) = 50 + ce^{-0.196t}$$

10. First, divide through by the t to get the differential equation into the correct form.

$$y' + \frac{2}{t}y = t - 1 + \frac{1}{t}$$

Now let's get the integrating factor, $\mu(t)$.

$$\mu(t) = e^{\int \frac{2}{t} dt} = e^{2 \ln |t|}$$

We need to simplify $\mu(t)$, recall that

$$\ln x^r = r \ln x$$

and rewrite the integrating factor in a form that will allow us to simplify it.

$$\mu(t) = e^{2 \ln |t|} = e^{\ln |t|^2} = |t|^2 = t^2$$

Now, multiply the rewritten differential equation (remember we can't use the original differential equation here...) by the integrating factor.

$$(t^2y)' = t^3 - t^2 + t$$

Integrate both sides and solve for the solution.

$$\begin{aligned}
 t^2 y &= \int t^3 - t^2 + t \, dt \\
 &= \frac{1}{4} t^4 - \frac{1}{3} t^3 + \frac{1}{2} t^2 + c \\
 y(t) &= \frac{1}{4} t^2 - \frac{1}{3} t + \frac{1}{2} + \frac{c}{t^2}
 \end{aligned}$$

Finally, apply the initial condition to get the value of cc.

$$\frac{1}{2} = y(1) = \frac{1}{4} - \frac{1}{3} + \frac{1}{2} + c \quad \Rightarrow \quad c = \frac{1}{12}$$

The solution is then,

$$y(t) = \frac{1}{4} t^2 - \frac{1}{3} t + \frac{1}{2} + \frac{1}{12 t^2}$$

11. First separate and then integrate both sides.

$$\begin{aligned}
 y^{-3} dy &= x(1+x^2)^{-\frac{1}{2}} dx \\
 \int y^{-3} dy &= \int x(1+x^2)^{-\frac{1}{2}} dx \\
 -\frac{1}{2y^2} &= \sqrt{1+x^2} + c
 \end{aligned}$$

Apply the initial condition to get the value of cc.

$$-\frac{1}{2} = \sqrt{1} + c \quad c = -\frac{3}{2}$$

The implicit solution is then,

$$-\frac{1}{2y^2} = \sqrt{1+x^2} - \frac{3}{2}$$

Now let's solve for y(x).

$$\frac{1}{y^2} = 3 - 2\sqrt{1+x^2}$$

$$y^2 = \frac{1}{3 - 2\sqrt{1+x^2}}$$

$$y(x) = \pm \frac{1}{\sqrt{3 - 2\sqrt{1+x^2}}}$$

Reapplying the initial condition shows us that the “-” is the correct sign. The explicit solution is then,

$$y(x) = -\frac{1}{\sqrt{3 - 2\sqrt{1+x^2}}}$$

Let’s get the interval of validity. That’s easier than it might look for this problem.

First,

since $1+x^2 \geq 0$ the “inner” root will not be a problem. Therefore, all we need to worry about is division by zero and negatives under the “outer” root. We can take care of both by requiring

$$3 - 2\sqrt{1+x^2} > 0$$

$$3 > 2\sqrt{1+x^2}$$

$$9 > 4(1+x^2)$$

$$\frac{9}{4} > 1+x^2$$

$$\frac{5}{4} > x^2$$

Finally solving for x we see that the only possible range of x ’s that will not give division by zero or square roots of negative numbers will be,

$$-\frac{\sqrt{5}}{2} < x < \frac{\sqrt{5}}{2}$$

and nicely enough this also contains the initial condition $x=0, y=0$. This interval is therefore our interval of validity.

12. Here are the derivatives for this problem.

$$\begin{aligned}
 f^{(0)}(x) &= x^3 - 10x^2 + 6 & f^{(0)}(3) &= -57 \\
 f^{(1)}(x) &= 3x^2 - 20x & f^{(1)}(3) &= -33 \\
 f^{(2)}(x) &= 6x - 20 & f^{(2)}(3) &= -2 \\
 f^{(3)}(x) &= 6 & f^{(3)}(3) &= 6 \\
 f^{(n)}(x) &= 0 & f^{(4)}(3) &= 0 \quad n \geq 4
 \end{aligned}$$

This Taylor series will terminate after $n=3$. This will always happen when we are finding the Taylor Series of a polynomial.

Here is the Taylor Series for this one

$$\begin{aligned}
 x^3 - 10x^2 + 6 &= \sum_{n=0}^{\infty} \frac{f^{(n)}(3)}{n!} (x-3)^n \\
 &= f(3) + f'(3)(x-3) + \frac{f''(3)}{2!}(x-3)^2 + \frac{f'''(3)}{3!}(x-3)^3 + 0 \\
 &= -57 - 33(x-3) - (x-3)^2 + (x-3)^3
 \end{aligned}$$

13. Let $f(x) = (1+x)^\mu$ where μ is a real number and $x \neq -1$. Then we can write the derivatives as follows

$$f'(x) = \mu(1+x)^{\mu-1},$$

$$f''(x) = \mu(\mu-1)(1+x)^{\mu-2},$$

$$f'''(x) = \mu(\mu-1)(\mu-2) \cdot (1+x)^{\mu-3},$$

$$f^{(n)}(x) = \mu(\mu-1)(\mu-2) \cdots (\mu-n+1)(1+x)^{\mu-n}.$$

For $x=0$, we obtain

$$f(0) = 1, \quad f'(0) = \mu, \quad f''(0) = \mu(\mu-1), \dots, \quad f^{(n)}(0) = \mu(\mu-1) \cdots (\mu-n+1).$$

Hence, the series expansion can be written in the form

$$(1+x)^\mu = 1 + \mu x + \frac{\mu(\mu-1)}{2!}x^2 + \frac{\mu(\mu-1)(\mu-2)}{3!}x^3 + \dots$$

$$+ \frac{\mu(\mu-1)\cdots(\mu-n+1)}{n!}x^n + \dots$$

14. Using the binomial series found in the previous example and substituting $\mu = \frac{1}{2}$, we get

$$\sqrt{1+x} = (1+x)^{\frac{1}{2}} = 1 + \frac{x}{2} + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}x^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}x^3 + \dots$$

$$= 1 + \frac{x}{2} - \frac{1 \cdot x^2}{2^2 2!} + \frac{1 \cdot 3 \cdot x^3}{2^3 3!} - \frac{1 \cdot 3 \cdot 5 \cdot x^3}{2^4 4!} + \dots$$

$$+ (-1)^{n+1} \cdot \frac{1 \cdot 3 \cdot 5 \cdots (2n-3) x^n}{2^n n!}.$$

Keeping only the first three terms, we can write this series as

$$\sqrt{1+x} \approx 1 + \frac{x}{2} - \frac{x^2}{8}.$$

15. We'll need to start over from the beginning and start taking some derivatives of the function.

$$\begin{aligned} n=0: & \quad f(x) = e^{-6x} \\ n=1: & \quad f'(x) = -6e^{-6x} \\ n=2: & \quad f''(x) = (-6)^2 e^{-6x} \\ n=3: & \quad f^{(3)}(x) = (-6)^3 e^{-6x} \\ n=4: & \quad f^{(4)}(x) = (-6)^4 e^{-6x} \end{aligned}$$

It is now time to see if we can get a formula for the general term in the Taylor Series.

$$f^{(n)}(x) = (-6)^n e^{-6x} \quad n = 0, 1, 2, 3, \dots$$

Recall that we don't really want the general term at any x . We want the general term at $x = -4$. This is,

$$f^{(n)}(-4) = (-6)^n e^{24} \quad n = 0, 1, 2, 3, \dots$$

The Taylor Series for this problem.

$$e^{-6x} = \sum_{n=0}^{\infty} \frac{f^{(n)}(-4)}{n!} (x+4)^n = \boxed{\sum_{n=0}^{\infty} \frac{(-6)^n e^{24}}{n!} (x+4)^n}$$

16. We'll need to start off this problem by taking a few derivatives of the function.

$$n = 0 : \quad f(x) = \ln(3 + 4x)$$

$$n = 1 : \quad f'(x) = \frac{4}{3 + 4x} = 4(3 + 4x)^{-1}$$

$$n = 2 : \quad f''(x) = -4^2(3 + 4x)^{-2}$$

$$n = 3 : \quad f^{(3)}(x) = 4^3(2)(3 + 4x)^{-3}$$

$$n = 4 : \quad f^{(4)}(x) = -4^4(2)(3)(3 + 4x)^{-4}$$

$$n = 5 : \quad f^{(5)}(x) = 4^5(2)(3)(4)(3 + 4x)^{-5}$$

The general term is given by,

$$f^{(0)}(x) = \ln(3 + 4x)$$

$$n = 0$$

$$f^{(n)}(x) = (-1)^{n+1} 4^n (n-1)! (3 + 4x)^{-n}$$

$$n = 1, 2, 3, \dots$$

We want the general term at $x=0$. This is,

$$f^{(0)}(0) = \ln(3)$$

$$n = 0$$

$$f^{(n)}(0) = (-1)^{n+1} 4^n (n-1)! (3)^{-n}$$

$$= (-1)^{n+1} 4^n (n-1)! \frac{1}{3^n}$$

$$= (-1)^{n+1} \left(\frac{4}{3}\right)^n (n-1)! \quad n = 1, 2, 3, \dots$$

The Taylor Series for this problem.

$$\begin{aligned}
\ln(3+4x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \\
&= f(0) + \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \\
&= \ln(3) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \left(\frac{4}{3}\right)^n (n-1)!}{n!} x^n \\
&= \ln(3) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \left(\frac{4}{3}\right)^n}{n} x^n
\end{aligned}$$

Answer 17:

20. Answer. $\int \frac{x^4 - 6x^3 + e^x \sqrt{x}}{\sqrt{x}} dx.$

The integral can be rewritten as

$$\int (x^{7/2} - 6x^{5/2} + e^x) dx$$

which equals $\frac{2}{9}x^{9/2} - \frac{12}{7}x^{7/2} + e^x + C.$

INSTITUTE OF ACTUARIAL
& QUANTITATIVE STUDIES

Answer 18:

Given the differential equation: $x(dy/dx) - y = 2x^2$

Now, divide both sides of the equation by x , we get

$$(x/x)(dy/dx) - (y/x) = (2x^2/x)$$

$$(dy/dx) - (y/x) = 2x \dots (1)$$

Hence, the differential equation is of the form:

$$(dy/dx) + Py = Q \dots (2)$$

Comparing the equations (1) and (2), we get

$$P = -1/x, \text{ and } Q = 2x.$$

We know that the Integrating factor, $IF = e^{\int P dx}$

Now, substitute the obtained values in the above formula, and we get

$$IF = e^{\int (-1/x) dx}$$

$$IF = e^{-\log x}$$

$$\text{As, } x \log y = \log y^x,$$

$$IF = e^{\log x^{-1}}$$

$$\text{Hence, } IF = x^{-1} \text{ [Since } e^{\log x} = x]$$

$$IF = 1/x \text{ [As, } x^{-1} = 1/x]$$

Therefore, the integrating factor (IF) of the differential equation $x(dy/dx) - y = 2x^2$ is $1/x$.

Answer 19:

$$n = 0 : \quad f(x) = \frac{7}{x^4} = 7x^{-4}$$

$$n = 1 : \quad f'(x) = -7(4)x^{-5}$$

$$n = 2 : \quad f''(x) = 7(4)(5)x^{-6}$$

$$n = 3 : \quad f^{(3)}(x) = -7(4)(5)(6)x^{-7}$$

$$n = 4 : \quad f^{(4)}(x) = 7(4)(5)(6)(7)x^{-8}$$

$$\begin{aligned}
 f^{(n)}(x) &= 7(-1)^n \frac{(2)(3)}{(2)(3)} (4)(5)(6) \cdots (n+3) x^{-(n+4)} \\
 &= 7(-1)^n \frac{(2)(3)(4)(5)(6) \cdots (n+3)}{6} x^{-(n+4)} \\
 &= \frac{7}{6} (-1)^n (n+3)! x^{-(n+4)} \quad n = 0, 1, 2, 3, \dots
 \end{aligned}$$

$$\begin{aligned}
 f^{(n)}(-3) &= \frac{7}{6} (-1)^n (n+3)! (-3)^{-(n+4)} \\
 &= \frac{7(-1)^n (n+3)!}{6(-3)^{n+4}} \\
 &= \frac{7(-1)^n (n+3)!}{6(-1)^{n+4} (3)^{n+4}} \\
 &= \frac{7(n+3)!}{6(-1)^4 (3)^{n+4}} \\
 &= \frac{7(n+3)!}{6(3)^{n+4}} \quad n = 1, 2, 3, \dots
 \end{aligned}$$

INSTITUTE OF ACTUARIAL
INITIATIVE STUDIES

Okay, at this point we can formally write down the Taylor Series for this problem.

$$\frac{7}{x^4} = \sum_{n=0}^{\infty} \frac{f^{(n)}(-3)}{n!} (x+3)^n = \sum_{n=0}^{\infty} \frac{7(n+3)!}{6(3)^{n+4} n!} (x+3)^n = \boxed{\sum_{n=0}^{\infty} \frac{7(n+3)(n+2)(n+1)}{6(3)^{n+4}} (x+3)^n}$$

Don't forget to simplify/cancel where we can in the final answer. In this case we could do some simplifying with the factorials.

Answer 20:

Recalling that the derivative of the exponential function is

$$f'(x) = e^x$$

In fact, all the derivatives are

$$e^x$$

.

$$f'(0) = e^0 = 1$$

$$f''(0) = e^0 = 1$$

$$f'''(0) = e^0 = 1$$

We see that all the derivatives, when evaluated at $x = 0$, give us the value 1.

Also, $f(0) = 1$, so we can conclude the Maclaurin Series expansion will be simply:

$$e^x \approx 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \dots$$