INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Subject: Calculus

Chapter: 5,6,7 (Unit 3 & 4)

Category: Assignment 2 Solution

1. Here it is,

$$u = \frac{2}{w} \qquad du = -\frac{2}{w^2}\,dw \qquad \Rightarrow \qquad \frac{1}{w^2}\,dw = -\frac{1}{2}\,du$$

$$w = 2 \qquad \Rightarrow \qquad u = 1 \qquad w = \frac{1}{50} \qquad \Rightarrow \qquad u = 100$$

Here is the integral.

$$\begin{aligned}\int_{\frac{1}{50}}^{2} \frac{\mathbf{e}^{\frac{2}{w}}}{w^2}\,dw &= -\frac{1}{2}\int_{100}^{1} \mathbf{e}^u\,du \\ &= -\frac{1}{2}\mathbf{e}^u\Big|_{100}^{1} \\ &= -\frac{1}{2}\left(\mathbf{e}^1 - \mathbf{e}^{100}\right)\end{aligned}$$

2. Clearly the derivative of the denominator, ignoring the exponent, differs from the numerator only by a multiplicative constant and so the substitution is,

$$u = t^4 + 2t \qquad du = \left(4t^3 + 2\right)dt = 2\left(2t^3 + 1\right)dt \qquad \Rightarrow \qquad \left(2t^3 + 1\right)dt = \frac{1}{2}\,du$$

After a little manipulation of the differential we get the following integral.

$$\begin{aligned}\int \frac{2t^3 + 1}{\left(t^4 + 2t\right)^3}\,dt &= \frac{1}{2}\int \frac{1}{u^3}\,du \\ &= \frac{1}{2}\int u^{-3}\,du \\ &= \frac{1}{2}\left(-\frac{1}{2}\right)u^{-2} + c \\ &= -\frac{1}{4}\left(t^4 + 2t\right)^{-2} + c\end{aligned}$$

3. The solution is;

$$f(x) = \int f'(x)\,dx = \int x^4 + 3x - 9\,dx = \frac{1}{5}x^5 + \frac{3}{2}x^2 - 9x + c$$

4. In this case, x=1 is between the limits of integration. Thus we can write the integral as follows,

$$\int_{-2}^{3} f(x)\,dx = \int_{-2}^{1} f(x)\,dx + \int_{1}^{3} f(x)\,dx$$

The mathematics is then,

$$\begin{aligned}
\int_{-2}^{3} f(x)\, dx &= \int_{-2}^{1} f(x)\, dx + \int_{1}^{3} f(x)\, dx \\
&= \int_{-2}^{1} 3x^2\, dx + \int_{1}^{3} 6\, dx \\
&= x^3\Big|_{-2}^{1} + 6x\Big|_{1}^{3} \\
&= 1 - (-8) + (18 - 6) \\
&= 21
\end{aligned}$$

5. It looks like the substitution should be,

$$u = 4y^2 - y \qquad du = (8y - 1)\, dy$$

The integral is then,

$$\begin{aligned}
\int 3(8y-1)\,\mathbf{e}^{4y^2-y}\, dy &= 3\int \mathbf{e}^u\, du \\
&= 3\mathbf{e}^u + c \\
&= 3\mathbf{e}^{4y^2-y} + c
\end{aligned}$$

6. Let's first get the most general possible first derivative by integrating the second derivative.

$$\begin{aligned}
f'(x) &= \int f''(x)\, dx \\
&= \int 15x^{\frac{1}{2}} + 5x^3 + 6\, dx \\
&= 15\left(\frac{2}{3}\right)x^{\frac{3}{2}} + \frac{5}{4}x^4 + 6x + c \\
&= 10x^{\frac{3}{2}} + \frac{5}{4}x^4 + 6x + c
\end{aligned}$$

We can now find the most general possible function by integrating the first derivative which we found above.

$$\begin{aligned}
f(x) &= \int 10x^{\frac{3}{2}} + \frac{5}{4}x^4 + 6x + c\, dx \\
&= 4x^{\frac{5}{2}} + \frac{1}{4}x^5 + 3x^2 + cx + d
\end{aligned}$$

Now, plug in the two values of the function that we've got.

$$-\frac{5}{4}=f(1)=4+\frac{1}{4}+3+c+d=\frac{29}{4}+c+d$$

$$404=f(4)=4(32)+\frac{1}{4}(1024)+3(16)+c(4)+d=432+4c+d$$

This gives us a system of two equations in two unknowns that we can solve.

$$\begin{array}{lcl} -\frac{5}{4}=\frac{29}{4}+c+d & \Rightarrow & c=-\frac{13}{2} \\ 404=432+4c+d & & d=-2 \end{array}$$

The function is then,

$$f(x)=4x^{\frac{5}{2}}+\frac{1}{4}x^5+3x^2-\frac{13}{2}x-2$$

7. Given z = f (x + iy) + g (x – iy)

Diff. partially w.r.t. x and y we get

$$\frac{\partial z}{\partial x}=p=f'(x+iy)+g'(x-iy)$$

$$\frac{\partial z}{\partial y}=q=if'(x+iy)-ig'(x-iy)$$

Again differentiating w.r.t x and y we get

$$\frac{\partial^2 z}{\partial x^2}=f''(x+iy)+g''(x-iy)$$

$$\frac{\partial^2 z}{\partial y^2}=-f''(x+iy)-g''(x-iy)$$

Adding the above equations we get

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0$$

8. Get things separated out and then integrate.

$$\frac{1}{r^2}dr=\frac{1}{\theta}d\theta$$

$$\int\frac{1}{r^2}dr=\int\frac{1}{\theta}d\theta$$

$$-\frac{1}{r}=\ln|\theta|+c$$

Now, apply the initial condition to find c.

$$-\frac{1}{2}=\ln(1)+c \quad c=-\frac{1}{2}$$

So, the implicit solution is then,

$$-\frac{1}{r}=\ln|\theta|-\frac{1}{2}$$

Solving for r gets us our explicit solution.

$$r=\frac{1}{\frac{1}{2}-\ln|\theta|}$$

9. First, we need to get the differential equation in the correct form.

$$\frac{dv}{dt}+0.196v=9.8$$

From this we can see that p(t)=0.196 and so μ(t) is then.

$$\mu\left(t\right)=\mathbf{e}^{\int 0.196dt}=\mathbf{e}^{0.196t}$$

Now multiply all the terms in the differential equation by the integrating factor and do some simplification.

$$\begin{aligned}\mathbf{e}^{0.196t}\frac{dv}{dt}+0.196\mathbf{e}^{0.196t}v&=9.8\mathbf{e}^{0.196t}\\ \left(\mathbf{e}^{0.196t}v\right)'&=9.8\mathbf{e}^{0.196t}\end{aligned}$$

Integrate both sides and don't forget the constants of integration that will arise from both integrals.

$$\begin{aligned}\int\left(\mathbf{e}^{0.196t}v\right)'dt&=\int 9.8\mathbf{e}^{0.196t}\,dt\\ \mathbf{e}^{0.196t}v+k&=50\mathbf{e}^{0.196t}+c\end{aligned}$$

We can subtract k from both sides to get.

$$\mathbf{e}^{0.196t}v=50\mathbf{e}^{0.196t}+c-k$$

Both c and k are unknown constants and so the difference is also an unknown constant. We will therefore write the difference as c. So, we now have

$$\mathbf{e}^{0.196t}v=50\mathbf{e}^{0.196t}+c$$

The final step in the solution process is then to divide both sides by $e^{-0.196t}$ or to multiply both sides by $e^{-0.196t}$. Either will work, but we usually prefer the multiplication route. Doing this gives the general solution to the differential equation.

$$v(t) = 50 + ce^{-0.196t}$$

10. First, divide through by the t to get the differential equation into the correct form.

$$y' + \frac{2}{t}y = t - 1 + \frac{1}{t}$$

Now let's get the integrating factor, μ(t).

$$\mu(t) = \mathbf{e}^{\int \frac{2}{t}dt} = \mathbf{e}^{2\ln|t|}$$

We need to simplify μ(t), recall that

$$\ln x^r = r\ln x$$

and rewrite the integrating factor in a form that will allow us to simplify it.

$$\mu(t) = \mathbf{e}^{2\ln|t|} = \mathbf{e}^{\ln|t|^2} = |t|^2 = t^2$$

Now, multiply the rewritten differential equation (remember we can't use the original differential equation here...) by the integrating factor.

$$(t^2y)' = t^3 - t^2 + t$$

Integrate both sides and solve for the solution.

$$\begin{aligned} t^2y &= \int t^3 - t^2 + t\,dt \\ &= \frac{1}{4}t^4 - \frac{1}{3}t^3 + \frac{1}{2}t^2 + c \\ y(t) &= \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2} + \frac{c}{t^2} \end{aligned}$$

Finally, apply the initial condition to get the value of cc.

$$\frac{1}{2} = y(1) = \frac{1}{4} - \frac{1}{3} + \frac{1}{2} + c \quad \Rightarrow \quad c = \frac{1}{12}$$

The solution is then,

$$y(t) = \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2} + \frac{1}{12t^2}$$

11. First separate and then integrate both sides.

$$y^{-3}dy = x\left(1+x^2\right)^{-\frac{1}{2}}dx$$

$$\int y^{-3}dy = \int x\left(1+x^2\right)^{-\frac{1}{2}}dx$$

$$-\frac{1}{2y^2} = \sqrt{1+x^2}+c$$

Apply the initial condition to get the value of cc.

$$-\frac{1}{2} = \sqrt{1}+c \qquad c = -\frac{3}{2}$$

The implicit solution is then,

$$-\frac{1}{2y^2} = \sqrt{1+x^2}-\frac{3}{2}$$

Now let's solve for y(x).

$$\frac{1}{y^2} = 3-2\sqrt{1+x^2}$$

$$y^2 = \frac{1}{3-2\sqrt{1+x^2}}$$

$$y(x) = \pm\frac{1}{\sqrt{3-2\sqrt{1+x^2}}}$$

Reapplying the initial condition shows us that the "–" is the correct sign. The explicit solution is then,

$$y(x) = -\frac{1}{\sqrt{3-2\sqrt{1+x^2}}}$$

Let's get the interval of validity. That's easier than it might look for this problem. First, since $1+x^2 \geq 0$ the "inner" root will not be a problem. Therefore, all we need to worry about is division by zero and negatives under the "outer" root. We can take care of both by requiring

$$
\begin{aligned}
3-2\sqrt{1+x^2} &> 0\\
3 &> 2\sqrt{1+x^2}\\
9 &> 4\left(1+x^2\right)\\
\frac{9}{4} &> 1+x^2\\
\frac{5}{4} &> x^2
\end{aligned}
$$

Finally solving for xx we see that the only possible range of xx's that will not give division by zero or square roots of negative numbers will be,

$$-\frac{\sqrt{5}}{2} < x < \frac{\sqrt{5}}{2}$$

and nicely enough this also contains the initial condition x=0x=0. This interval is therefore our interval of validity.

12. Here are the derivatives for this problem.

$$
\begin{aligned}
f^{(0)}(x) &= x^3-10x^2+6 & f^{(0)}(3) &= -57\\
f^{(1)}(x) &= 3x^2-20x & f^{(1)}(3) &= -33\\
f^{(2)}(x) &= 6x-20 & f^{(2)}(3) &= -2\\
f^{(3)}(x) &= 6 & f^{(3)}(3) &= 6\\
f^{(n)}(x) &= 0 & f^{(4)}(3) &= 0 \quad n \geq 4
\end{aligned}
$$

This Taylor series will terminate after n=3n=3. This will always happen when we are finding the Taylor Series of a polynomial.
Here is the Taylor Series for this one

$$
\begin{aligned}
x^3-10x^2+6 &= \sum_{n=0}^{\infty} \frac{f^{(n)}(3)}{n!}(x-3)^n\\
&= f(3)+f'(3)(x-3)+\frac{f''(3)}{2!}(x-3)^2+\frac{f'''(3)}{3!}(x-3)^3+0\\
&= -57-33(x-3)-(x-3)^2+(x-3)^3
\end{aligned}
$$

13. Let f(x) = $(1 + x)^{\mu}$ where μ is a real number and x≠–1. Then we can write the derivatives as follows

$$f'(x) = \mu(1+x)^{\mu-1},$$

$$f''(x) = \mu(\mu-1)(1+x)^{\mu-2},$$

$$f'''(x) = \mu(\mu-1)(\mu-2)\cdot(1+x)^{\mu-3},$$

$$f^{(n)}(x) = \mu(\mu-1)(\mu-2)\cdots(\mu-n+1)(1+x)^{\mu-n}.$$

For x=0, we obtain

$$f(0) = 1,\ f'(0) = \mu,\ f''(0) = \mu(\mu-1), \ldots\ f^{(n)}(0) = \mu(\mu-1)\cdots(\mu-n+1).$$

Hence, the series expansion can be written in the form

$$(1+x)^{\mu} = 1 + \mu x + \frac{\mu(\mu-1)}{2!}x^2 + \frac{\mu(\mu-1)(\mu-2)}{3!}x^2 + \ldots + \frac{\mu(\mu-1)\cdots(\mu-n+1)}{n!}x^n + \ldots$$

14. Using the binomial series found in the previous example and substituting μ = $\frac{1}{2}$, we get

$$\sqrt{1+x} = (1+x)^{\frac{1}{2}} = 1 + \frac{x}{2} + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}x^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}x^3 + \ldots$$
$$= 1 + \frac{x}{2} - \frac{1\cdot x^2}{2^2 2!} + \frac{1\cdot 3\cdot x^3}{2^3 3!} - \frac{1\cdot 3\cdot 5\cdot x^3}{2^4 4!} + \ldots$$
$$+ (-1)^{n+1}\cdot\frac{1\cdot 3\cdot 5\cdots(2n-3)\,x^n}{2^n n!}.$$

Keeping only the first three terms, we can write this series as

$$\sqrt{1+x} \approx 1 + \frac{x}{2} - \frac{x^2}{8}.$$

15. We'll need to start over from the beginning and start taking some derivatives of the function.

$$n=0: \quad f(x)=\mathbf{e}^{-6x}$$
$$n=1: \quad f'(x)=-6\mathbf{e}^{-6x}$$
$$n=2: \quad f''(x)=(-6)^2\mathbf{e}^{-6x}$$
$$n=3: \quad f^{(3)}(x)=(-6)^3\mathbf{e}^{-6x}$$
$$n=4: \quad f^{(4)}(x)=(-6)^4\mathbf{e}^{-6x}$$

It is now time to see if we can get a formula for the general term in the Taylor Series.

$$f^{(n)}(x)=(-6)^n\mathbf{e}^{-6x} \quad n=0,1,2,3,\ldots$$

Recall that we don't really want the general term at any x. We want the general term at x = −4. This is,

$$f^{(n)}(-4)=(-6)^n\mathbf{e}^{24} \quad n=0,1,2,3,\ldots$$

The Taylor Series for this problem.

$$\mathbf{e}^{-6x}=\sum_{n=0}^{\infty}\frac{f^{(n)}(-4)}{n!}(x+4)^n=\boxed{\sum_{n=0}^{\infty}\frac{(-6)^n\mathbf{e}^{24}}{n!}(x+4)^n}$$

16. We'll need to start off this problem by taking a few derivatives of the function.

$$n=0: \quad f(x)=\ln(3+4x)$$
$$n=1: \quad f'(x)=\frac{4}{3+4x}=4(3+4x)^{-1}$$
$$n=2: \quad f''(x)=-4^2(3+4x)^{-2}$$
$$n=3: \quad f^{(3)}(x)=4^3(2)(3+4x)^{-3}$$
$$n=4: \quad f^{(4)}(x)=-4^4(2)(3)(3+4x)^{-4}$$
$$n=5: \quad f^{(5)}(x)=4^5(2)(3)(4)(3+4x)^{-5}$$

The general term is given by,

$$f^{(0)}(x)=\ln(3+4x) \qquad n=0$$
$$f^{(n)}(x)=(-1)^{n+1}4^n(n-1)!(3+4x)^{-n} \qquad n=1,2,3,..$$

We want the general term at x=0x=0. This is,

$$
\begin{aligned}
f^{(0)}(0) &= \ln(3) & n&=0\\
f^{(n)}(0) &= (-1)^{n+1}4^n\,(n-1)!(3)^{-n}\\
&= (-1)^{n+1}4^n\,(n-1)!\frac{1}{3^n}\\
&= (-1)^{n+1}\left(\frac{4}{3}\right)^n(n-1)! & n&=1,2,3,\ldots
\end{aligned}
$$

The Taylor Series for this problem.

$$
\begin{aligned}
\ln(3+4x) &= \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n\\
&= f(0)+\sum_{n=1}^{\infty}\frac{f^{(n)}(0)}{n!}x^n\\
&= \ln(3)+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(\frac{4}{3}\right)^n(n-1)!}{n!}x^n\\
&= \boxed{\ln(3)+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(\frac{4}{3}\right)^n}{n}x^n}
\end{aligned}
$$

Answer 17:

20. Answer. $\displaystyle\int \frac{x^4-6x^3+e^x\sqrt{x}}{\sqrt{x}}\,dx.$

The integral can be rewritten as

$$\int (x^{7/2}-6x^{5/2}+e^x)\,dx$$

which equals $\frac{2}{9}x^{9/2}-\frac{12}{7}x^{7/2}+e^x+C.$

Answer 18:

Given the differential equation: $x(dy/dx) - y = 2x^2$

Now, divide both sides of the equation by x, we get

$(x/x)(dy/dx) - (y/x) = (2x^2/x)$

$(dy/dx) - (y/x) = 2x$ ...(1)

Hence, the differential equation is of the form:

$(dy/dx) + Py = Q$ ...(2)

Comparing the equations (1) and (2), we get

$P = -1/x$, and $Q = 2x$.

We know that the Integrating factor, $IF = e^{\int P dx}$

Now, substitute the obtained values in the above formula, and we get

$IF = e^{\int (-1/x) dx}$

$IF = e^{-\log x}$

As, $x \log y = \log y^x$,

$$IF = e^{\log x^{-1}}$$

Hence, $IF = x^{-1}$ [Since $e^{\log x} = x$]

$IF = 1/x$ [As, $x^{-1} = 1/x$]

Therefore, the integrating factor (IF) of the differential equation $x(dy/dx) - y = 2x^2$ is $1/x$.

Answer 19:

$$\begin{aligned}
n = 0: &\qquad f(x) = \frac{7}{x^4} = 7x^{-4} \\
n = 1: &\qquad f'(x) = -7(4)x^{-5} \\
n = 2: &\qquad f''(x) = 7(4)(5)x^{-6} \\
n = 3: &\qquad f^{(3)}(x) = -7(4)(5)(6)x^{-7} \\
n = 4: &\qquad f^{(4)}(x) = 7(4)(5)(6)(7)x^{-8}
\end{aligned}$$

$$f^{(n)}(x) = 7(-1)^n \frac{(2)(3)}{(2)(3)}(4)(5)(6)\cdots(n+3)\,x^{-(n+4)}$$
$$= 7(-1)^n \frac{(2)(3)(4)(5)(6)\cdots(n+3)}{6}x^{-(n+4)}$$
$$= \frac{7}{6}(-1)^n (n+3)!x^{-(n+4)} \qquad n = 0,1,2,3,\ldots$$

$$f^{(n)}(-3) = \frac{7}{6}(-1)^n (n+3)!(-3)^{-(n+4)}$$
$$= \frac{7(-1)^n (n+3)!}{6(-3)^{n+4}}$$
$$= \frac{7(-1)^n (n+3)!}{6(-1)^{n+4}(3)^{n+4}}$$
$$= \frac{7(n+3)!}{6(-1)^4(3)^{n+4}}$$
$$= \frac{7(n+3)!}{6(3)^{n+4}} \qquad n = 1,2,3,\ldots$$

ıkay, at this point we can formally write down the Taylor Series for this problem.

$$\frac{7}{x^4} = \sum_{n=0}^{\infty} \frac{f^{(n)}(-3)}{n!}(x+3)^n = \sum_{n=0}^{\infty} \frac{7(n+3)!}{6(3)^{n+4}n!}(x+3)^n = \boxed{\sum_{n=0}^{\infty} \frac{7(n+3)(n+2)(n+1)}{6(3)^{n+4}}(x+3)^n}$$

on't forget to simplify/cancel where we can in the final answer. In this case we could do some simplifying with the factorials.

Answer 20:

Recalling that the derivative of the exponential function is

$f'(x) = e^x$

In fact, all the derivatives are

e^x

.

$f'(0) = e^0 = 1$

$f''(0) = e^0 = 1$

$f'''(0) = e^0 = 1$

We see that all the derivatives, when evaluated at $x = 0$, give us the value 1.

Also, $f(0)=1$, so we can conclude the Maclaurin Series expansion will be simply:

$$e^x \approx 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \ldots.$$