INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Subject: **Calculus**

Chapter: **Unit 1**

Category: **Practice questions**

Chapter 1

1. Construct a table of at least 4 ordered pairs of points on the graph of the function and use the ordered pairs from the table to sketch the graph of the function.

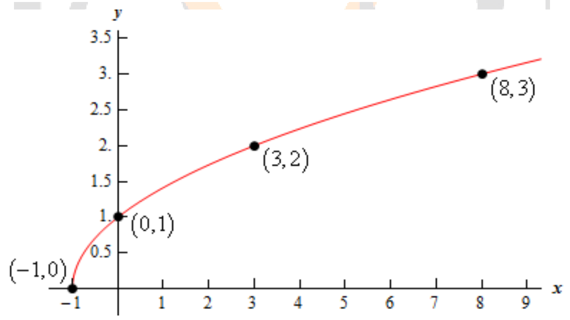
$$f(x) = \sqrt{x+1}$$

Answer :

1. Here is the table of points we'll use for this problem.

x	$f(x)$	(x,y)
-1	0	$(-1,0)$
0	1	$(0,1)$
3	2	$(3,2)$
8	3	$(8,3)$

Here is a sketch of the function.



2. Given A(x) = $\sqrt[5]{2x+11}$ find $A^{-1}(x)$

Answer:

Given that

$$x = \sqrt[5]{2y+11}$$

Solve the equation from above for y.

$$\begin{aligned} x^5 &= 2y+11 \\ x^5 - 11 &= 2y \\ \frac{x^5-11}{2} &= y \end{aligned}$$

Replace y with $A^{-1}(x)$.

$$A^{-1}(x) = \frac{x^5 - 11}{2}$$

Checking once,

$$\left(A \circ A^{-1}\right)(x) = A\left[A^{-1}(x)\right] = A\left[\frac{x^5 - 11}{2}\right]$$
$$= \sqrt[5]{2\left(\frac{x^5 - 11}{2}\right) + 11} = \sqrt[5]{x^5 - 11 + 11} = \sqrt[5]{x^5} = x$$

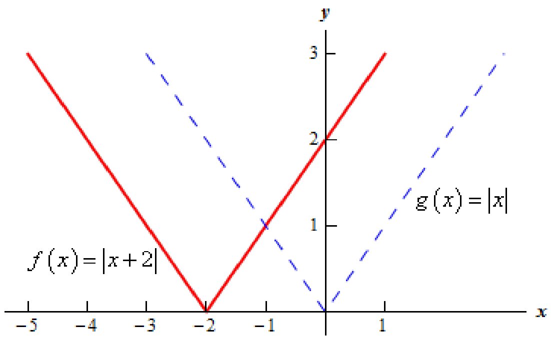
The check works out so we know we did the work correctly and have inverse.

3. Use transformations to sketch the graph of the following function.
f(x) = |x+2|

Answer:
Let's first identify the "base" function (*i.e.* the function we are transforming). In this case it looks like we are transforming g(x)=|x|
The function that have here looks like it can be written as,

$$f(x) = |x + 2| = g(x + 2)$$

Therefore, we can see that the graph of f(x) is simply going to be the graph of g(x) shifted left by 2.
Here is a sketch of both the base function (blue dashed curve) and the function we were asked to graph (red solid curve).



4. Given f(x) = 6x+2 and g(x) = 10−7x compute each of the following.
(a) (f-g)(2)
(b) (g-f)(2)

(c) f g

(d) $(\frac{f}{g})$ (x)

Answer 4:

a)

$$(f-g)(2) = f(2) - g(2) = 14 - (-4) = \boxed{18}$$

b)

$$(g-f)(2) = g(2) - f(2) = -4 - 14 = \boxed{-18}$$

c)

$$f\,g = f(x)\,g(x) = (6x+2)(10-7x) = \boxed{-42x^2 + 46x + 20}$$

d)

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \boxed{\frac{6x+2}{10-7x}}$$

5. Using ordered pairs of points sketch the graph of the following function.

$$f(x) = \begin{cases} 10 - 2x & if\ x < 2 \\ x^2 + 2 & if\ x \geq 2 \end{cases}$$

Answer:

We have a piecewise function here so we need to make sure and pick points from each range of x when building our table of points.

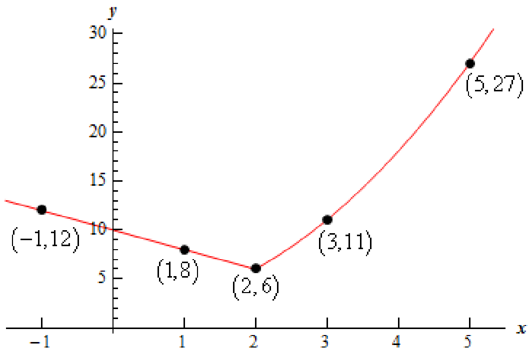
Also, we'll want to plug the "cutoff point",x=2, into the top equation as well so we know where it will be at the end of the range of x's for which it is valid.

Here is the table of points we'll use for this problem.

x	$10-2x$	(x,y)
-1	12	$(-1,12)$
1	8	$(1,8)$
2	6	$(2,6)$

x	x^2+2	(x,y)
2	6	$(2,6)$
3	11	$(3,11)$
5	27	$(5,27)$

Here is a sketch of the function.



Note that in this case the two pieces "met" at the point (2,6). Sometimes this will happen with these problems and sometimes it won't. This is the reason we always evaluate both equations at the "cutoff point". Without doing that we wouldn't know if the pieces meet or not.

6. Use transformations to sketch the graph of the following function.

$$f(x) = |x - 7| + 2$$

Answer 6:

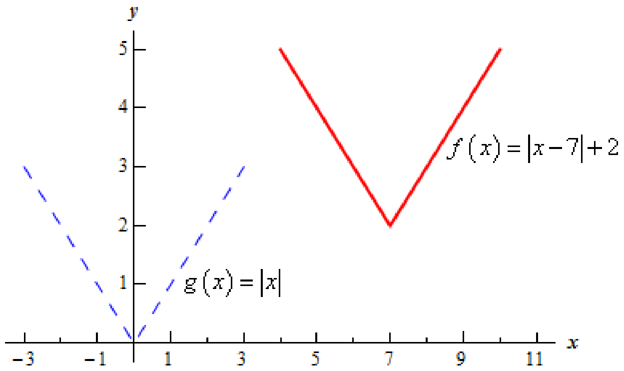
Let's first identify the "base" function (*i.e.* the function we are transforming). In this case it looks like we are transforming g(x)=|x| .

The function that have here looks like it can be written as,

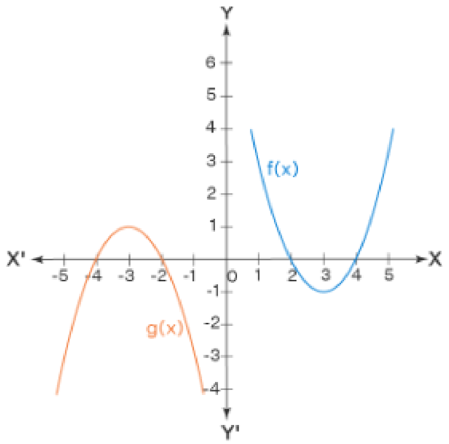
$$f(x) = |x - 7| + 2 = g(x - 7) + 2$$

Therefore, we can see that the graph of f(x) is simply going to be the graph of g(x) shifted right by 7 and up by 2.

Here is a sketch of both the base function (blue dashed curve) and the function we were asked to graph (red solid curve).



7. Write the function corresponding to the graph of g(x) that transformed from the graph f(x) by using the function transformation rules



Answer

CHAPTER NAME

PRACTICE/NOTES/ASSIGNMENT

Solution:

Take f(x) as the original function and observe how it is moving/transforming to give g(x). Observe the vertex of both graphs to get an idea. It is very clear that

- it moved 6 units to the left and so the function is f(x + 6).
- it then reflected with respect to the x-axis, so the function is -f(x + 6).

Answer: g(x) = - f(x + 6).

8. If a function f(x) is translated 3 units left and 4 units upward, then which of the following is the new function?

Answer:

- ○ f(x - 3) - 4
- ○ f(x - 3) + 4
- ◉ f(x + 3) + 4 ✔
- ○ f(x + 3) - 4

9. Find the inverse function of the function f(x) = 5x + 4.

Answer:

The given function is f(x) = 5x + 4

we rewrite it as y = 5x + 4 and simplify it to find the value of x.

y = 5x + 4

y - 4 = 5x

x = (y - 4)/5

$f^{-1}(x) = (x - 4)/5$

Answer: Therefore the inverse function is $f^{-1}(x) = (x - 4)/5$

10. Which of the following functions are even function?.

A. Cosx

B. x^2

C. Sin x

D. x

Answer: B

11. Perform the indicated function evaluations for $f(x)=3-5x-2x^2$

1. f(−3)
2. f(7−4x)

Answer:

1. $f(-3) = 3 - 5(-3) - 2(-3)^2 = 0$
2.

$$\begin{aligned} f(7-4x) &= 3 - 5(7-4x) - 2(7-4x)^2 \\ &= 3 - 5(7-4x) - 2\left(49 - 56x + 16x^2\right) \\ &= 3 - 35 + 20x - 98 + 112x - 32x^2 \\ &= -130 + 132x - 32x^2 \end{aligned}$$

Chapter 2

For problems 1 – 4 evaluate the limit, if it exists.

1. $\lim_{x \to 2}(8 - 3x + 12x^2)$

Answer:

We know that the first thing that we should try to do is simply plug in the value and see if we can compute the limit.

$$\lim_{x \to 2}\left(8 - 3x + 12x^2\right) = 8 - 3\left(2\right) + 12\left(4\right) = \boxed{50}$$

2. $\lim_{t \to -3} \frac{6+4t}{t^2+1}$

Answer:

$$\lim_{t \to -3} \frac{6 + 4t}{t^2 + 1} = \frac{-6}{10} = \boxed{-\frac{3}{5}}$$

3. $\lim_{z \to 8} \frac{2z^2-17z+8}{8-z}$

Answer:

In this case we see that if we plug in the value we get 0/0. Recall that this DOES NOT mean that the limit doesn't exist. We'll need to do some more work before we make that conclusion. All we need to do here is some simplification and then we'll reach a point where we can plug in the value.

$$\lim_{z \to 8} \frac{2z^2 - 17z + 8}{8 - z} = \lim_{z \to 8} \frac{\left(2z - 1\right)\left(z - 8\right)}{-\left(z - 8\right)} = \lim_{z \to 8} \frac{2z - 1}{-1} = \boxed{-15}$$

4. $\lim_{h \to 0} \frac{(6+h^2)-36}{h}$

Answer:

$$\lim_{h \to 0} \frac{\left(6 + h\right)^2 - 36}{h} = \lim_{h \to 0} \frac{36 + 12h + h^2 - 36}{h} = \lim_{h \to 0} \frac{h\left(12 + h\right)}{h} = \lim_{h \to 0}(12 + h) = \boxed{12}$$

5. Given the function

$$f(x) = \begin{cases} 7 - 4x & if\ x < 1 \\ x^2 + 2 & if\ x \geq 1 \end{cases}$$

Evaluate the following limits, if they exist.

$(a) \lim_{x \to -6} f(x)$ (b) $\lim_{x \to 1} f(x)$

Answer:

a) For this part we know that $-6 < 1$ and so there will be values of x on both sides of -6 in the range x<1 and so we can assume that, in the limit, we will have x<1. This will allow us to use the piece of the function in that range and then just use standard limit techniques to compute the limit.

$$\lim_{x \to -6} f(x) = \lim_{x \to -6} (7 - 4x) = \boxed{31}$$

b) his part is going to be different from the previous part. We are looking at the limit at x=1 and that is the "cut–off" point in the piecewise functions. Recall from the discussion in the section, that this means that we are going to have to look at the two one sided limits.

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (7 - 4x) = \underline{3} \qquad \text{because } x \to 1^- \text{ implies that } x < 1$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x^2 + 2) = \underline{3} \qquad \text{because } x \to 1^+ \text{ implies that } x > 1$$

So, in this case, we can see that,

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = 3$$

and so we know that the overall limit must exist and,

$$\lim_{x \to 1} f(x) = \boxed{3}$$

6. Below is the graph of f(x). For each of the given points determine the value of f(a),

$$\lim_{x \to a^-} f(x) \text{ , } \lim_{x \to a^+} f(x) \text{ and } \lim_{x \to a} f(x)$$

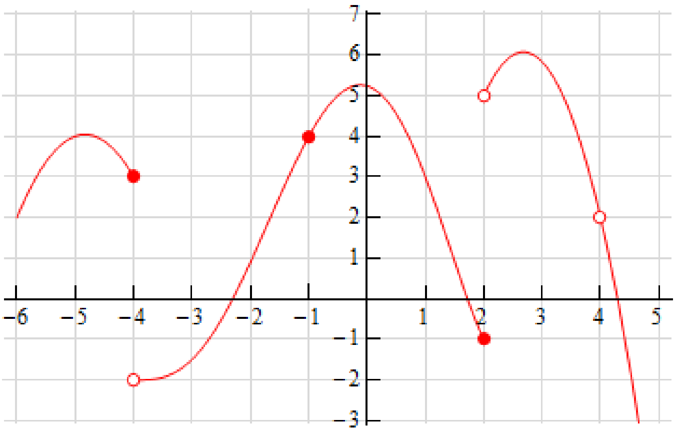
If any of the quantities do not exist clearly explain why.

a) a = −4

b) a = −1

c) a = 2

d) a = 4



Answer:

a)

From the graph we can see that,

$$\boxed{f(-4) = 3}$$

because the closed dot is at the value of $y = 3$.

We can also see that as we approach $x = -4$ from the left the graph is approaching a value of 3 and as we approach from the right the graph is approaching a value of -2. Therefore, we get,

$$\boxed{\lim_{x \to -4^-} f(x) = 3 \qquad \& \qquad \lim_{x \to -4^+} f(x) = -2}$$

Now, because the two one-sided limits are different we know that,

$$\boxed{\lim_{x \to -4} f(x) \text{ does not exist}}$$

b)

From the graph we can see that,

$$\boxed{f(-1) = 4}$$

because the closed dot is at the value of $y = 4$.

We can also see that as we approach $x = -1$ from both sides the graph is approaching the same value, 4, and so we get,

$$\boxed{\lim_{x \to -1^-} f(x) = 4 \qquad \& \qquad \lim_{x \to -1^+} f(x) = 4}$$

The two one-sided limits are the same so we know,

$$\boxed{\lim_{x \to -1} f(x) = 4}$$

c)

From the graph we can see that,

$$\boxed{f(2) = -1}$$

because the closed dot is at the value of $y = -1$.

We can also see that as we approach $x = 2$ from the left the graph is approaching a value of -1 and as we approach from the right the graph is approaching a value of 5. Therefore, we get,

$$\boxed{\lim_{x \to 2^-} f(x) = -1 \qquad \& \qquad \lim_{x \to 2^+} f(x) = 5}$$

Now, because the two one-sided limits are different we know that,

$$\boxed{\lim_{x \to 2} f(x) \text{ does not exist}}$$

d)

Because there is no closed dot for $x = 4$ we can see that,

$$\boxed{f(4) \text{ does not exist}}$$

We can also see that as we approach $x = 4$ from both sides the graph is approaching the same value, 2, and so we get,

$$\boxed{\lim_{x \to 4^-} f(x) = 2 \qquad \& \qquad \lim_{x \to 4^+} f(x) = 2}$$

The two one-sided limits are the same so we know,

$$\boxed{\lim_{x \to 4} f(x) = 2}$$

Always recall that the value of a limit (including one-sided limits) does not actually depend upon the value of the function at the point in question. The value of a limit only depends on the values of the function around the point in question. Therefore, even though the function doesn't exist at this point the limit and one-sided limits can still have a value.

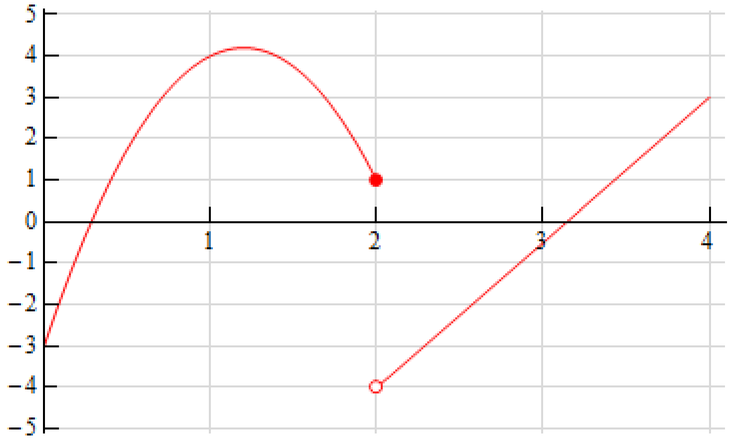
7. Sketch a graph of a function that satisfies each of the following conditions.

$\lim_{x \to 2^-} f(x) = 1$, $\lim_{x \to 2^+} f(x) = -4$ and $f(2) = 1$

Answer:

There are literally an infinite number of possible graphs that we could give here for an answer. However, all of them must have a closed dot on the graph at the point (2,1), the graph must be approaching a value of 1 as it approaches x=2 from the left (as indicated by the left-hand limit) and it must be approaching a value of -4 as it approaches x=2 from the right (as indicated by the right-hand limit).

Here is a sketch of one possible graph that meets these conditions.



8. Evaluate $\lim\limits_{x\to\infty} \frac{x}{\sqrt{1+2x^2}}$

Answer:

$\lim\limits_{x\to\infty} \frac{x}{\sqrt{1+2x^2}}$ [Form $\frac{\infty}{\infty}$]

This limit is of the form $\frac{\infty}{\infty}$, Here, We can cancel a factor going to ∞ out of the numerator and denominator.

$$\lim_{x\to\infty} \frac{x}{\sqrt{1+2x^2}}$$

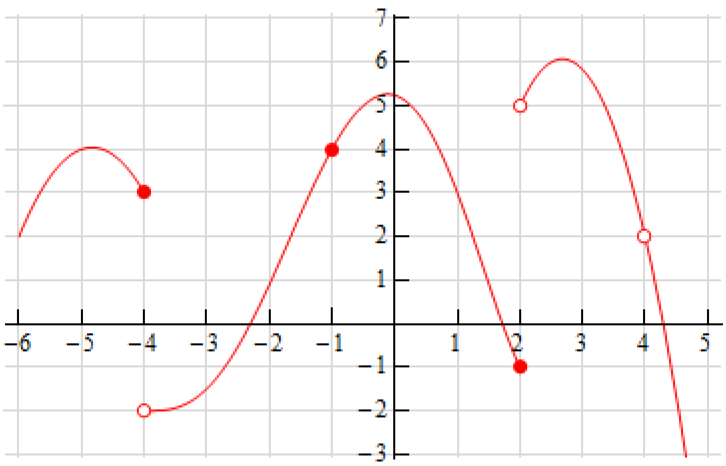
$$= \lim_{x\to\infty} \frac{x}{x\sqrt{\frac{1}{x^2}+2}}$$

Factor x becomes ∞ at x tends to ∞, So we need to cancel this factor from numerator and denominator.

$$= \lim_{x\to\infty} \frac{1}{\sqrt{\frac{1}{x^2}+2}}$$

$$= \frac{1}{\sqrt{\frac{1}{\infty^2}+2}} = \frac{1}{\sqrt{0+2}} = \frac{1}{\sqrt{2}}$$

9. The graph of f(x) is given below. Based on this graph determine where the function is discontinuous.



Answer:

Before starting the solution recall that in order for a function to be continuous at $x = a$ both $f(a)$ and $\lim_{x \to a} f(x)$ must exist and we must have,

$$\lim_{x \to a} f(x) = f(a)$$

Using this idea it should be fairly clear where the function is not continuous.

First notice that at $x = -4$ we have,

$$\lim_{x \to -4^-} f(x) = 3 \neq -2 = \lim_{x \to -4^+} f(x)$$

and therefore, we also know that $\lim_{x \to -4} f(x)$ doesn't exist. We can therefore conclude that $f(x)$ is **discontinuous** at $x = -4$ because the limit does not exist.

Likewise, at $x = 2$ we have,

$$\lim_{x \to 2^-} f(x) = -1 \neq 5 = \lim_{x \to 2^+} f(x)$$

and therefore, we also know that

$$\lim_{x \to 2} f(x)$$

doesn't exist. So again, because the limit does not exist, we can see that $f(x)$ is **discontinuous** at $x = 2$.

Finally let's take a look at $x = 4$. Here we can see that,

$$\lim_{x \to 4^-} f(x) = 2 = \lim_{x \to 4^+} f(x)$$

and therefore, we also know that $\lim_{x \to 4} f(x) = 2$. However, we can also see that $f(4)$ doesn't exist and so once again $f(x)$ is **discontinuous** at $x = 4$ because this time the function does not exist at $x = 4$.

All other points on this graph will have both the function and limit exist and we'll have $\lim_{x \to a} f(x) = f(a)$ and so will be continuous.

In summary then the points of discontinuity for this graph are : $x = -4$, $x = 2$ and $x = 4$.

10. For the following questions using only the limit properties, one-sided limit properties (if needed) and the definition of continuity determine if the given function is continuous or discontinuous at the indicated points.

i. $f(x) = \frac{4x+5}{9-3x}$

(a) x = -1, (b) x = 0 , (c) x = 3

Answer:

(a)

So, here we go.

$$\lim_{x \to -1} f(x) = \lim_{x \to -1} \frac{4x+5}{9-3x} = \frac{\lim_{x \to -1}(4x+5)}{\lim_{x \to -1}(9-3x)} = \frac{4\lim_{x \to -1} x + \lim_{x \to -1} 5}{\lim_{x \to -1} 9 - 3\lim_{x \to -1} x} = \frac{4(-1)+5}{9-3(-1)} = f(-1)$$

So, we can see that $\lim_{x \to -1} f(x) = f(-1)$ and so the function **is continuous** at $x = -1$.

(b)

For justification on why we can't just plug in the number here check out the comment at the beginning of the solution to **(a)**.

Here is the work for this part.

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{4x+5}{9-3x} = \frac{\lim_{x \to 0}(4x+5)}{\lim_{x \to 0}(9-3x)} = \frac{4\lim_{x \to 0} x + \lim_{x \to 0} 5}{\lim_{x \to 0} 9 - 3\lim_{x \to 0} x} = \frac{4(0)+5}{9-3(0)} = f(0)$$

So, we can see that $\lim_{x \to 0} f(x) = f(0)$ and so the function **is continuous** at $x = 0$.

(c)

For practice you might want to verify that,

$$\lim_{x\to 3^-} f(x) = \infty \qquad \lim_{x\to 3^+} f(x) = -\infty$$

and so $\lim_{x\to 3} f(x)$ also doesn't exist.