INSTITUTE OF ACTUARIAL
& QUANTITATIVE STUDIES

Class: FY BSc

Subject : Calculus

Chapter: Unit 2 – Chapter 2

Chapter Name: Application of Derivatives

Today's Agenda

1. Application of derivatives
    1. Rate of change of a quantity
    2. Approximation Value
    3. Tangent & Normal to the curve
    4. Minima, Maxima and Point of inflexion
    5. Increasing & Decreasing functions

1 Applications of Derivative

Derivatives have wide usage. They are used in many situations like finding maxima or minima of a function, finding the slope of the curve, and even inflection point.

The most common usage of application of derivatives is seen in:

- Finding Rate of Change of a Quantity
- Finding the Approximation Value
- Finding the equation of a Tangent and Normal To a Curve
- Finding Maxima and Minima, and Point of Inflection
- Determining Increasing and Decreasing Functions

1.1 Rate of Change of a Quantity

By using the application of derivatives we can find the approximate change in one quantity with respect to the change in the other quantity.
Assume we have a function y = f(x), which is defined in the interval [a, a+h],

Then ,
the average rate of change in the function in the given interval is: $\frac{f(a+h)-f(a)}{h}$

Now using the definition of derivative, we can write:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$$

Hence the derivative gives the instantaneous rate of change of a function within the given limits and can be used to find the estimated change in the function f(x) for the small change in the other variable(x).

1.1 Example

1. Find the rate of change of the area of a circle per second with respect to its radius r when r = 5 cm

Solution:
The area A of a circle with radius r is given by A = π r^2
the rate of change of the area A with respect to its radius r is given by:

$$\frac{dA}{dr} = \frac{d}{dr}(\pi r^2) = 2\pi r$$

When r = 5 cm

$$\frac{dA}{dr} = 10\pi$$

Thus, the area of the circle is changing at the rate of 10π cm2/s.

Question

1. The total cost $C(x)$ in Rupees, associated with the production of x units of an item is given by $C(x) = 0.005x^3 - 0.02x^2 + 30x + 5000$

Find the marginal cost when 3 units are produced, where by marginal cost we mean the instantaneous rate of change of total cost at any level of output.

Solution

Since marginal cost is the rate of change of total cost with respect to the output, we have

Marginal cost = dC/dx = $0.005(3x^2) - 0.02(2x) + 30$

When $x = 3$,

$MC = 0.015(3^2) - 0.04(3) + 30$
$= 0.135 - 0.12 + 30$
$MC = 30.015$

1.2 Approximation Value

The linear approximation method was given by Newton and he suggested finding the value of the function at the given point and then finding the equation of the tangent line to find the approximately close value to the function.

The equation of the function of the tangent is : $\mathbf{L(x) = f(a) + f'(a)(x-a)}$

1.2 Example

1. **Find the linear approximation of $f(x)= \sqrt{x}$ at $x=9$ and use the approximation to estimate $\sqrt{9.1}$**

Solution:
Since we are looking for the linear approximation at $x=9$, we know the linear approximation is given by
$L(x)= f(9)+ f'(9)(x-9)$.

We need to find $f(9)$ and $f'(9)$

$f(x)= \sqrt{x} \Rightarrow f(9)= \sqrt{9} = 3$

$f'(x)= \frac{1}{2\sqrt{x}} \Rightarrow f'(9)= \frac{1}{2\sqrt{9}}= \frac{1}{6}$

Therefore, the linear approximation is given

$L(x)= 3+ \frac{1}{6}(x-9)$

1.2 Example

Using the linear approximation, we can estimate $\sqrt{9.1}$

By writing ,

$$\sqrt{9.1} = f(9.1) \approx L(9.1)$$

$$= 3 + \frac{1}{6}(9.1 - 9)$$

$$\approx 3.0167$$

Question

1. Find the approximate value of f (3.02), where,

   $f(x) = 3x^2 + 5x + 3.$

Solution

Solution Let $x = 3$ and $\Delta x = 0.02$. Then

$$f(3.02) = f(x + \Delta x) = 3(x + \Delta x)^2 + 5(x + \Delta x) + 3$$

Note that $\Delta y = f(x + \Delta x) - f(x)$. Therefore

$$f(x + \Delta x) = f(x) + \Delta y$$

$$\approx f(x) + f'(x)\,\Delta x \qquad \text{(as } dx = \Delta x\text{)}$$

or

$$f(3.02) \approx (3x^2 + 5x + 3) + (6x + 5)\,\Delta x$$

$$= (3(3)^2 + 5(3) + 3) + (6(3) + 5)\,(0.02) \qquad \text{(as } x = 3, \Delta x = 0.02\text{)}$$

$$= (27 + 15 + 3) + (18 + 5)\,(0.02)$$

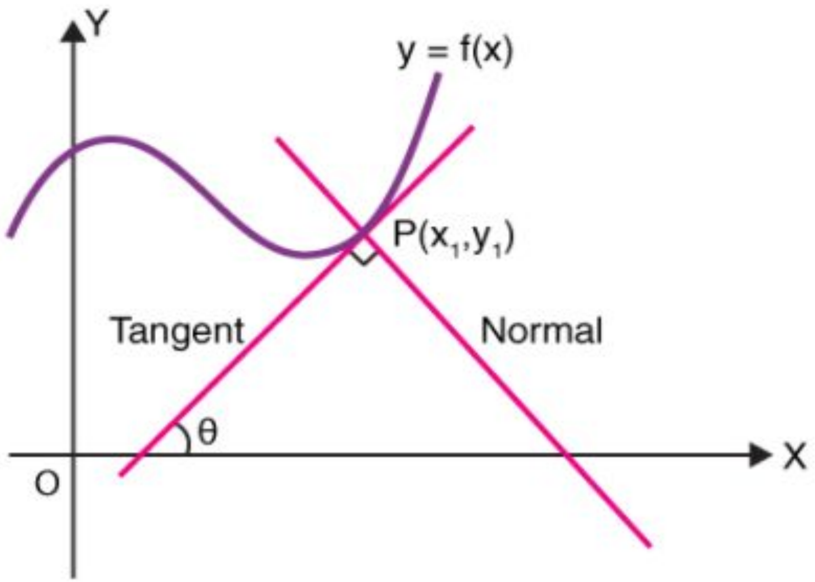
$$= 45 + 0.46 = 45.46$$

Hence, approximate value of $f(3.02)$ is 45.46.

1.3 Tangent and Normal To a Curve

Tangent is the line that touches the curve at a point and doesn't cross it, whereas normal is the perpendicular to that tangent.

Let the tangent meet the curve at $P(x_1, y_1)$.



1.3 Tangent and Normal To a Curve

Now the straight-line equation which passes through a point having slope m could be written as;
$y - y_1 = m(x - x_1)$

We can see from the above equation, the slope of the tangent to the curve $y = f(x)$ and at the point $P(x_1, y_1)$, it is given as dy/dx at $P(x_1, y_1) = f'(x)$.

Therefore,

Equation of the tangent to the curve at $P(x_1, y_1)$ can be written as:
$y - y_1 = f'(x_1)(x - x_1)$

Equation of normal to the curve is given by;
$y - y_1 = [-1/ f'(x_1)] (x - x_1)$
Or
$(y - y_1) f'(x_1) + (x-x_1) = 0$

1.3 Example

Example 19 Find the equations of the tangent and normal to the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 2$ at (1, 1).

Solution Differentiating $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 2$ with respect to x, we get

$$\frac{2}{3}x^{\frac{-1}{3}} + \frac{2}{3}y^{\frac{-1}{3}}\frac{dy}{dx} = 0$$

or
$$\frac{dy}{dx} = -\left(\frac{y}{x}\right)^{\frac{1}{3}}$$

Therefore, the slope of the tangent at (1, 1) is $\left.\frac{dy}{dx}\right]_{(1,1)} = -1$.

So the equation of the tangent at (1, 1) is

$$y - 1 = -1\,(x - 1) \qquad \text{or} \qquad y + x - 2 = 0$$

Also, the slope of the normal at (1, 1) is given by

$$\frac{-1}{\text{slope of the tangent at }(1,1)} = 1$$

Therefore, the equation of the normal at (1, 1) is

$$y - 1 = 1\,(x - 1) \qquad \text{or} \qquad y - x = 0$$

Question

Example 17 Find points on the curve $\frac{x^2}{4}+\frac{y^2}{25}=1$ at which the tangents are (i) parallel to x-axis (ii) parallel to y-axis.

Solution

Solution Differentiating $\frac{x^2}{4}+\frac{y^2}{25}=1$ with respect to x, we get

$$\frac{x}{2}+\frac{2y}{25}\frac{dy}{dx}=0$$

or

$$\frac{dy}{dx}=\frac{-25}{4}\frac{x}{y}$$

(i) Now, the tangent is parallel to the x-axis if the slope of the tangent is zero which gives $\frac{-25}{4}\frac{x}{y}=0$. This is possible if $x = 0$. Then $\frac{x^2}{4}+\frac{y^2}{25}=1$ for $x = 0$ gives $y^2 = 25$, i.e., $y = \pm 5$.

Thus, the points at which the tangents are parallel to the x-axis are (0, 5) and (0, – 5).

(ii) The tangent line is parallel to y-axis if the slope of the normal is 0 which gives $\frac{4y}{25x}=0$, i.e., $y = 0$. Therefore, $\frac{x^2}{4}+\frac{y^2}{25}=1$ for $y = 0$ gives $x = \pm 2$. Hence, the points at which the tangents are parallel to the y-axis are (2, 0) and (–2, 0).

1.4 Maxima and Minima, and Point of Inflection

To calculate the highest and lowest point of the curve in a graph or to know its turning point, the derivative function is used.

- When $x = a$, if $f(x) \leq f(a)$ for every x in the domain, then $f(x)$ has an Absolute Maximum value and the point a is the point of the maximum value of f.
- When $x = a$, if $f(x) \leq f(a)$ for every x in some open interval (p, q) then $f(x)$ has a Relative Maximum value.
- When $x = a$, if $f(x) \geq f(a)$ for every x in the domain then $f(x)$ has an Absolute Minimum value and the point a is the point of the minimum value of f.
- When $x = a$, if $f(x) \geq f(a)$ for every x in some open interval (p, q) then $f(x)$ has a Relative Minimum value.

1.4 Example

1.Find local maximum and local minimum values of the function f given by
$f(x) = 3x^4 + 4x^3 - 12x^2 + 12$

Solution:

$$f(x) = 3x^4 + 4x^3 - 12x^2 + 12$$
$$f'(x) = 12x^3 + 12x^2 - 24x = 12x(x-1)(x+2)$$
$$f'(x) = 0 \text{ at } x = 0,\ x = 1 \text{ and } x = -2.$$
$$f''(x) = 36x^2 + 24x - 24 = 12(3x^2 + 2x - 2)$$

$$\begin{cases} f''(0) & = -24 < 0 \\ f''(1) & = 36 > 0 \\ f''(-2) & = 72 > 0 \end{cases}$$

Therefore, by second derivative test, $x = 0$ is a point of local maxima and local maximum value of f at $x = 0$ is f (0) = 12 while $x = 1$ and $x = -2$ are the points of local minima and local minimum values of f at $x = -1$ and -2 are f (1) = 7 and f (–2) = –20, respectively.

Question

1. Find two positive numbers whose sum is 15 and the sum of whose squares is minimum.

Solution

Let one of the numbers be x. Then the other number is (15 – x).
Let S(x) denote the sum of the squares of these numbers.

Then,
$S(x) = x^2 + (15 - x^2)$
$= 2x^2 - 30x + 225$
$S'(x) = 4x - 30$
$S''(x) = 4$

Now $S'(x) = 0$
We get $x = 15/2$

Also $S''(x) > 0$
Therefore, by second derivative test, $x = 15/2$ is the point of local minima of S.
Hence the sum of squares of numbers is minimum when the numbers are 15/2 and 15/2.

1.5 Increasing and Decreasing Functions

To find that a given function is increasing or decreasing or constant, say in a graph, we use derivatives. If f is a function which is continuous in [p, q] and differentiable in the open interval (p, q), then,

- f is increasing at [p, q] if $f'(x) > 0$ for each $x \in (p, q)$
- f is decreasing at [p, q] if $f'(x) < 0$ for each $x \in (p, q)$
- f is constant function in [p, q], if $f'(x)=0$ for each $x \in (p, q)$

1.5 Example

1. Find the intervals in which the function f given by $f(x) = 4x^3 - 6x^2 - 72x + 30$
(a) increasing (b) decreasing

Solution:
$f(x) = 4x^3 - 6x^2 - 72x + 30$

$f'(x) = 12x^2 - 12x - 72$
$= 12(x^2 - x - 6)$
$= 12(x-3)(x+2)$

Therefore, $f'(x) = 0$ gives $x = -2, 3$
The points $x = -2$ and $x = 3$ divides the real line into three disjoint intervals, namely, $(-\infty, -2)$, $(-2, 3)$ and $(3, \infty)$.

1.5 Example

In the intervals (– ∞, – 2) and (3, ∞), f ′(x) is positive while in the interval (– 2, 3), f ′(x) is negative.

Consequently, the function f is increasing in the intervals (– ∞, – 2) and (3, ∞) while the function is decreasing in the interval (– 2, 3). However, f is neither increasing nor decreasing in R.

Interval	Sign of $f'(x)$	Nature of function f
$(-\infty, -2)$	$(-)\,(-) > 0$	f is increasing
$(-2, 3)$	$(-)\,(+) < 0$	f is decreasing
$(3, \infty)$	$(+)\,(+) > 0$	f is increasing

Question

1. Discuss the increasing and decreasing nature of the function $f(x) = x \ln(x)$

Solution

$f(x) = x \ln(x)$
$f'(x) = 1 + \ln(x)$

For a function to be increasing $f'(x) > 0$
$1 + \ln(x) > 0$
$\ln(x) > -1$
$\ln(x) > -\ln(e)$
$\ln(x) > \ln(e^{-1})$

We know that $\ln(x)$ is increasing function, so for $\ln(x) > \ln(e^{-1})$ to be hold
$x > e^{-1}$
$x > 1/e$

Thus, function $f(x) = x \ln(x)$ to be increasing $x \in (1/e, \infty)$ and for function $f(x) = x \ln(x)$ to be decreasing $x \in (0, 1/e)$.