INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Subject: Calculus

Chapter: Unit 2

Category: PQ- Solution

1.

$$\boxed{f'(x) = 18x^2 - 9}$$

2.

$$\boxed{\frac{dy}{dt} = 8t^3 - 20t + 13}$$

3.

$$\boxed{g'(z) = 28z^6 + 21z^{-8} + 9}$$

4. There isn't much to do here other than take the derivative using the rules we discussed in this section.
Remember that you'll need to convert the roots to fractional exponents before you start taking the derivative. Here is the rewritten function.

$$f(x) = 10\left(x^3\right)^{\frac{1}{5}} - \left(x^7\right)^{\frac{1}{2}} + 6\left(x^8\right)^{\frac{1}{3}} - 3 = 10x^{\frac{3}{5}} - x^{\frac{7}{2}} + 6x^{\frac{8}{3}} - 3$$

$$f'(x) = 10\left(\frac{3}{5}\right)x^{-\frac{2}{5}} - \frac{7}{2}x^{\frac{5}{2}} + 6\left(\frac{8}{3}\right)x^{\frac{5}{3}} = \boxed{6x^{-\frac{2}{5}} - \frac{7}{2}x^{\frac{5}{2}} + 16x^{\frac{5}{3}}}$$

5. There isn't much to do here other than take the derivative using the rules we discussed in this section.
Remember that you'll need to rewrite the terms so that each of the tt's are in the numerator with negative exponents before taking the derivative. Here is the rewritten function.

$$f(t) = 4t^{-1} - \frac{1}{6}t^{-3} + 8t^{-5}$$

$$\boxed{f'(t) = -4t^{-2} + \frac{1}{2}t^{-4} - 40t^{-6}}$$

6.

$$R(z) = 6z^{-\frac{3}{2}} + \frac{1}{8}z^{-4} - \frac{1}{3}z^{-10}$$

$$R'(z) = 6\left(-\frac{3}{2}\right)z^{-\frac{5}{2}} + \frac{1}{8}(-4)z^{-5} - \frac{1}{3}(-10)z^{-11} = \boxed{-9z^{-\frac{5}{2}} - \frac{1}{2}z^{-5} + \frac{10}{3}z^{-11}}$$

7.

$$z = 3x^3 - 9x$$

$$\boxed{\frac{dz}{dx} = 9x^2 - 9}$$

8.

$$g(y) = y^3 - 2y^2 - 8y$$

$$\boxed{g'(y) = 3y^2 - 4y - 8}$$

9.

$$h(x) = \frac{4x^3}{x} - \frac{7x}{x} + \frac{8}{x} = 4x^2 - 7 + 8x^{-1}$$

$$\boxed{h'(x) = 8x - 8x^{-2}}$$

10.

$$f'(t) = (8t-1)\left(t^3 - 8t^2 + 12\right) + \left(4t^2 - t\right)\left(3t^2 - 16t\right) = 20t^4 - 132t^3 + 24t^2 + 96t - 12$$

11.

$$y = \left(1 + x^{\frac{3}{2}}\right)\left(x^{-3} - 2x^{\frac{1}{3}}\right)$$

$$\frac{dy}{dx} = \left(\frac{3}{2}x^{\frac{1}{2}}\right)\left(x^{-3} - 2x^{\frac{1}{3}}\right) + \left(1 + x^{\frac{3}{2}}\right)\left(-3x^{-4} - \frac{2}{3}x^{-\frac{2}{3}}\right) = -3x^{-4} - \frac{3}{2}x^{-\frac{5}{2}} - \frac{2}{3}x^{-\frac{2}{3}} - \frac{11}{3}x^{\frac{5}{6}}$$

12.

$$g'(x) = \frac{12x(2-x) - 6x^2(-1)}{(2-x)^2} = \boxed{\frac{24x - 6x^2}{(2-x)^2}}$$

13.

$$R'(w) = \frac{\left(3 + 4w^3\right)\left(2w^2 + 1\right) - \left(3w + w^4\right)(4w)}{\left(2w^2 + 1\right)^2} = \boxed{\frac{4w^5 + 4w^3 - 6w^2 + 3}{\left(2w^2 + 1\right)^2}}$$

14.

$$\boxed{R'(w) = 3^w \ln(3)\log(w) + \frac{3^w}{w\ln(10)}}$$

Recall that log(x) is the common logarithm and so is really log10(x).

15.

$$\boxed{y' = 5z^4 - \mathbf{e}^z \ln(z) - \frac{\mathbf{e}^z}{z}}$$

16.

$$f'(x) = 4\left(6x^2 + 7x\right)^3 (12x + 7) = \boxed{4(12x+7)\left(6x^2 + 7x\right)^3}$$

17.

$$\boxed{f'(t) = \left(4 + 7t^6\right) \mathbf{e}^{4t+t^7}}$$

18.

The first step is to write down equations describing this situation.

Let's call the two numbers x and y and we are told that the product is 750 (this is the constraint for the problem) or,

$$xy = 750$$

We are then being asked to minimize the sum of one and 10 times the other,

$$S = x + 10y$$

Note that it really doesn't worry which is x and which is y in the sum so we simply chose the y to be multiplied by 10.

We now need to solve the constraint for x or y (and it really doesn't matter which variable we solve for in this case) and plug this into the product equation.

$$x = \frac{750}{y} \qquad \Rightarrow \qquad S(y) = \frac{750}{y} + 10y$$

The next step is to determine the critical points for this equation.

$$S'(y) = -\frac{750}{y^2} + 10 \qquad \rightarrow \qquad -\frac{750}{y^2} + 10 = 0 \qquad \rightarrow \qquad y = \pm\sqrt{75} = 5\sqrt{3}$$

Because we are told that y must be positive we can eliminate the negative value and so the only value we really get out of this step is : $y = \sqrt{75} = 5\sqrt{3}$.

Now for the step many neglect as unnecessary. Just because we got a single value we can't just assume that this will give a minimum sum. We need to do a quick check to see if it does give a minimum.

As discussed in notes there are several methods for doing this, but in this case we can quickly see that,

$$S''(y) = \frac{1500}{y^3}$$

From this we can see that, provided we recall that y is positive, then the second derivative will always be positive. Therefore, $S(y)$ will always be concave up and so the single critical point from Step 3 that we can use must be a relative minimum and hence must be the value that gives a minimum sum.

Finally, let's actually answer the question. We need to give both values. We already have y so we need to determine x and that is easy to do from the constraint.

$$x = \frac{750}{5\sqrt{3}} = 50\sqrt{3}$$

The final answer is then,

$$\boxed{x = 50\sqrt{3} \qquad y = 5\sqrt{3}}$$

19.

$$f'(x) = 5\mathbf{e}^{9-2x} + 5x(-2)\mathbf{e}^{9-2x} = 5\mathbf{e}^{9-2x}(1 - 2x)$$

Recall that critical points are simply where the derivative is zero and/or doesn't exist. This derivative exists everywhere and so we don't need to worry about that. Therefore, all we need to do is determine where the derivative is zero.
Notice as well that because we know that exponential functions are never zero and so the derivative will only be zero if,

$$1 - 2x = 0 \quad \rightarrow \quad x = \frac{1}{2}$$

So, we have a single critical point, $\boxed{x = \frac{1}{2}}$, for this function.

20. We can easily see from the graph where the function in increasing/decreasing and so all we need to do is write down the intervals.

$$\boxed{\text{Increasing}: (-3,1) \quad \& \quad (7,\infty) \qquad \text{Decreasing}: (-\infty,-3) \quad \& \quad (1,7)}$$

Note as well that we don't include the end points in the interval. For this problem that is important because at the end points we are at infinity or the function is either not increasing or decreasing.

21. a.

$$\boxed{\frac{dy}{dx} = 30x^{-\frac{7}{2}} - \frac{3}{2}x^{-3}}$$

b.

$$\boxed{\frac{dy}{dx} = -\frac{1}{3}x^{-2} + \frac{5}{3}x^{\frac{3}{2}} - \frac{1}{6}x^{-\frac{3}{2}}}$$

22.

$$y = x^2 - 9x + 13$$

$$\frac{dy}{dx} = 2x - 9$$

• when $x = 6$

$$y\Big|_{x=6} = 36 - 54 + 13 = -5$$

$$\frac{dy}{dx}\Big|_{x=6} = 12 - 9 = 3$$

$\therefore m = 3 \quad (6, -5)$

• $y - y_0 = m(x - x_0)$

$$y + 5 = 3(x - 6)$$

$$y + 5 = 3x - 18$$

$$y = 3x - 23$$

23.

(c)

• $y = x - 4\sqrt{x} = x - 4x^{\frac{1}{2}}$

• $\frac{dy}{dx} = 1 - 2x^{-\frac{1}{2}} = 1 - \frac{2}{\sqrt{x}}$

• $\frac{d^2y}{dx^2} = x^{-\frac{3}{2}} = \frac{1}{x^{3/2}}$

FOR MIN/MAX $\frac{dy}{dx} = 0$

$$1 - \frac{2}{\sqrt{x}} = 0$$

$$1 = \frac{2}{\sqrt{x}}$$

$$\sqrt{x} = 2$$

$x = 4 \quad \therefore y = 4 - 4\sqrt{4}$

$y = -4$

$$\frac{d^2y}{dx^2}\Big|_{x=4} = \frac{1}{4^{3/2}} = \frac{1}{8} > 0$$

$\therefore (4, -4)$ MIN

24.

(a) $f(x) = x^3 - 3x^2 - 9x + 10$

$f'(x) = 3x^2 - 6x - 9$

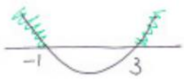
• INCREASING $\Rightarrow f'(x) > 0$

$3x^2 - 6x - 9 > 0$

$x^2 - 2x - 3 > 0$

$(x+1)(x-3) > 0$

C.V = -1, 3



$x < -1$ OR $x > 3$

25.

$f(x) = 4x^3 - 3x^2 - 6x$

$f'(x) = 12x^2 - 6x - 6$

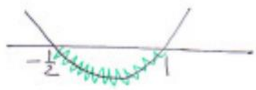
• DECREASING $\Rightarrow f'(x) < 0$

$12x^2 - 6x - 6 < 0$

$2x^2 - x - 1 < 0$

$(2x+1)(x-1) < 0$

C.V = 1, $-\frac{1}{2}$



$-\frac{1}{2} < x < 1$