INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Subject: Calculus

Chapter: Unit 3

Category: PQ - Solutions

1.

$$\int z^7-48z^{11}-5z^{16}\,dz=\frac{1}{8}z^8-\frac{48}{12}z^{12}-\frac{5}{17}z^{17}+c=\boxed{\frac{1}{8}z^8-4z^{12}-\frac{5}{17}z^{17}+c}$$

2. We first need to convert the roots to fractional exponents

$$\int \sqrt{x^7}-7\sqrt[6]{x^5}+17\sqrt[3]{x^{10}}\,dx=\int x^{\frac{7}{2}}-7\left(x^5\right)^{\frac{1}{6}}+17\left(x^{10}\right)^{\frac{1}{3}}\,dx=\int x^{\frac{7}{2}}-7x^{\frac{5}{6}}+17x^{\frac{10}{3}}\,dx$$

$$\begin{aligned}\int \sqrt{x^7}-7\sqrt[6]{x^5}+17\sqrt[3]{x^{10}}\,dx&=\int x^{\frac{7}{2}}-7x^{\frac{5}{6}}+17x^{\frac{10}{3}}\,dx\\&=\frac{2}{9}x^{\frac{9}{2}}-7\left(\frac{6}{11}\right)x^{\frac{11}{6}}+17\left(\frac{3}{13}\right)x^{\frac{13}{3}}+c=\boxed{\frac{2}{9}x^{\frac{9}{2}}-\frac{42}{11}x^{\frac{11}{6}}+\frac{51}{13}x^{\frac{13}{3}}+c}\end{aligned}$$

3.

$$\int \frac{7}{3y^6}+\frac{1}{y^{10}}-\frac{2}{\sqrt[3]{y^4}}\,dy=\int \frac{7}{3y^6}+\frac{1}{y^{10}}-\frac{2}{y^{\frac{4}{3}}}\,dy=\int \frac{7}{3}y^{-6}+y^{-10}-2y^{-\frac{4}{3}}\,dy$$

$$\begin{aligned}\int \frac{7}{3y^6}+\frac{1}{y^{10}}-\frac{2}{\sqrt[3]{y^4}}\,dy&=\int \frac{7}{3}y^{-6}+y^{-10}-2y^{-\frac{4}{3}}\,dy\\&=\frac{7}{3}\left(\frac{1}{-5}\right)y^{-5}+\left(\frac{1}{-9}\right)y^{-9}-2\left(-\frac{3}{1}\right)y^{-\frac{1}{3}}+c\\&=\boxed{-\frac{7}{15}y^{-5}-\frac{1}{9}y^{-9}+6y^{-\frac{1}{3}}+c}\end{aligned}$$

4.

$$\int \frac{z^8-6z^5+4z^3-2}{z^4}\,dz=\int \frac{z^8}{z^4}-\frac{6z^5}{z^4}+\frac{4z^3}{z^4}-\frac{2}{z^4}\,dz=\int z^4-6z+\frac{4}{z}-2z^{-4}\,dz$$

$$\int \frac{z^8-6z^5+4z^3-2}{z^4}\,dz=\int z^4-6z+\frac{4}{z}-2z^{-4}\,dz=\boxed{\frac{1}{5}z^5-3z^2+4\ln|z|+\frac{2}{3}z^{-3}+c}$$

5.

$$\int t^3 - \frac{\mathbf{e}^{-t} - 4}{\mathbf{e}^{-t}}\,dt = \int t^3 - \frac{\mathbf{e}^{-t}}{\mathbf{e}^{-t}} + \frac{4}{\mathbf{e}^{-t}}\,dt = \int t^3 - 1 + 4\mathbf{e}^{t}\,dt$$

$$\int t^3 - \frac{\mathbf{e}^{-t} - 4}{\mathbf{e}^{-t}}\,dt = \int t^3 - 1 + 4\mathbf{e}^{t}\,dt = \boxed{\frac{1}{4}t^4 - t + 4\mathbf{e}^{t} + c}$$

6.

$$f(x) = \int f'(x)\ dx$$

To arrive at a general formula for f(x)f(x) all we need to do is integrate the derivative that we've been given in the problem statement.

$$f(x) = \int 12x^2 - 4x\ dx = 4x^3 - 2x^2 + c$$

Because we have the condition that f(−3)=17 we can just plug x=−3 into our answer from the previous step, set the result equal to 17 and solve the resulting equation for c.

Doing this gives,

$$17 = f(-3) = -126 + c \qquad \Rightarrow \qquad c = 143$$

$$\boxed{f(x) = 4x^3 - 2x^2 + 143}$$

7. In this case it looks like we should use the following as our substitution.

$$u = y^4 - 7y^2 \qquad du = \left(4y^3 - 14y\right)dy$$

To help with the substitution let's do a little rewriting of this to get,

$$du = \left(4y^3 - 14y\right)dy = -2\left(7y - 2y^3\right)dy \qquad \Rightarrow \qquad \left(7y - 2y^3\right)dy = -\frac{1}{2}du$$

Doing the substitution and evaluating the integral gives,

$$\int (7y - 2y^3)\,\mathbf{e}^{y^4 - 7y^2}\,dy = -\frac{1}{2}\int \mathbf{e}^u\,du = -\frac{1}{2}\mathbf{e}^u + c$$

Finally, don't forget to go back to the original variable!

$$\int (7y - 2y^3)\,\mathbf{e}^{y^4 - 7y^2}\,dy = \boxed{-\frac{1}{2}\mathbf{e}^{y^4 - 7y^2} + c}$$

8.

$$\int_{-2}^{1} 5z^2 - 7z + 3\,dz = \left(\frac{5}{3}z^3 - \frac{7}{2}z^2 + 3z\right)\Big|_{-2}^{1}$$

$$\int_{-2}^{1} 5z^2 - 7z + 3\,dz = \frac{7}{6} - \left(-\frac{100}{3}\right) = \boxed{\frac{69}{2}}$$

9.

$$\int_{1}^{4} \frac{8}{\sqrt{t}} - 12\sqrt{t^3}\,dt = \int_{1}^{4} 8t^{-\frac{1}{2}} - 12t^{\frac{3}{2}}\,dt = \left(16t^{\frac{1}{2}} - \frac{24}{5}t^{\frac{5}{2}}\right)\Big|_{1}^{4}$$

$$= -\frac{608}{5} - \frac{56}{5} = \boxed{-\frac{664}{5}}$$

10. Breaking up the integral at t=1 gives,

$$\int_{0}^{4} f(t)\;dt = \int_{0}^{1} f(t)\;dt + \int_{1}^{4} f(t)\;dt$$

$$\int_{0}^{4} f(t)\;dt = \int_{0}^{1} 1 - 3t^2\,dt + \int_{1}^{4} 2t\,dt$$

$$\int_0^4 f(t)\,dt = \int_0^1 1-3t^2\,dt + \int_1^4 2t\,dt = \left(t-t^3\right)\Big|_0^1 + t^2\Big|_1^4 = [0-0]+[16-1] = \boxed{15}$$

11. The first step that we need to do is do the substitution

$$\int_{-1}^{2} x^3 + \mathbf{e}^{\frac{1}{4}x}\,dx = \int_{-1}^{2} x^3\,dx + \int_{-1}^{2} \mathbf{e}^{\frac{1}{4}x}\,dx$$

$$u = \frac{1}{4}x$$

Here is the actual substitution work for this integral.

$$du = \frac{1}{4}dx \qquad \rightarrow \qquad dx = 4du$$

$$x = -1 : u = -\frac{1}{4} \qquad\qquad x = 2 : u = \frac{1}{2}$$

Here is the integral after the substitution.

$$\int_{-1}^{2} x^3 + \mathbf{e}^{\frac{1}{4}x}\,dx = \int_{-1}^{2} x^3\,dx + 4\int_{-\frac{1}{4}}^{\frac{1}{2}} \mathbf{e}^{u}\,du$$

$$\int_{-1}^{2} x^3 + \mathbf{e}^{\frac{1}{4}x}\,dx = \frac{1}{4}x^4\Big|_{-1}^{2} + 4\mathbf{e}^{u}\Big|_{-\frac{1}{4}}^{\frac{1}{2}} = \left(4-\frac{1}{4}\right) + \left(4\mathbf{e}^{\frac{1}{2}} - 4\mathbf{e}^{-\frac{1}{4}}\right) = \boxed{\frac{15}{4} + 4\mathbf{e}^{\frac{1}{2}} - 4\mathbf{e}^{-\frac{1}{4}}}$$

12.

Solution. Substituting $u = x - 2$, $u + 3 = x + 1$ and $du = dx$, you get

$$\begin{aligned}\int (x+1)(x-2)^9 dx &= \int (u+3)u^9 du = \int (u^{10} + 3u^9) du = \\ &= \frac{1}{11}u^{11} + \frac{3}{10}u^{10} + C = \\ &= \frac{1}{11}(x-2)^{11} + \frac{3}{10}(x-2)^{10} + C.\end{aligned}$$

13.

Solution. Since the factor e^{2x} is easy to integrate and the factor x^3 is simplified by differentiation, try integration by parts with

$$g(x) = e^{2x} \quad \text{and} \quad f(x) = x^3.$$

Then,

$$G(x) = \int e^{2x} dx = \frac{1}{2}e^{2x} \quad \text{and} \quad f'(x) = 3x^2$$

and so

$$\int x^3 e^{2x} dx = \frac{1}{2}x^3 e^{2x} - \frac{3}{2}\int x^2 e^{2x} dx.$$

To find $\int x^2 e^{2x} dx$, you have to integrate by parts again, but this time with

$$g(x) = e^{2x} \quad \text{and} \quad f(x) = x^2.$$

Then,

$$G(x) = \frac{1}{2}e^{2x} \quad \text{and} \quad f'(x) = 2x$$

and so

$$\int x^2 e^{2x} dx = \frac{1}{2}x^2 e^{2x} - \int x e^{2x} dx.$$

To find $\int x e^{2x} dx$, you have to integrate by parts once again, this time with

$$g(x) = e^{2x} \quad \text{and} \quad f(x) = x.$$

Then,

$$G(x) = \frac{1}{2}e^{2x} \quad \text{and} \quad f'(x) = 1$$

and so

$$\int x e^{2x} dx = \frac{1}{2}x e^{2x} - \frac{1}{2}\int e^{2x} dx = \frac{1}{2}x e^{2x} - \frac{1}{4}e^{2x}.$$

Finally,

$$\begin{aligned}\int x^3e^{2x}dx &= \frac{1}{2}x^3e^{2x}-\frac{3}{2}\left[\frac{1}{2}x^2e^{2x}-\left(\frac{1}{2}xe^{2x}-\frac{1}{4}e^{2x}\right)\right]+C=\\ &= \left(\frac{1}{2}x^3-\frac{3}{4}x^2+\frac{3}{4}x-\frac{3}{8}\right)e^{2x}+C.\end{aligned}$$

14.

Solution. First rewrite the integrand as

$$\frac{1}{\sqrt{x^2+2x-3}}=\frac{1}{\sqrt{(x+1)^2-4}}$$

and then substitute $u=x+1$ and $du=dx$ to get

$$\begin{aligned}\int\frac{dx}{\sqrt{x^2+2x-3}} &= \int\frac{dx}{\sqrt{(x+1)^2-4}}=\int\frac{du}{\sqrt{u^2-4}}=\\ &= \ln\left|u+\sqrt{u^2-4}\right|+C=\ln\left|x+1+\sqrt{(x+1)^2-4}\right|+C=\\ &= \ln\left|x+1+\sqrt{x^2+2x-3}\right|+C.\end{aligned}$$

15.

Solution.

$$\begin{aligned}\int_{\ln\frac{1}{2}}^{2}\left(e^t-e^{-t}\right)dt &= \left(e^t+e^{-t}\right)\Big|_{\ln\frac{1}{2}}^{2}=e^2+e^{-2}-e^{\ln\frac{1}{2}}-e^{-\ln\frac{1}{2}}=\\ &= e^2+e^{-2}-e^{\ln\frac{1}{2}}-e^{\ln 2}=e^2+e^{-2}-\frac{1}{2}-2=\\ &= e^2+e^{-2}-\frac{5}{2}.\end{aligned}$$

16.

Question	Scheme
	$\int_k^9 \frac{6}{\sqrt{x}}\,dx=\left[ax^{\frac{1}{2}}\right]_k^9=20\Rightarrow 36-12\sqrt{k}=20$
	Correct method of solving Eg. $36-12\sqrt{k}=20\Rightarrow k=$
	$\Rightarrow k=\frac{16}{9}$ oe

17.

$\int(4x+12x^{1/2}+9)dx = 2x^2+8x^{3/2}+9x$ (ft dep. on 3 terms)

$[.....]_1^2 = (8+(8\times 2^{3/2})+18)-(2+8+9)$

$2^{3/2} = 2\sqrt{2}$ (seen or implied)

$= 7+16\sqrt{2}$

18.

$x\sqrt{x} = x^{\frac{3}{2}}$ (Seen, or implied by correct integration)

$x^{-1/2} \to kx^{1/2}$ or $x^{3/2} \to kx^{5/2}$ (k a non-zero constant)

$(y=)\dfrac{5x^{1/2}}{1/2}....+\dfrac{x^{5/2}}{5/2}(+C)$ ("$y=$" and "$+C$" are not required for these marks)

$35 = \dfrac{5\times 4^{1/2}}{1/2}+\dfrac{4^{5/2}}{5/2}+C$ An equation in C is required (see conditions below).

(With their terms simplified or unsimplified).

$C=\dfrac{11}{5}$ or equivalent $2\dfrac{1}{5}$, 22

$y=10x^{1/2}+\dfrac{2x^{5/2}}{5}+\dfrac{11}{5}$ (Or equivalent <u>simplified</u>)

I.s.w. if necessary, e.g. $y=10x^{1/2}+\dfrac{2x^{5/2}}{5}+\dfrac{11}{5} = 50x^{1/2}+2x^{1/2}+11$

19.

$$\frac{5x^2+2}{x^{\frac{1}{2}}} = 5x^{\frac{3}{2}} + 2x^{-\frac{1}{2}}$$ One ter

A1: Both ter

$$f(x) = 3x + \frac{5x^{\frac{5}{2}}}{\left(\frac{5}{2}\right)} + \frac{5x^{\frac{1}{2}}}{\left(\frac{1}{2}\right)} (+C)$$

$6 = 3 + 2 + 4 + C$ Use of $x = 1$

$C = -3$

$3x + 2x^{\frac{5}{2}} + 4x^{\frac{1}{2}} - 3$

[or: $3x + 2\sqrt{x^5} + 4\sqrt{x} - 3$ or equiv.]

20.

$x = u^2 + 1 \Rightarrow dx = 2u\,du$ oe	
Full substitution $\int \frac{3dx}{(x-1)(3+2\sqrt{x-1})} = \int \frac{3 \times 2u\,du}{(u^2+1-1)(3+2u)}$	
Finds correct limits e.g. $p = 2, q = 3$	
$= \int \frac{3 \times 2\cancel{u}\,du}{u^{\cancel{2}}(3+2u)} = \int \frac{6\,du}{u(3+2u)}$ *	
$\frac{6}{u(3+2u)} = \frac{A}{u} + \frac{B}{3+2u} \Rightarrow A = ..., B = ...$	
Correct PF. $\frac{6}{u(3+2u)} = \frac{2}{u} - \frac{4}{3+2u}$	
$\int \frac{6\,du}{u(3+2u)} = 2\ln u - 2\ln(3+2u)$ $(+c)$	
Uses limits $u = "3", u = "2"$ with some correct ln work leading to $k \ln b$ E.g. $(2\ln 3 - 2\ln 9) - (2\ln 2 - 2\ln 7) = 2\ln\frac{7}{6}$	
$\ln\frac{49}{36}$	

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