INSTITUTE OF ACTUARIAL
& QUANTITATIVE STUDIES

Subject: Calculus

Chapter: Unit 4

Category: PQ - Solutions

1.
a) First order
b) Third order

2. We multiply through by dt and divide through by 5x−3:

$$\frac{dx}{5x-3} = dt.$$

We integrate both sides

$$\int \frac{dx}{5x-3} = \int dt$$
$$\frac{1}{5}\log|5x-3| = t + C_1$$
$$5x-3 = \pm\exp(5t+5C_1)$$
$$x = \pm\frac{1}{5}\exp(5t+5C_1) + 3/5.$$

Letting $C = \frac{1}{5}\exp(5C_1)$, we can write the solution as

$$x(t) = Ce^{5t} + \frac{3}{5}.$$

We check to see that $x(t)$ satisfies the ODE:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 5Ce^{5t}$$
$$5x-3 = 5Ce^{5t} + 3 - 3 = 5Ce^{5t}.$$

Both expressions are equal, verifying our solution.

3.

Solution: We multiply both sides of the ODE by dx, divide both sides by y^2, and integrate:

$$\int y^{-2}dy = \int 7x^3dx$$
$$-y^{-1} = \frac{7}{4}x^4 + C$$
$$y = \frac{-1}{\frac{7}{4}x^4 + C}.$$

The general solution is

$$y(x) = \frac{-1}{\frac{7}{4}x^4 + C}.$$

Verify the solution:

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{-1}{\frac{7}{4}x^4 + C}\right)$$
$$= \frac{7x^3}{(\frac{7}{4}x^4 + C)^2}.$$

Given our solution for y, we know that

$$y(x)^2 = \left(\frac{-1}{\frac{7}{4}x^4 + C}\right)^2 = \frac{1}{(\frac{7}{4}x^4 + C)^2}.$$

Therefore, we see that indeed

$$\frac{dy}{dx} = \frac{7x^3}{(\frac{7}{4}x^4 + C)^2} = 7x^3y^2.$$

The solution satisfies the ODE.

To determine the constant C, we plug the solution into the equation for the initial conditions $y(2) = 3$:

$$3 = \frac{-1}{\frac{7}{4}2^4 + C}.$$

The constant C is

$$C = -28\frac{1}{3} = -\frac{85}{3},$$

and the final solution is

$$y(x) = \frac{-1}{\frac{7}{4}x^4 - \frac{85}{3}}.$$

4. C

UNIT 4

PRACTICE QUESTION SOLUTIONS

5. We'll need to start over from the beginning and start taking some derivatives of the function.

$$\begin{aligned}
n=0: &\quad f(x)=\mathbf{e}^{-6x}\\
n=1: &\quad f'(x)=-6\mathbf{e}^{-6x}\\
n=2: &\quad f''(x)=(-6)^2\mathbf{e}^{-6x}\\
n=3: &\quad f^{(3)}(x)=(-6)^3\mathbf{e}^{-6x}\\
n=4: &\quad f^{(4)}(x)=(-6)^4\mathbf{e}^{-6x}
\end{aligned}$$

It is now time to see if we can get a formula for the general term in the Taylor Series.
In this case, it is (hopefully) pretty simple to catch the pattern in the derivatives above. The general term is given by,

$$f^{(n)}(x)=(-6)^n\mathbf{e}^{-6x} \qquad n=0,1,2,3,\ldots$$

Now, recall that we don't really want the general term at any x. We want the general term at x=−4. This is,

$$f^{(n)}(-4)=(-6)^n\mathbf{e}^{24} \qquad n=0,1,2,3,\ldots$$

Okay, at this point we can formally write down the Taylor Series for this problem.

$$\mathbf{e}^{-6x}=\sum_{n=0}^{\infty}\frac{f^{(n)}(-4)}{n!}(x+4)^n=\boxed{\sum_{n=0}^{\infty}\frac{(-6)^n\mathbf{e}^{24}}{n!}(x+4)^n}$$

6. Okay, we'll need to start off this problem by taking a few derivatives of the function.

$$\begin{aligned}
n=0: &\quad f(x)=\frac{7}{x^4}=7x^{-4}\\
n=1: &\quad f'(x)=-7(4)x^{-5}\\
n=2: &\quad f''(x)=7(4)(5)x^{-6}\\
n=3: &\quad f^{(3)}(x)=-7(4)(5)(6)x^{-7}\\
n=4: &\quad f^{(4)}(x)=7(4)(5)(6)(7)x^{-8}
\end{aligned}$$

It is now time to see if we can get a formula for the general term in the Taylor Series.
Hopefully you can see the pattern in the derivatives above. The general term is given by,

$$\begin{aligned}
f^{(n)}(x)&=7(-1)^n\frac{(2)(3)}{(2)(3)}(4)(5)(6)\cdots(n+3)x^{-8}\\
&=7(-1)^n\frac{(2)(3)(4)(5)(6)\cdots(n+3)}{6}x^{-8}\\
&=\frac{7}{6}(-1)^n(n+3)!x^{-(n+4)} \qquad n=0,1,2,3,\ldots
\end{aligned}$$

Now, recall that we don't really want the general term at any x. We want the general term at x=−3. This is,

$$\begin{aligned} f^{(n)}(-3) &= \frac{7}{6}(-1)^n(n+3)!(-3)^{-(n+4)} \\ &= \frac{7(-1)^n(n+3)!}{6(-3)^{n+4}} \\ &= \frac{7(-1)^n(n+3)!}{6(-1)^{n+4}(3)^{n+4}} \\ &= \frac{7(n+3)!}{6(-1)^4(3)^{n+4}} \\ &= \frac{7(n+3)!}{6(3)^{n+4}} \qquad n=1,2,3,\ldots \end{aligned}$$

Okay, at this point we can formally write down the Taylor Series for this problem.

$$\frac{7}{x^4} = \sum_{n=0}^{\infty}\frac{f^{(n)}(-3)}{n!}(x+3)^n = \sum_{n=0}^{\infty}\frac{7(n+3)!}{6(3)^{n+4}n!}(x+3)^n = \boxed{\sum_{n=0}^{\infty}\frac{7(n+3)(n+2)(n+1)}{6(3)^{n+4}}(x+3)^n}$$

Don't forget to simplify/cancel where we can in the final answer. In this case we could do some simplifying with the factorials.

7. We'll start off by taking a few derivatives of the function and evaluating them at x=2

$$\begin{array}{lll} n=0: & f(x)=7x^2-6x+1 & f(2)=17 \\ n=1: & f'(x)=14x-6 & f'(2)=22 \\ n=2: & f''(x)=14 & f''(2)=14 \\ n\geq 3: & f^{(n)}(x)=0 & f^{(n)}(2)=0 \end{array}$$

Because all the derivatives are zero after some point we don't need a formula for the general term. All we need are the values of the non-zero derivative terms.

Once we have the values from the previous step all we need to do is write down the Taylor Series. To do that all we need to do is strip all the non-zero terms from the series and then acknowledge that the remainder will just be zero (all the remaining terms are zero after all!).

Doing this gives,

$$\begin{aligned} 7x^2-6x+1 &= \sum_{n=0}^{\infty}\frac{f^{(n)}(2)}{n!}(x-2)^n \\ &= f(2)+f'(2)(x-2)+\frac{1}{2}f''(2)(x-2)^2+\sum_{n=3}^{\infty}\frac{f^{(n)}(2)}{n!}(x-2)^n \\ &= \boxed{17+22(x-2)+7(x-2)^2} \end{aligned}$$

It looks a little strange but there it is. Do not multiply/simplify this out. This really is the answer we are looking for.

8.

Calculate the derivatives:

$$f'(x) = \left(e^{kx}\right)' = ke^{kx}, \quad f''(x) = \left(ke^{kx}\right)' = k^2 e^{kx}, \ldots \quad f^{(n)}(x) = k^n e^{kx}.$$

Then, at $x = 0$ we have

$$f(0) = e^0 = 1, \quad f'(0) = ke^0 = k, \quad f''(0) = k^2 e^0 = k^2, \ldots \quad f^{(n)}(0) = k^n e^0 = k^n.$$

Hence, the Maclaurin expansion for the given function is

$$e^{kx} = \sum_{n=0}^{\infty} f^{(n)}(0) \frac{x^n}{n!} = 1 + kx + \frac{k^2 x^2}{2!} + \frac{k^3 x^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{k^n x^n}{n!}.$$

9.

Let $f(x) = (1+x)^{\mu}$, where μ is a real number and $x \neq -1$. Then we can write the derivatives as follows

$$f'(x) = \mu(1+x)^{\mu-1},$$

$$f''(x) = \mu(\mu-1)(1+x)^{\mu-2},$$

$$f'''(x) = \mu(\mu-1)(\mu-2) \cdot (1+x)^{\mu-3},$$

$$f^{(n)}(x) = \mu(\mu-1)(\mu-2) \cdots (\mu-n+1)(1+x)^{\mu-n}.$$

For $x = 0$, we obtain

$$f(0) = 1, \quad f'(0) = \mu, \quad f''(0) = \mu(\mu-1), \ldots \quad f^{(n)}(0) = \mu(\mu-1) \cdots (\mu-n+1).$$

Hence, the series expansion can be written in the form

$$(1+x)^{\mu} = 1 + \mu x + \frac{\mu(\mu-1)}{2!} x^2 + \frac{\mu(\mu-1)(\mu-2)}{3!} x^2 + \ldots + \frac{\mu(\mu-1) \cdots (\mu-n+1)}{n!} x^n +$$ so on.

This series is called the binomial series.

10. D

UNIT 4

PRACTICE QUESTION SOLUTIONS

11.

Given, dv / dx + $(1+y^2)$ / y = 0.

Or dv / dx = - $(1+y^2)$ / y.

Or y / $(1+y^2)$dy = - dx.

Now, 2y / $(1+y^2)$dy = - 2dx

Or log $(1+y^2)$ = -2x + c.

So, we have the solution to the differential equation to be ½ log $(1+y^2)$ + x = c.

12.

Given that, dy / dx = $(1 + y^2) / y^3$.

So, $y^3 / (1 + y^2)$ dy = dx.

Or, [y–y / $(1 + y^2)$] dy = dx.

Ydy–½ 2y / $(1 + y^2)$dy = dx.

Or, y^2 / 2–½ log $(y^2 + 1)$ = x + c.

Thus, we have the solution to the differential equation to be y^2 / 2 – ½ log $(y^2 + 1)$ = x + c.

13.

Solution The equation is of the type $\frac{dy}{dx}+\mathrm{P}y=\mathrm{Q}$, which is a linear differential equation.

Now I.F. = $\int \frac{1}{x} dx = e^{\log x} = x$.

Therefore, solution of the given differential equation is

$$y.x = \int x x^2\, dx \text{, i.e. } yx = \frac{x^4}{4}+c$$

Hence $y = \frac{x^3}{4}+\frac{c}{x}$.

14.

UNIT 4

PRACTICE QUESTION SOLUTIONS

We know that Taylor Series Formula is

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots \ldots + \frac{f^n(x)}{n!}(x-a)^n$$

Given function, $f(x) = x^3$
Centered at $a = 2$
$f'(x) = 3x^2$
$f''(x) = 6x$
$f'''(x) = 6$
$f''''(x) = 0$

Now the Taylor Series expansion of $\mathbf{f(x) = x^3}$ *is,*

$$f(x) = f(2) + [\frac{f'(2)'}{1!}](x-2) + [\frac{f''(2)}{2!}](x-2)^2 + [\frac{f'''(2)}{3!}](x-2)^3 + [\frac{f''''(a)}{4!}](x-2)^4$$

$f(x) = 8 + 12(x-2) + 6(x-2)^2 + (x-2)^3 + 0$
$f(x) = (x-2)^3 + 6(x-2)^2 + 12x - 16$

15.

Given function, $f(x) = 4x$
Centered at $a = 1$
$f'(x) = 4$
$f''(x) = 0$

Now the Taylor Series expansion of $\mathbf{f(x) = 4x}$ *is,*

$$f(x) = f(1) + [\frac{f'(1)'}{1!}](x-1) + [\frac{f''(1)}{2!}](x-1)^2$$

$f(x) = 4 + 4(x-1) + 0$
$f(x) = 4x$