Lecture 1

INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Class: FY BSc

Subject : Calculus

Subject Code: PUSASQF1.1

Chapter: Unit 1 – Chapter 1

Chapter Name: Pre-Calculus and Graphs

Syllabus Objectives

Unit 1

- Pre-Calculus and Graphs
- Limits and Continuity

Unit 2

- Differentiation
- Application of Derivatives

Unit 3

- Definite and Indefinite Integrals

Unit 4

- Differential Equations
- Taylor and Maclaurin series

Paper Pattern

External Examination break up – 60 marks

Question No.	Particulars	Total Marks	Question type
Q.1	10 Multiple choice questions of 1.5 mark each	15 marks	Concept checking questions
Q.2	3 questions of 5 marks each	15 marks	Knowledge Application oriented
Q.3	3 questions of 5 marks each	15 marks	Knowledge Application oriented
Option in Q4 (Part A or Part B)			
Q.4 Part A	One question or can be divided into multiple qts	15 marks	Higher order Application oriented
Q.4 Part B	One question or can be divided into multiple qts	15 marks	Higher order Application oriented

Study Pattern

- Concept checkers – After each chapter
- Quiz – Before lectures
- Class task – Once in a while
- PPT – Read for revision
- Reference notes/ Class notes – Regular reading, understanding concepts
- Practice questions
- Assignment* - For mandatory practice (at least 2)
- Project/ Case study
- Mock exam paper – Additional practice under time constraint and exam difficulty

* Considered in internal marking

Today's Agenda

1. Calculus
2. Functions
3. Types of Functions
4. Transformations of Functions

1 Calculus

- What is calculus?
- What do you think is the scope of calculus?
- Why do we study calculus?

Watch Video – The Essence of Calculus
https://www.youtube.com/watch?v=WUvTyaaNkzM

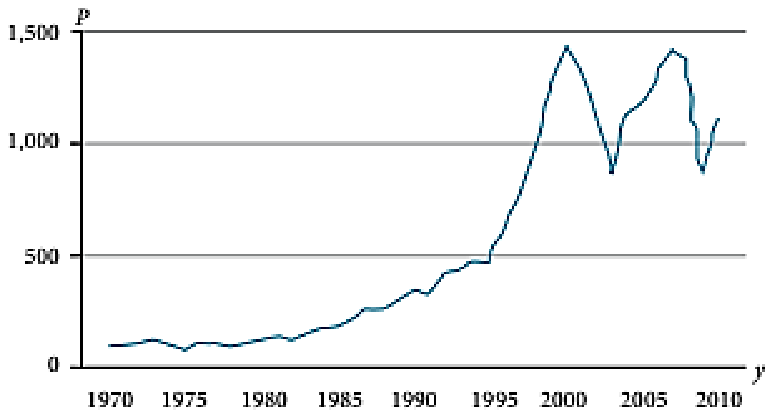
2 Functions

What are functions?
Or simply state, what do you understand by functions?

Do they have a practical use?

Motivation

Toward the end of the twentieth century, the values of stocks of internet and technology companies rose dramatically. As a result, the Standard and Poor's stock market average rose as well. Figure below tracks the value of that initial investment of just under $100 over the 40 years. It shows that an investment that was worth less than $500 until about 1995 skyrocketed up to about $1,100 by the beginning of 2000. That five-year period became known as the "dot-com bubble" because so many internet startups were formed. As bubbles tend to do, though, the dot-com bubble eventually burst. Many companies grew too fast and then suddenly went out of business. The result caused the sharp decline represented on the graph beginning at the end of 2000.



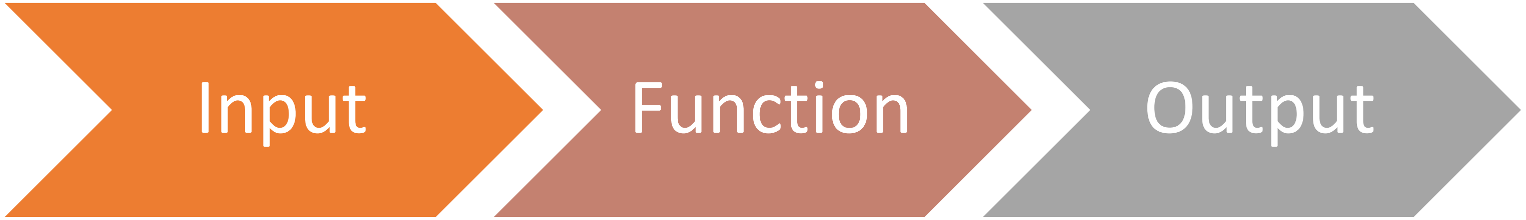
Notice, as we consider this example, that there is a definite relationship between the year and stock market average. For any year we choose, we can determine the corresponding value of the stock market average. In this chapter, we will explore these kinds of relationships and their properties.

2 Functions

Functions are used all the time in mathematics to describe relationships between two variables.

The concept of function is one of the most important in mathematics. It is explained as follows:
Given two sets A and B, a set with elements that are ordered pairs (x, y), where x is an element of A and y is an element of B, is a relation from A to B. A relation from A to B defines a relationship between those two sets. A function is a special type of relation in which each element of the first set is related to exactly one element of the second set. The element of the first set is called the input; the element of the second set is called the output.

A function f consists of a set of inputs, a set of outputs, and a rule for assigning each input to exactly one output.

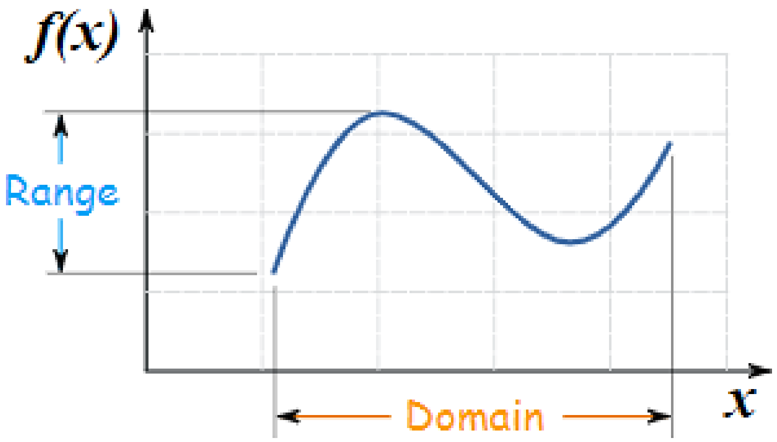


2 Functions & their graphs

- Let two sets X and Y be given. If for every element x in the set X there is exactly one element (an image) y=f(x) in the set Y, then it is said that the function f is defined on the set X.

- If we consider the number sets X⊂R, Y⊂R (where R is the set of real numbers), then the function y=f(x) can be represented as a graph in a Cartesian coordinate system Oxy.

- The input values make up the domain, and the output values make up the range.



Question

Which table, Table 6 or Table 8, represents a function (if any)?

Input	Output
2	1
5	3
8	6

Table 6

Input	Output
1	0
5	2
5	4

Table 8

Solution

Table 6 defines a function. Each input value corresponds to exactly one output value.
Table 8 does not define a function because the input value of 5 corresponds to two different output values.

Functions

Watch Video – Function, Domain and Range
https://www.youtube.com/watch?v=vO5qqfsWzhc

3 Types of Functions

What are the different types of functions that you know?

3 Types of Functions

There are a wide range of functions we could think off:

- Even functions
- Odd functions
- Linear functions
- Polynomial functions
- Exponential functions
- Trigonometric functions
- Inverse functions
- Cubic functions
- Quadratic functions
- Logarithmic functions
- Identity functions

3 Types of Functions

What is the difference between an even and an odd function?

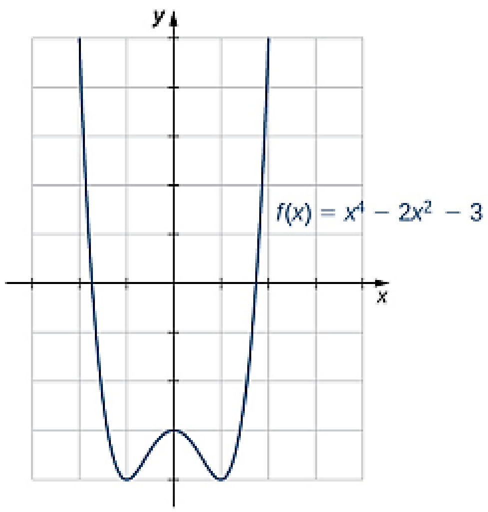
3.1 Even & Odd Function

If $f(x) = f(-x)$ for all x in the domain of f, then f is an even function. An even function is symmetric about the y-axis.

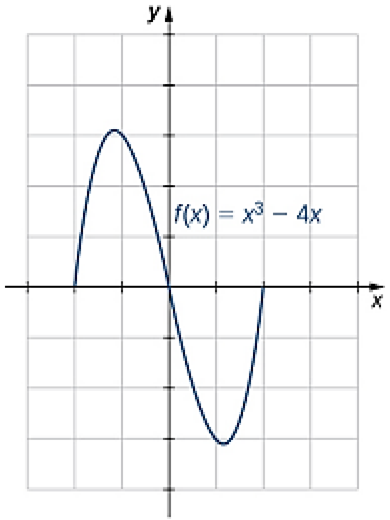
If $f(-x) = -f(x)$ for all x in the domain of f, then f is an odd function. An odd function is symmetric about the origin.

Even function has symmetry about the y-axis.

Odd function has symmetry about the origin.



(a) Symmetry about the y-axis



(b) Symmetry about the origin

3.1 Even & Odd Functions

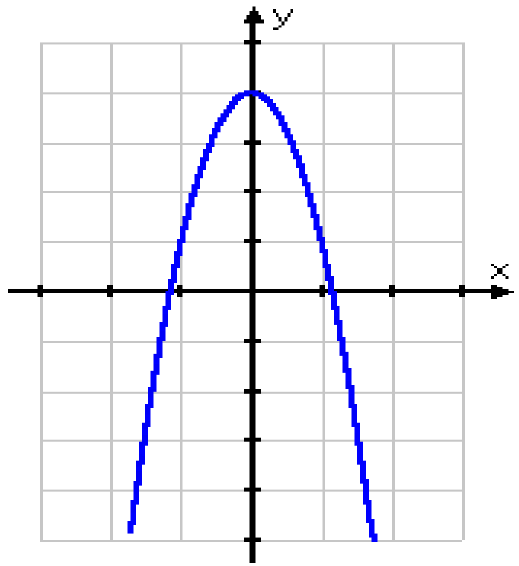
Ex 1: Determine algebraically whether $f(x) = -3x^2 + 4$ is even, odd, or neither

Solution:
I'll plug $-x$ in for x, and simplify:

$$f(-x) = -3(-x)^2 + 4$$
$$= -3(x^2) + 4$$
$$= -3x^2 + 4$$

Since, $f(x) = f(-x)$

$f(x)$ is even function.



3.1 Even & Odd Functions

Ex 2: Determine algebraically whether $f(x) = 2x^3 - 4x$ is even, odd, or neither.

Solution:

I'll plug $-x$ in for x, and simplify:

$f(-x) = 2(-x)^3 - 4(-x)$

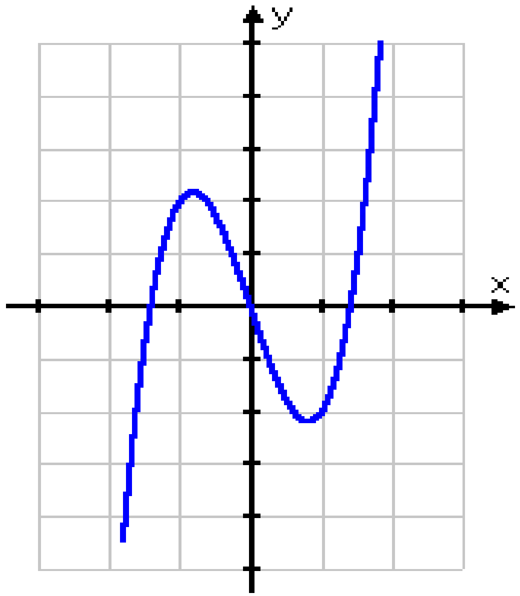
$= 2(-x^3) + 4x$

$= -2x^3 + 4x$

$f(-x) = -(2x^3 - 4x)$

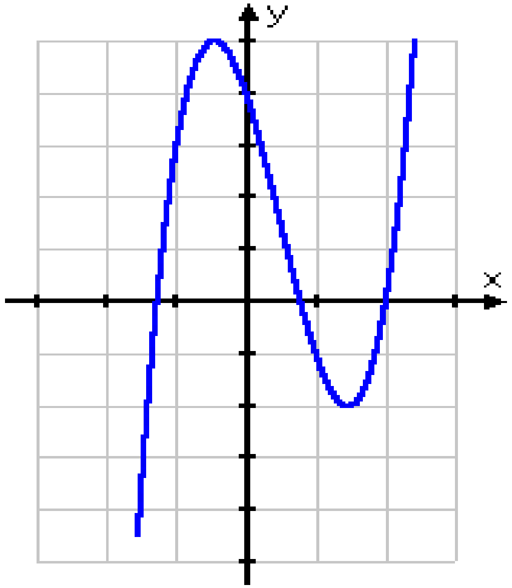
$f(-x) = -f(x)$

$f(x)$ is an odd function.



3.1 Even & Odd Functions

Ex 3: Determine algebraically whether $f(x) = 2x^3 - 3x^2 - 4x + 4$ is even, odd, or neither.



Solution:
I'll plug $-x$ in for x, and simplify:

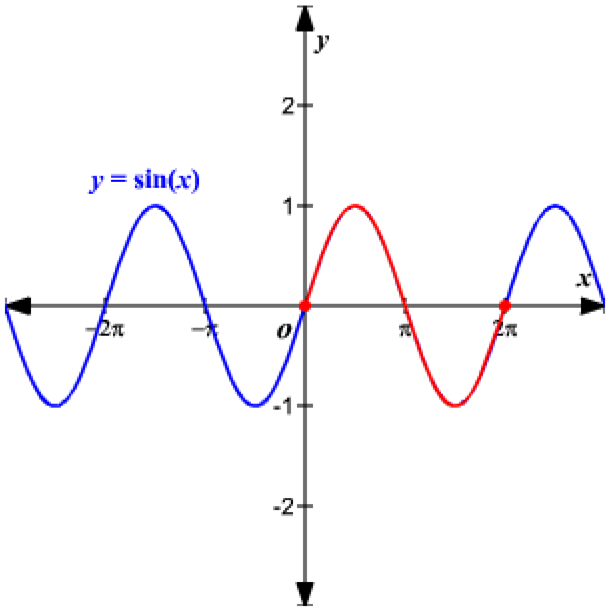
$$f(-x) = 2(-x)^3 - 3(-x)^2 - 4(-x) + 4$$
$$= 2(-x^3) - 3(x^2) + 4x + 4$$
$$= -2x^3 - 3x^2 + 4x + 4$$

$f(x)$ is neither even nor odd.

3.2 Periodic Function

Define a periodic function as $f(x+kT)=f(x)$, where k is an integer, T is the period of the function

Periodic functions are functions that behave in a cyclic (repetitive) manner over a specified interval (called a period). The graph repeats itself over and over as it is traced from left to right.



3.2 Types & uses of Periodic Function

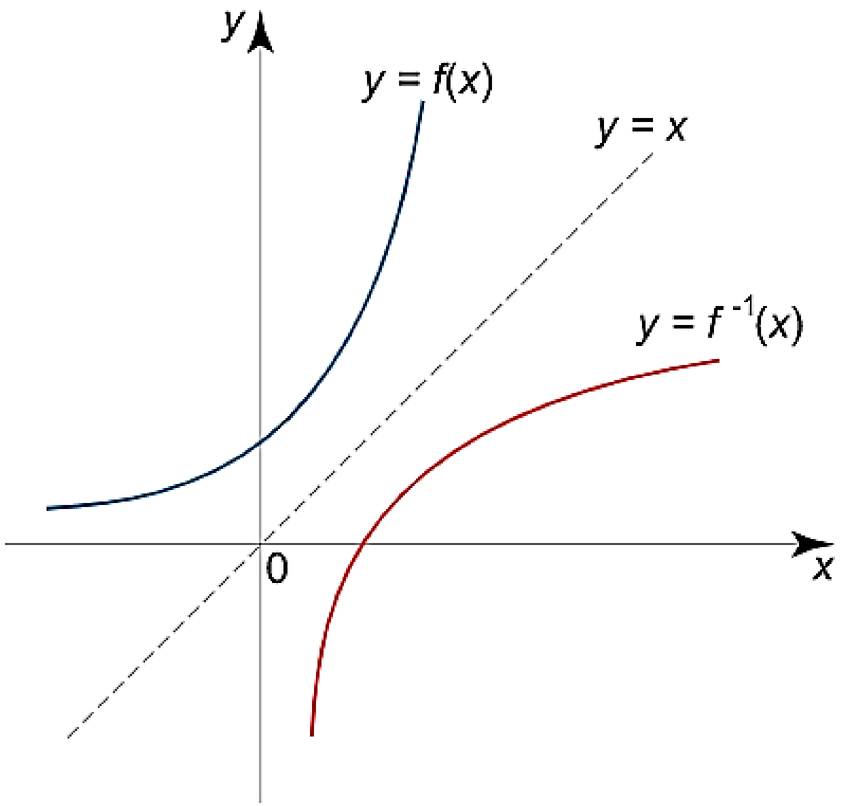
1. **Euler's Formula:**

2. **Jaccobi Elliptic Functions:** These functions are commonly used in the description of the motion of a pendulum.

3. **Fourier Series :** The Fourier series has applications in the representation of heatwaves , vibration analysis, quantum mechanics, electrical engineering, signal processing, image processing.

3.3 Inverse Function

- Given a function y=f(x).
- To find its inverse function of it, it is necessary solve the equation y=f(x) for x and then switch the variables x and y.
- The inverse function is often denoted as y= f^{-1}(x).
- The graphs of the original and inverse functions are symmetric about the line y=x.



3.3 Inverse Function

Ex 1: Find the inverse of h(x)=x^3+2

Solution:

To start, replace $h(x)$ with y. Then solve for x.

$$h(x) = x^3 + 2$$

$$y = x^3 + 2 \qquad \text{Replace h(x) with y}$$

$$y - 2 = x^3 \qquad \text{Subtract 2 from both sides}$$

$$\sqrt[3]{y-2} = x \qquad \text{Cube root of both sides}$$

$$\sqrt[3]{y-2} = h^{-1}(y) \qquad \text{Replace x with h\^{}\{-1\}(y)}$$

Since the choice of variable is arbitrary, we can now switch y to x to write the inverse in terms of x.

$$h^{-1}(x) = \sqrt[3]{x-2}$$

3.4 Composite Function

- Suppose that a function $y=f(u)$ depends on an intermediate variable u, which in turn is a function of the independent variable x: $u=g(x)$.
- In this case, the relationship between y and x represents a “function of a function” or a composite function, which can be written as $y=f(g(x))$.
- The two-layer composite functions can be easily generalized to an arbitrary number of “layers”.

3.4 Composite Function

Ex 1: Find g [f(x)] given that, $f(x) = 2x + 3$ and $g(x) = -x^2 + 5$

Solution:

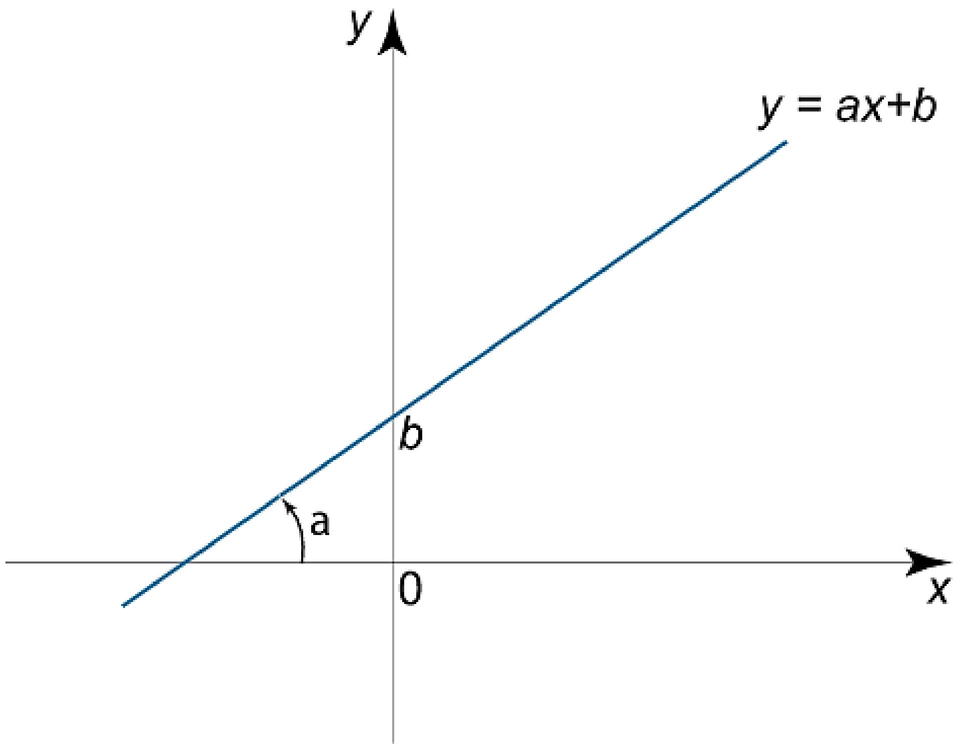
Replace x in $g(x) = -x^2 + 5$ with $2x + 3$

$$
\begin{aligned}
g[f(x)] &= -(2x + 3)^2 + 5 \\
&= -(4x^2 + 12x + 9) + 5 \\
&= -4x^2 - 12x - 9 + 5 \\
&= -4x^2 - 12x - 4
\end{aligned}
$$

3.5 Linear Function

Linear functions have the form y = ax + b, where a and b are constants.

- Here the number a is called the slope of the straight line.
- It is equal to the tangent of the angle between the straight line and the positive direction of the x-axis: a=tanα.
- The number b is the y-intercept.



JAQS

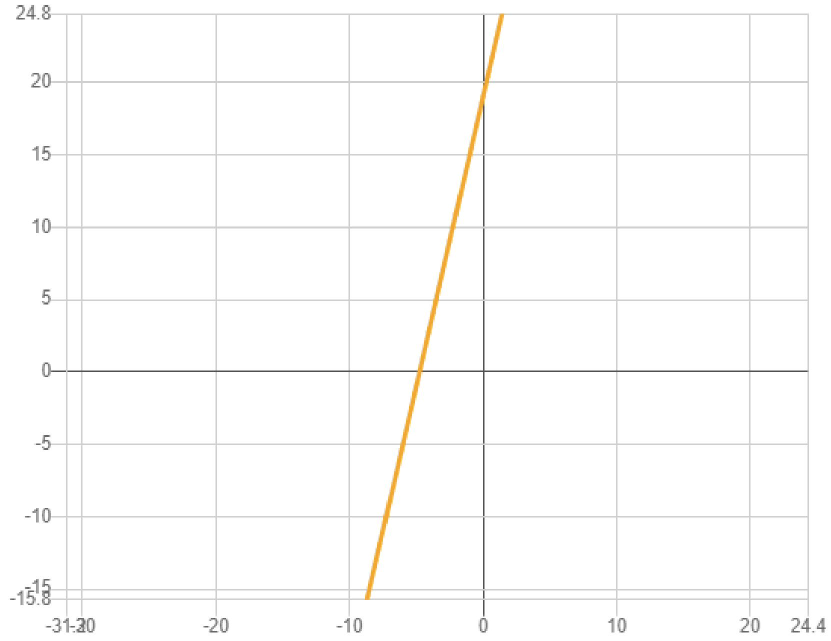
3.5 Linear Function

Ex 1: Find the linear function that has two points (-1, 15) and (2, 27) on it.

Solution:
The given points are $(x_1, y_1) = (-1, 15)$ and $(x_2, y_2) = (2, 27)$.

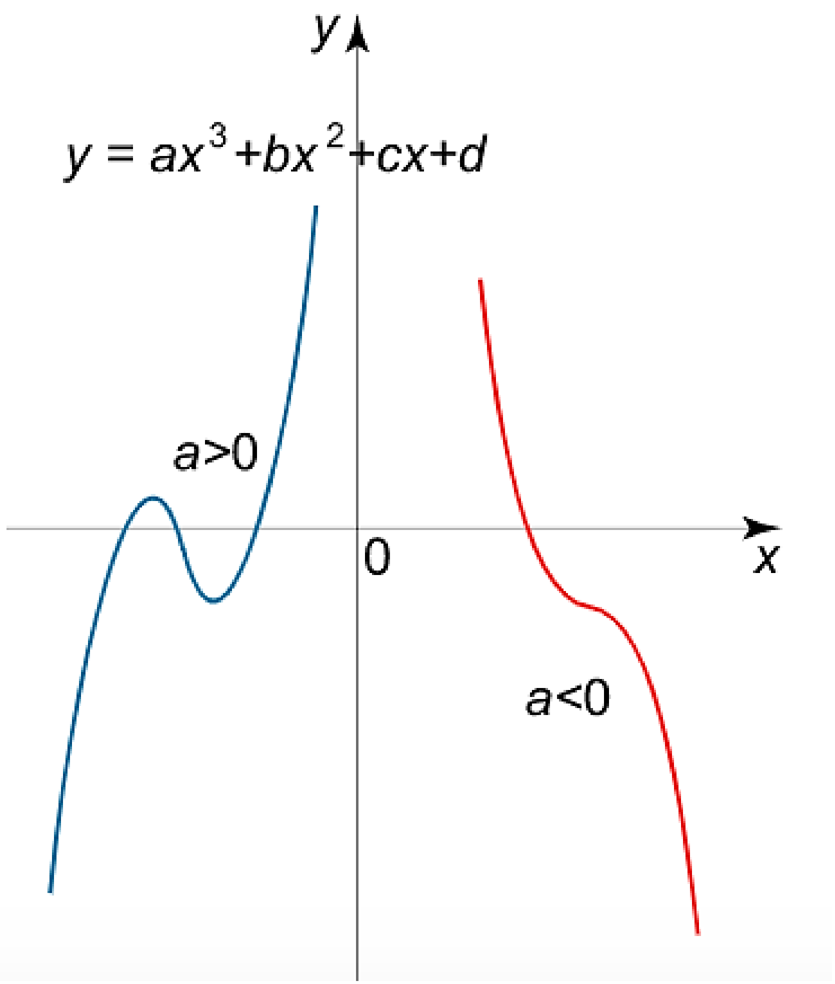
Find the slope of the function using the slope formula :
$m = (y_2 - y_1) / (x_2 - x_1) = (27 - 15) / (2 - (-1)) = 12/3 = 4$.

Find the equation of linear function using the point slope form.
$y - y_1 = m (x - x_1)$
$y - 15 = 4 (x - (-1))$
$y - 15 = 4 (x + 1)$
$y - 15 = 4x + 4$
$y = 4x + 19$
Therefore, the equation of the linear function is, $f(x) = 4x + 19$.



3.6 Cubic Function

- The simplest cubic function is given by
  $y = x^3$, $x \in R$.
- In general, a cubic function is described by the formula
  $y = ax^3 + bx^2 + cx + d$, $x \in R$,
  where a, b, c, d are real numbers ($a \neq 0$).
- The graph of a cubic function is called a cubic parabola.
- When $a>0$, the cubic function is increasing, and when $a<0$, the cubic function is, respectively, decreasing

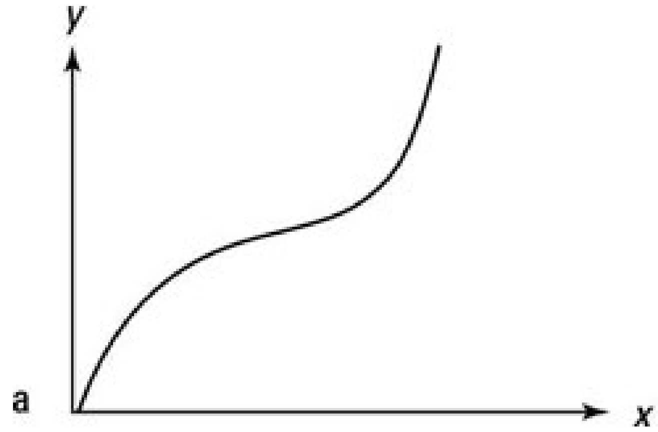


3.6 Uses of Cubic Function

Cubic Functions in Econometrics Are Good for Inflexion

With a cubic function, you allow the effect of the independent variable (*X*) on the dependent variable (*Y*) to change. As the value of *X* increases (or decreases), the impact of the dependent variable may increase or decrease. However, unlike a quadratic function, this relationship changes at some unique value of *X*.

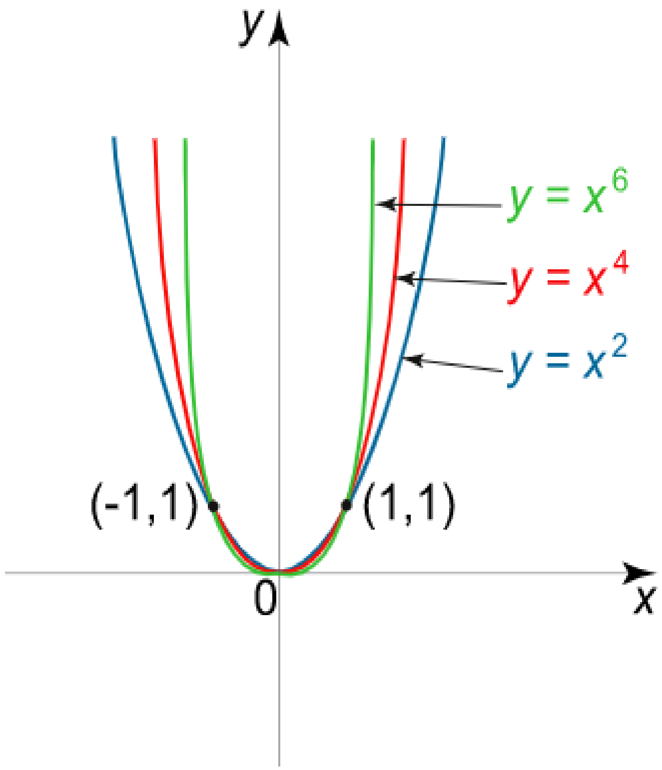
In other words, at some specific point, a decreasing effect becomes increasing or an increasing effect becomes decreasing. The point at which this occurs is called the **inflexion point**.

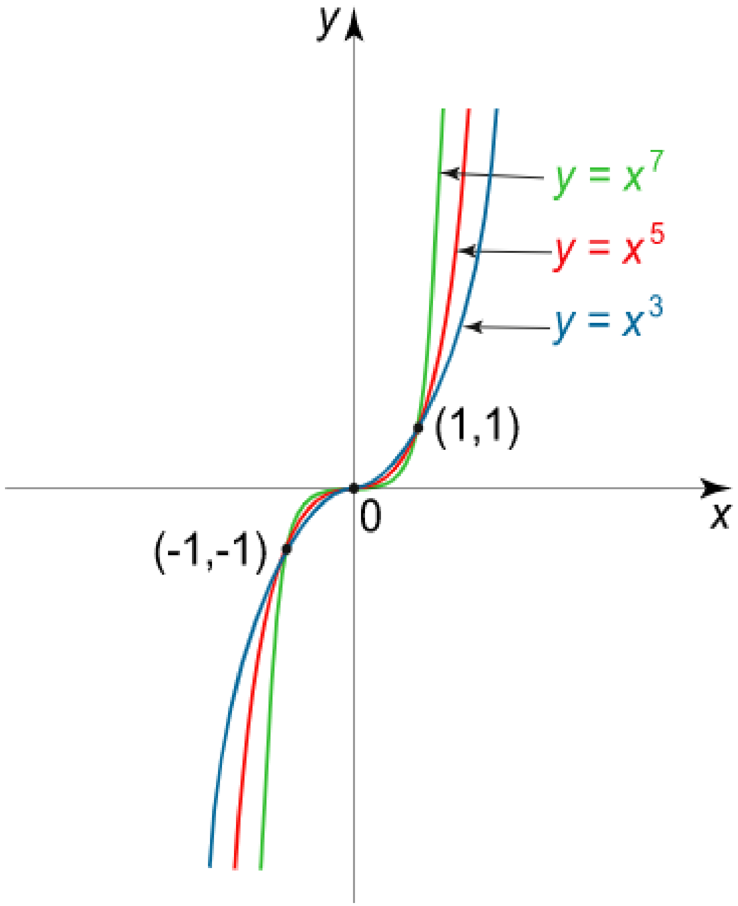


A decreasing slope followed by an increasing slope, as shown in part .

3.7 Power Function

$Y = x^n, X \in R, N \in N.$





3.7 Power Function

Ex 1: Sketch the graph of $f(x)=x^8$

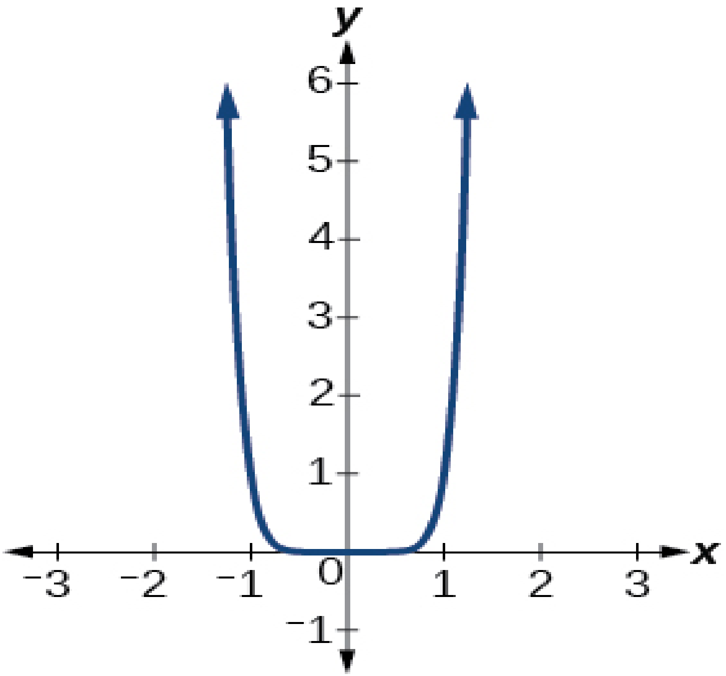
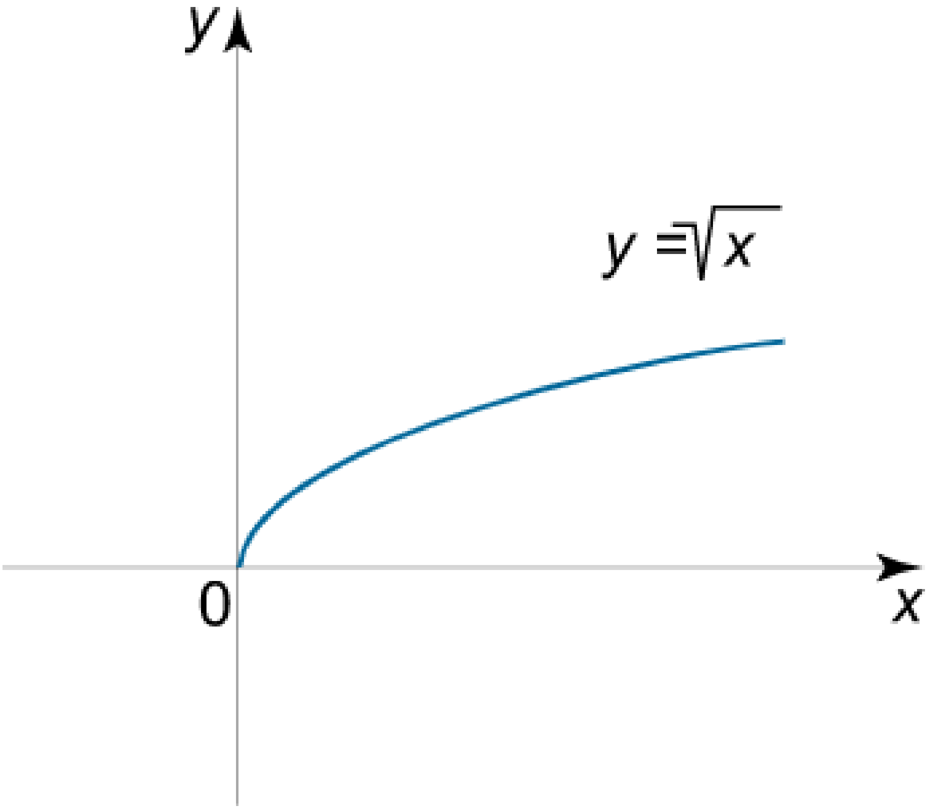


Figure 3.3.5: Graph of $f(x) = x^8$.

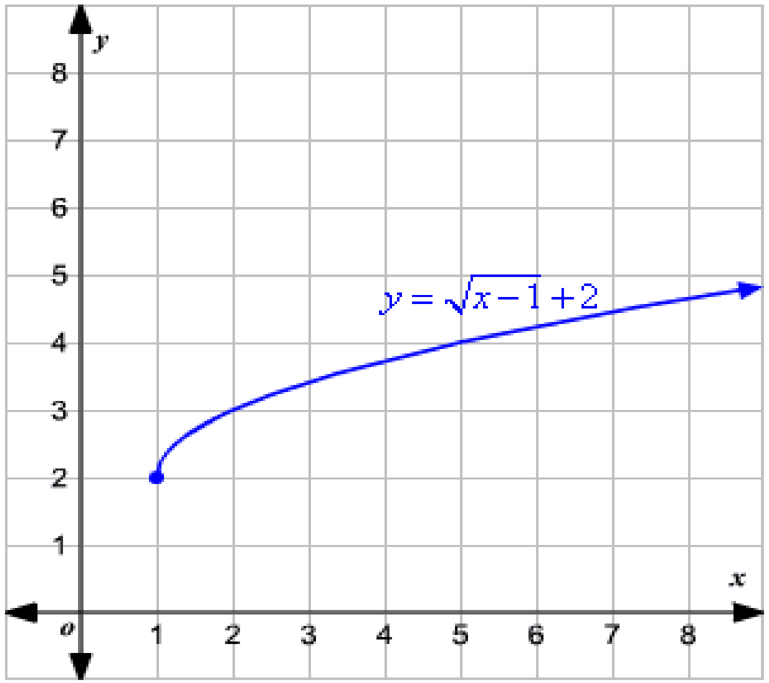
3.8 Square root Function

$y = \sqrt{x}, \quad x \in [0,\infty).$



3.8 Square root Function

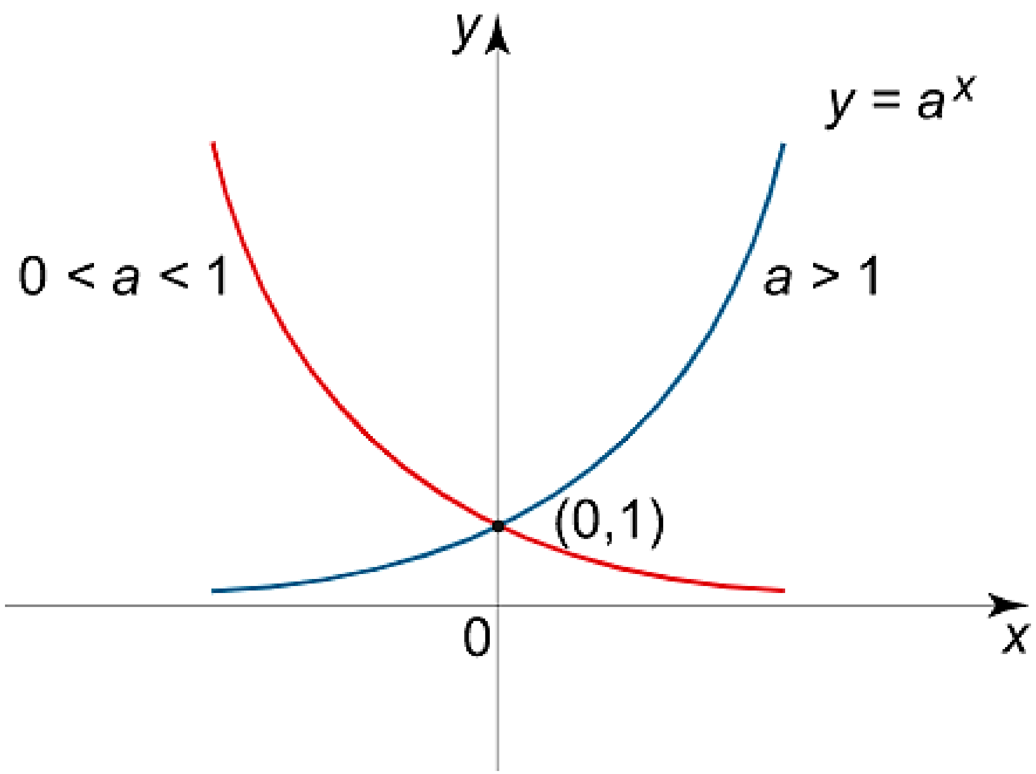
Ex 1: Sketch the graph of y=(√x−1) + 2



3.9 Exponential Function

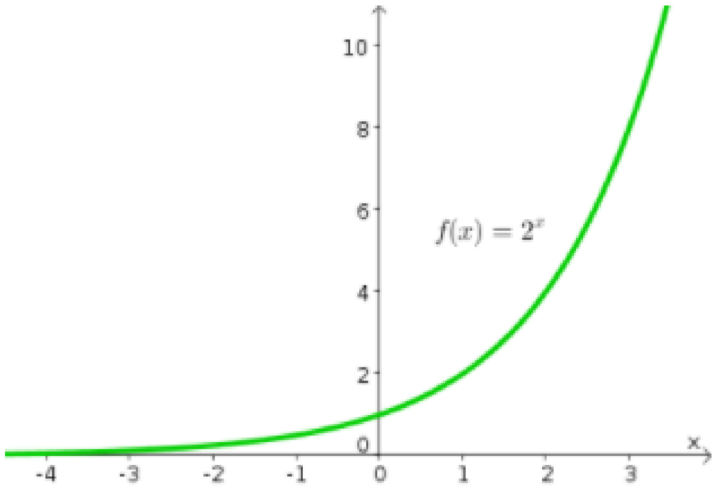
$Y = a^x$, x ∈ R, a>0, a≠1,
y = e^x when a = e ≈ 2.71828182846…

An exponential function increases when a>1 and decreases when 0<a<1.

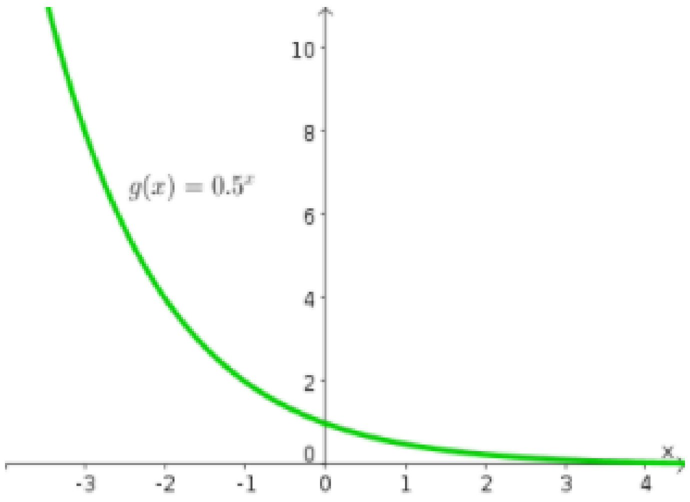


3.9 Exponential Function

Ex 1: $f(x) = 2^x$



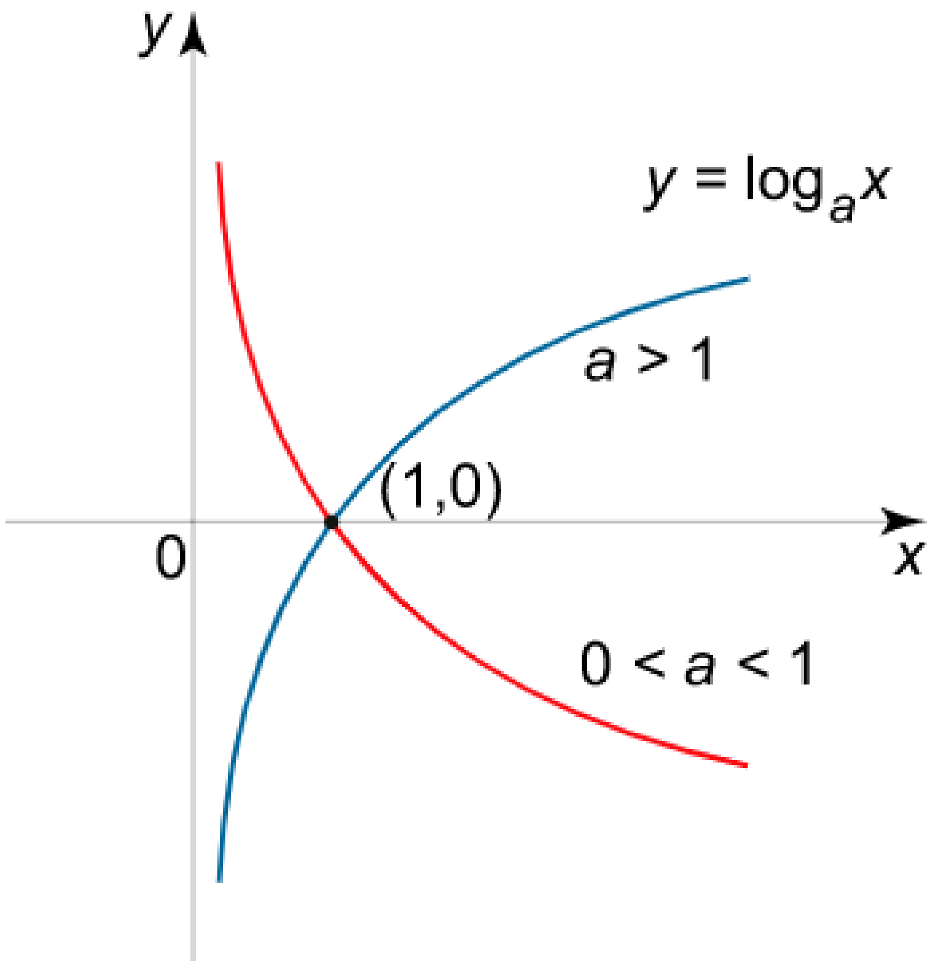
Ex 2: $g(x) = \left(\frac{1}{2}\right)^x$



3.10 Logarithmic Function

Y = logax , x ∈ (0,∞), a>0, a≠1,
y = ln x , when a = e, x ∈ (0,∞).

A logarithmic function increases if a>1 and decreases if 0<a<1.



3.10 Logarithmic Function

1) A derivative of a log variable with respect to time will give you the growth rate or percentage change of that variable.

2) Logarithmic functions are extremely convenient for Taylor Approximation. First order Taylor polynomial is enough to approximate a log function. This property comes in very handy in **log linearization**.

•This property helps in finding out speed of convergence in growth theories i.e. how fast an economy which has fallen out of a steady state will return to the steady state.

3) A third point is that logarithms help in scaling down values of variables. If your data has very large deviations then taking logs instead of the base value helps in making charts and graphs.

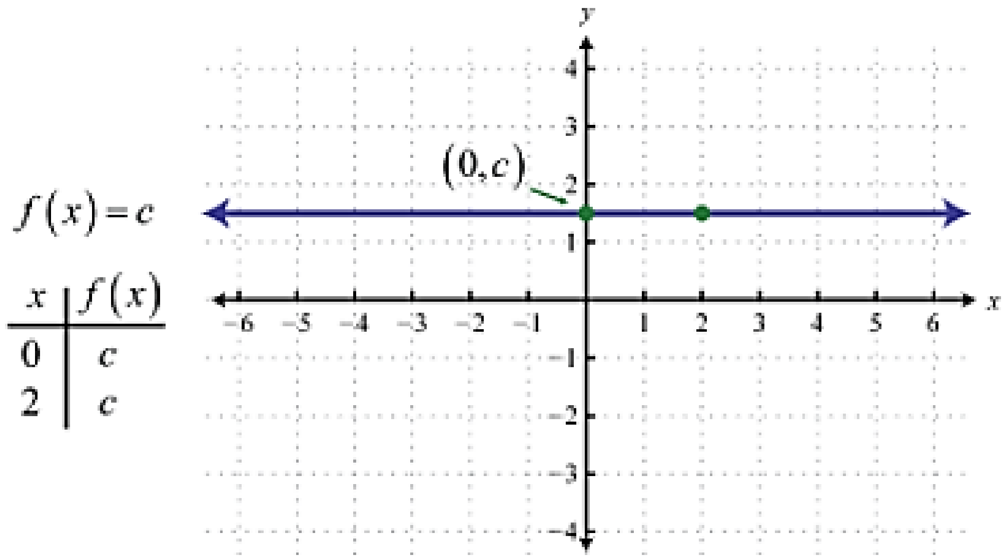
3.11 Constant Function

Any function of the form f(x) = c, where *c* is any real number, is called a **constant function**.

Constant functions are linear and can be written f(x) = 0x + c.

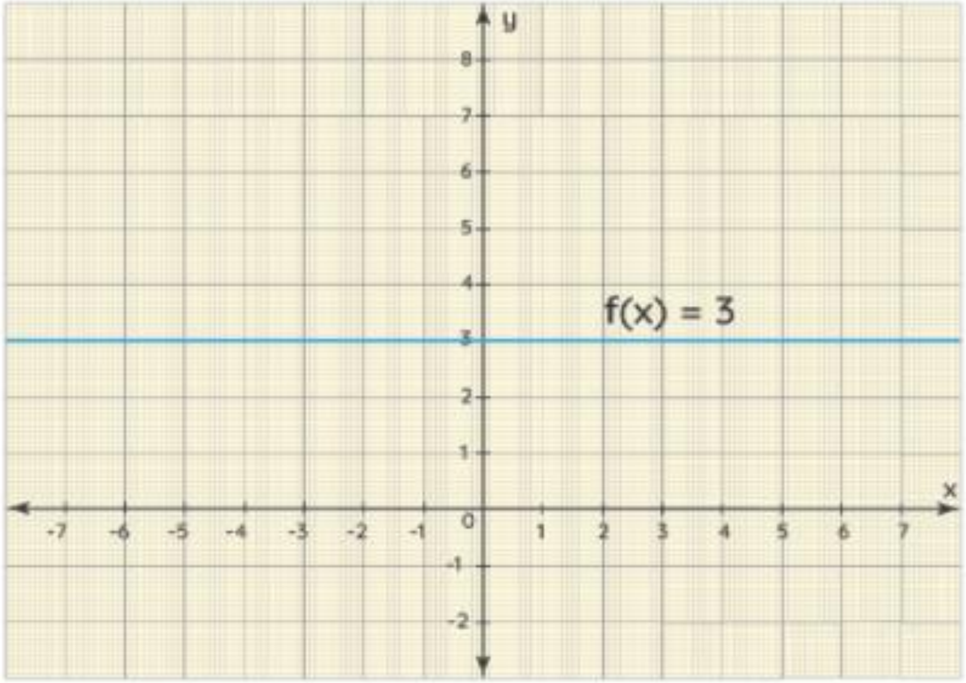
In this form, it is clear that the slope is 0 and the *y*-intercept is (0,c).

The graph of a constant function is a horizontal line



3.11 Constant Function

Ex 1: The graph of the constant function f(x) = 3



3.12 Identity Function

Define the **identity function** f(x) = x.

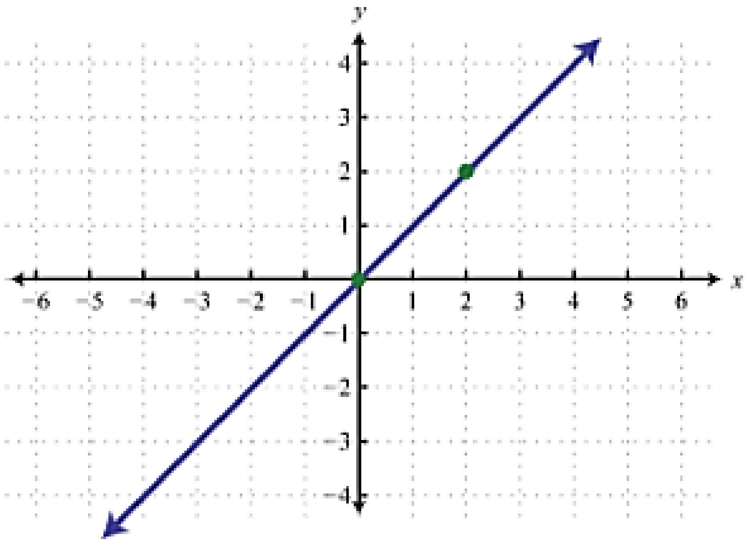
Evaluating any value for *x* will result in that same value.

For example, f(0) = 0 and f(2) = 2.

The identity function is linear, f(x) = 1x + 0, with slope m = 1 and *y*-intercept (0, 0).

$f(x) = x$

x	$f(x)$
0	0
2	2



3.12 Identity Function

Example 1: If g(y) = (2y+3)/(3y-2).Then prove that g ◦ g(y) is an identity function.

Solution:

$g \circ g(y) = g(g(y)) = g((2y+3)/(3y-2))$

$$= \frac{2(\frac{2y+3}{3y-2})+3}{3(\frac{2y+3}{3y-2})-2}$$

$= (4y + 6 + 9y - 6)/(6y + 9 - 6y + 4)$

$= 13y/13$

$= y$

g ◦ g(y) = y. Thus, g ◦ g(y) is an identity function

3.13 Absolute Value Function

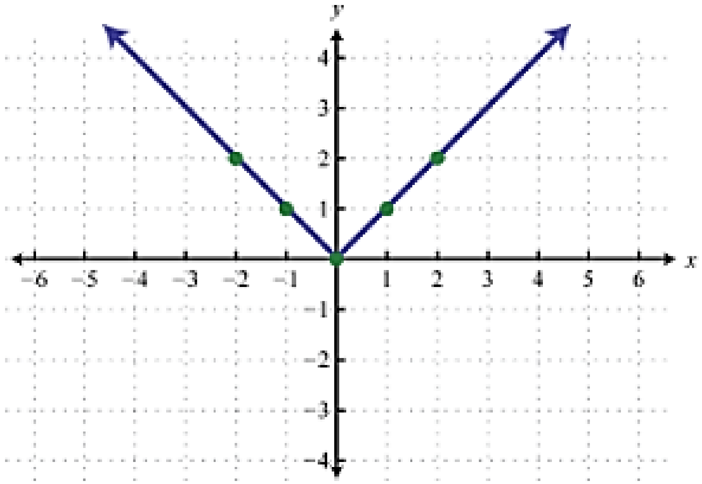
The **absolute value function**, defined by f(x) = |x|, is a function where the output represents the distance to the origin on a number line.

The result of evaluating the absolute value function for any nonzero value of *x* will always be positive.

For example, f(−2) = |−2| = 2 and f(2) = |2| = 2.

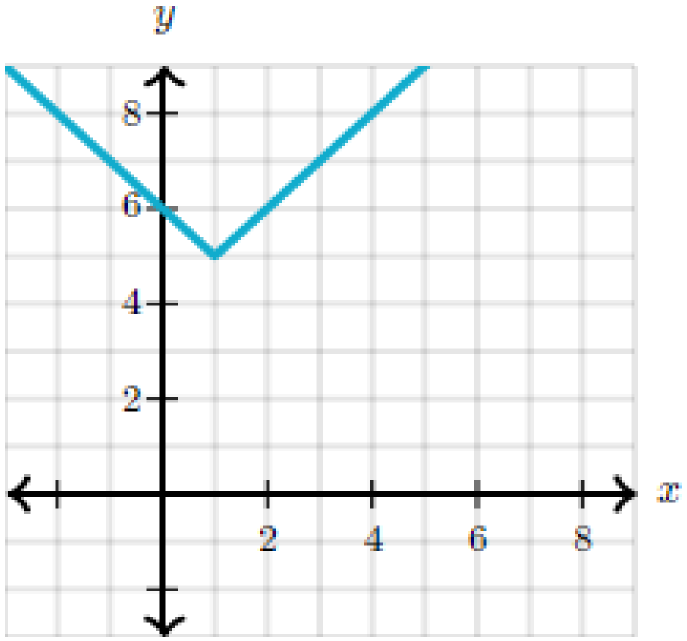
$f(x) = |x|$

x	$f(x)$
−2	2
−1	1
0	0
1	1
2	2



3.13 Absolute Value Function

Ex 1: Sketch the graph of $f(x)=|x-1|+5$



3.14 Functions

Polynomial Functions

$f(x) = a_0x^n + a_1x^{n-1} + ... + a^{n-1} x + a_n$

where $a_0, a_1, ..., a_n$ are real numbers, $a_0 \neq 0$.

Rational Functions

$f(x) = p(x)/q(x)$, where $p(x)$ and $q(x)$ are polynomials in x.

Domain is $R - \{x : q(x) = 0\}$

Questions

Identify the type of function :

1. $f(x) = |x - 1| + 3.$
2. $f(x) = \sqrt{(x^2 + 1)} + 2$
3. $f(g(x)) = -2x^2 + 5$
4. $g(x) = 6^{x+1}$
5. $f(x) = 3x - 1$

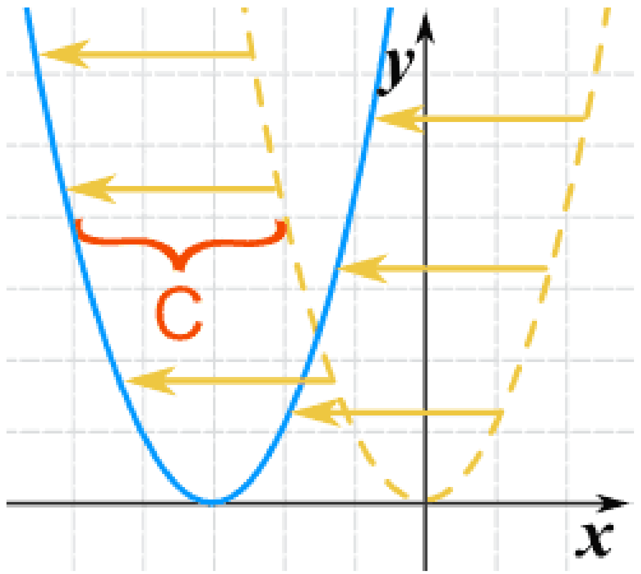
Solution

Identify the type of function :

1. f(x) =Absolute value function
2. f(x) =Square root function
3. f(g(x)) =Composite function
4. g(x) =Exponential function
5. f(*x*) =Linear function

4 Transformations of Functions

- Transformations alter a function while maintaining the original characteristics of that function.
- Transformations are ways that a function can be adjusted to create new functions.
- Transformations often preserve the original shape of the function.
- Common types of transformations include rotations, translations, reflections, and scaling (also known as stretching/shrinking).



4 Common Transformations

Translation: Shift of an entire function in a specific direction.

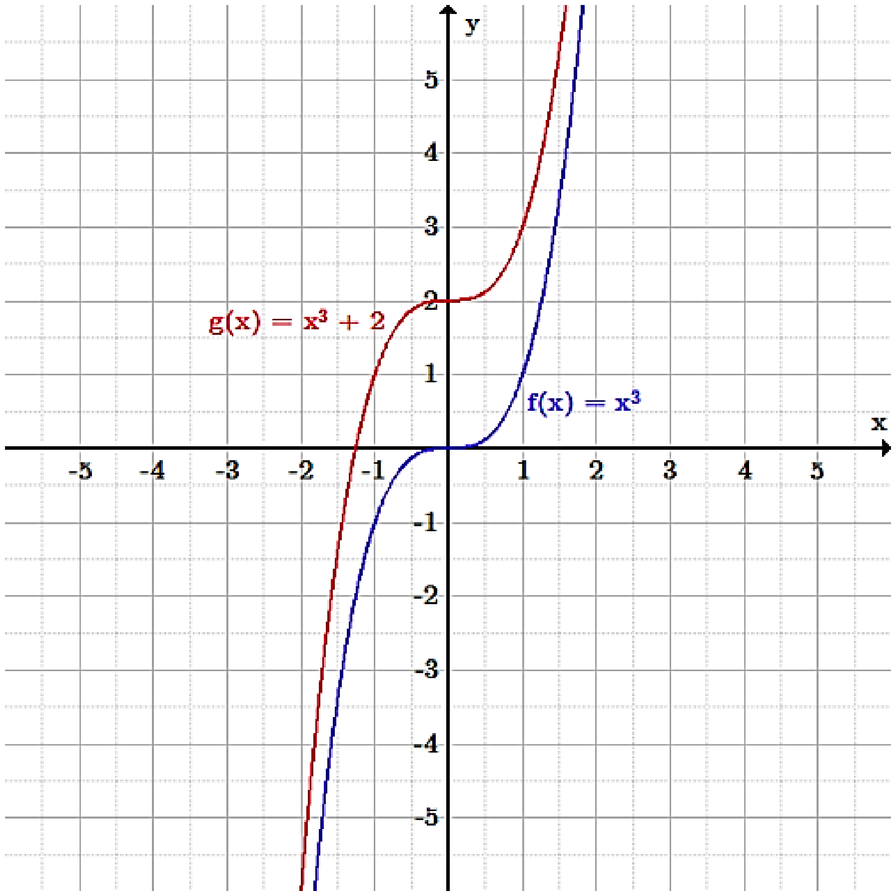
Scaling: Changes the size and/or the shape of the function.

Rotation: Spins the function around the origin.

Reflection: Mirror image of a function.

4.1 Translations

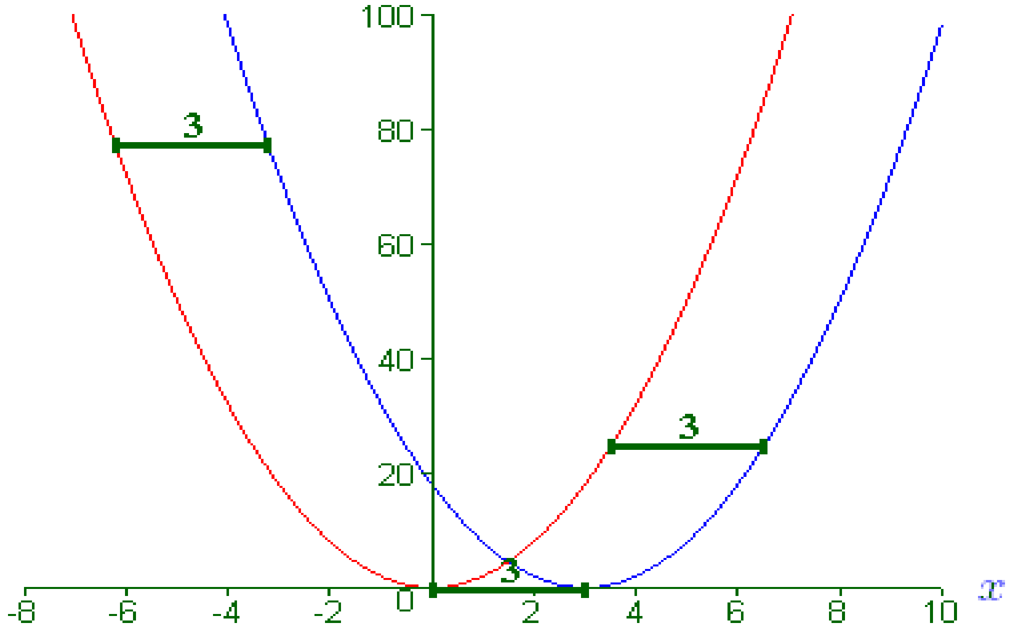
- A translation moves every point by a fixed distance in the same direction.
- The movement is caused by the addition or subtraction of a constant from a function.
- As an example, let $f(x)=x^3$.
- One possible translation of f(x) would be $x^3 + 2$.
- This would then be read as, "the translation of f(x) by two in the positive y direction".



4.1 Translations

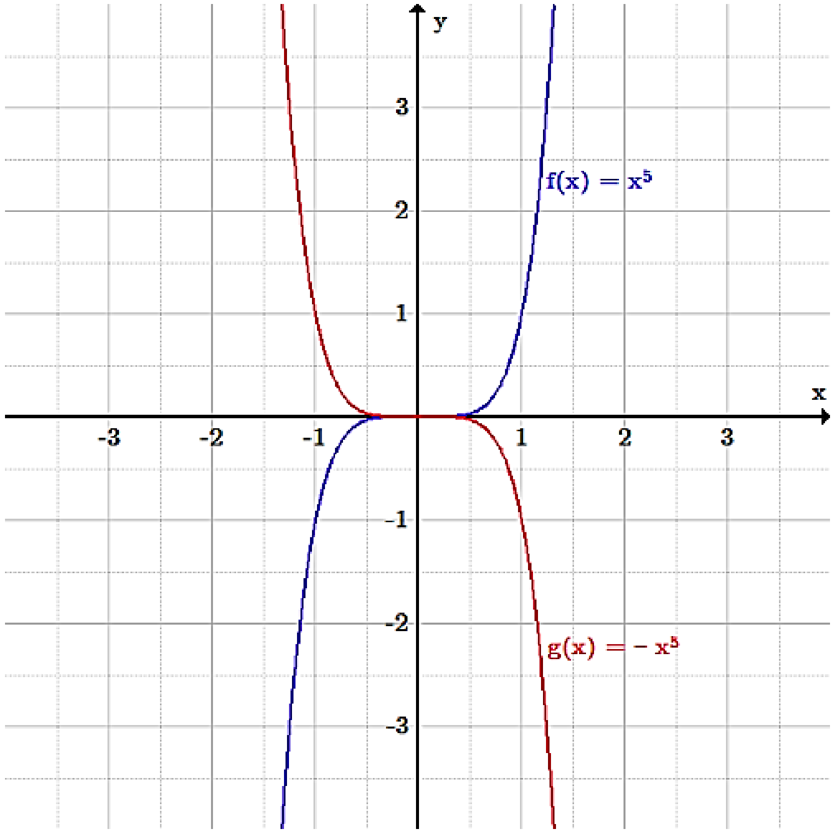
Example :

$$f(x) = 2x^2$$
$$y_1(x) = 2(x-3)^2$$



4.2 Reflection

- A reflection of a function causes the graph to appear as a mirror image of the original function.
- This can be achieved by switching the sign of the input going into the function.
- Let the function in question be $f(x) = x^5$.
- The mirror image of this function across the y-axis would then be $f(-x) = -x^5$. Therefore, we can say that $f(-x)$ is a reflection of $f(x)$ across the y-axis



4.2 Reflection

Example : $(i)\ y = x^2 \quad (ii)\ y = -x^2.$

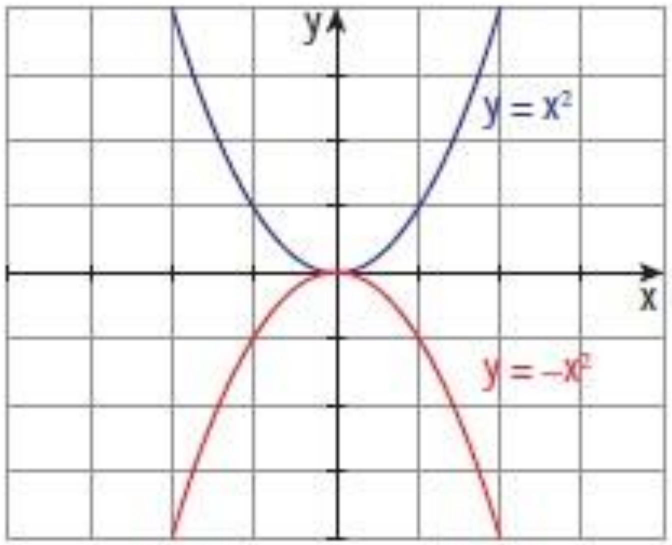


Figure 1.48

4.3 Rotation

A rotation is a transformation that is performed by "spinning" the object around a fixed point known as the centre of rotation.

Although the concept is simple, it has the most advanced mathematical process of the transformations discussed.

There are two formulas that are used:
$x_1 = x_0 \cos\theta - y_0 \sin\theta$y1
$y_1 = x_0 \sin\theta + y_0 \cos\theta$x1
where x_1 and y_1 are the new expressions for the rotated function, x_0 and y_0 are the original expressions from the function being transformed, and θ is the angle at which the function is to be rotated.

As an example, let y = x_2. If we rotate this function by 90 degrees, the new function reads:
[xsin(π/2)+ycos(π/2)]=[xcos(π/2)−ysin(π/2)]2

4.4 Scaling

- A transformation that changes the size and/or shape of the graph of the function.
- In algebra, equations can undergo scaling, meaning they can be stretched horizontally or vertically along an axis.
- This is accomplished by multiplying either x or y by a constant, respectively.

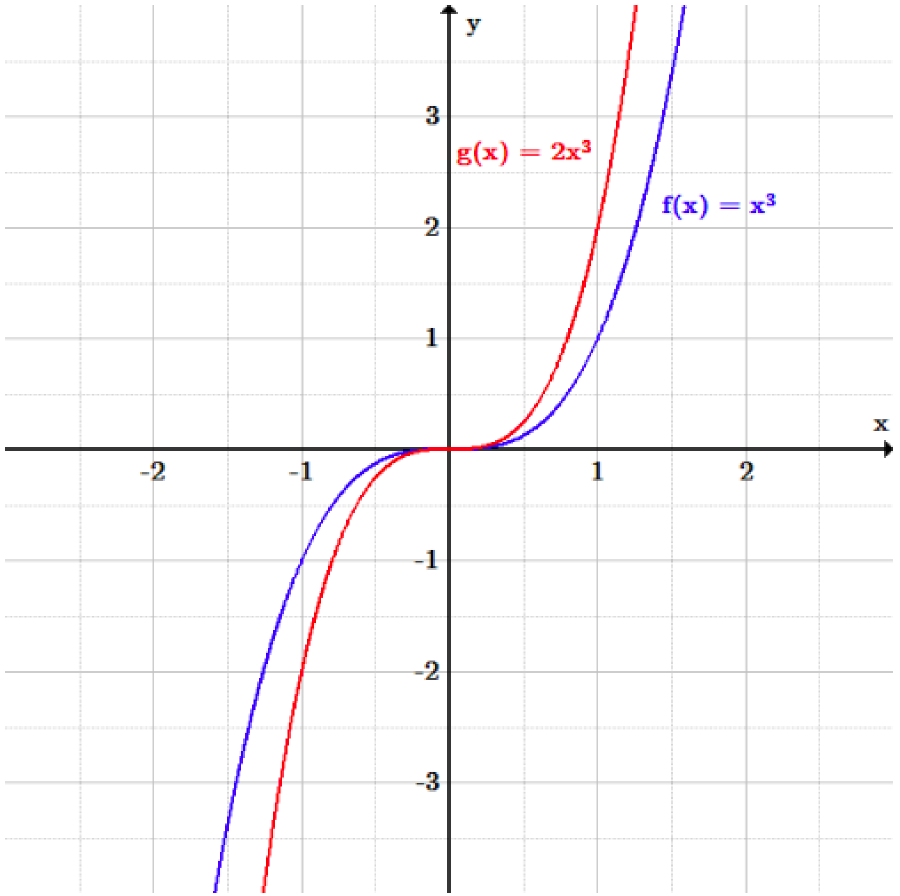
4.4 Scaling

- Scaling is a transformation that changes the size and/or the shape of the graph of the function.

- Note that until now, none of the transformations we discussed could change the size and shape of a function – they only moved the graphical output from one set of points to another set of points.

- As an example, let f(x)= x^3.
- Following from this, 2f(x)=2 x^3.

- The graph has now physically gotten "taller", with every point on the graph of the original function being multiplied by two.



4.5 Stretching & Shrinking

- When by either $f(x)$ or x is multiplied by a number, functions can "stretch" or "shrink" vertically or horizontally, respectively, when graphed.

- In general, a vertical stretch is given by the equation $y=bf(x)$. If $b>1$, the graph stretches with respect to the y-axis, or vertically. If $b<1$, the graph shrinks with respect to the y-axis.

- In general, a horizontal stretch is given by the equation $y=f(cx)$.

- If $c>1$, the graph shrinks with respect to the x-axis, or horizontally. If $c<1$, the graph stretches with respect to the x-axis.

4.5 Stretching & Shrinking

Example : (i) $f(x) = x^2$ (ii) $f(x) = \frac{1}{2}x^2$ (iii) $f(x) = 2x^2$

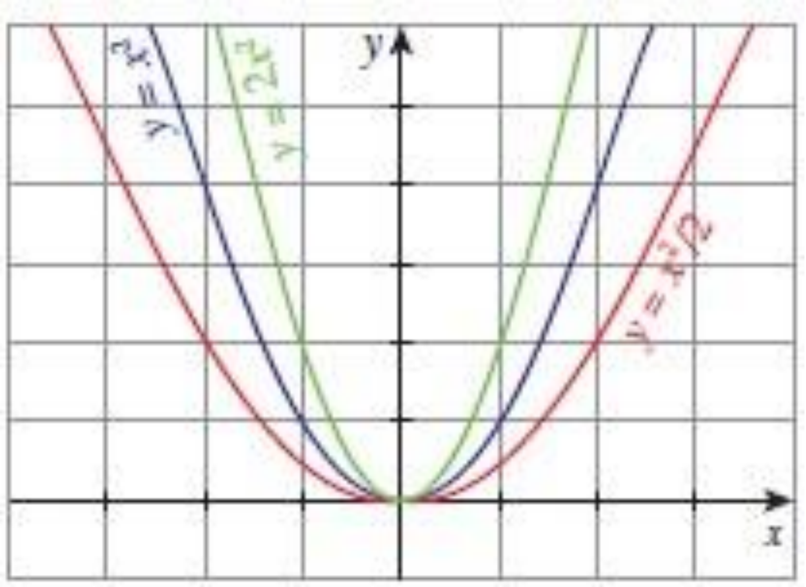


Figure 1.53

4.6 Vertical Scaling

- In vertical scaling, multiplying the entire function f(x) by a constant greater than one causes all the y values of an equation to increase, t]his leads to a “stretched” appearance in the vertical direction.

- If the function f(x) is multiplied by a value less than one, all the y values of the equation will decrease, leading to a “shrunken” appearance in the vertical direction.

- In general, the equation for vertical scaling is: $y=bf(x)$
where f(x) is some function and b is an arbitrary constant.

- If b is greater than one the function will undergo vertical stretching, and if b is less than one the function will undergo vertical shrinking.

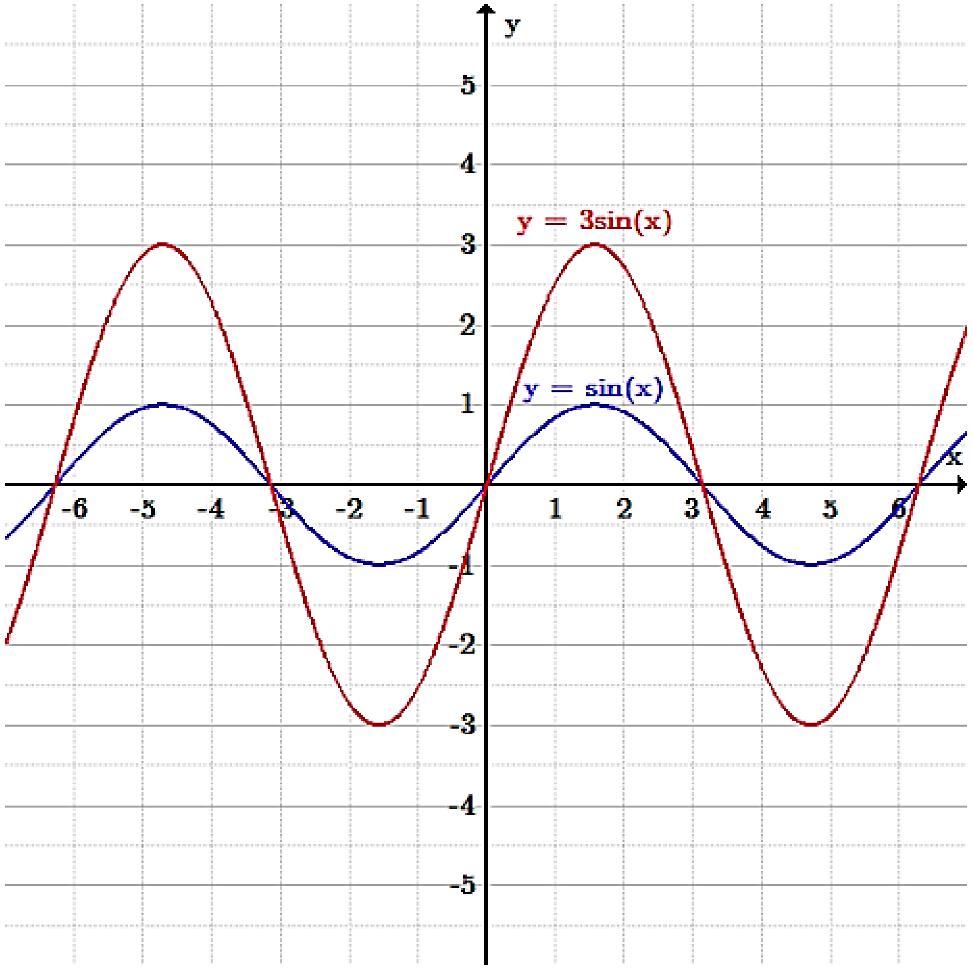
4.6 Vertical Scaling - Example

Consider y=sin(x)

If we want to vertically stretch the function by a factor of three, then the new function becomes:

Example: y=3f(x)

The graph looks like:



4.7 Horizontal Scaling

Now lets analyse horizontal scaling.

- Multiplying the independent variable x by a constant greater than one causes all the x values of an equation to increase. This leads to a “shrunken” appearance in the horizontal direction.

- If the independent variable x is multiplied by a value less than one, all the x values of the equation will decrease, leading to a “stretched” appearance in the horizontal direction.

4.7 Horizontal Scaling

In general, the equation for horizontal scaling is:

- Y = f(cx)
  where f(x) is some function and cc is an arbitrary constant.

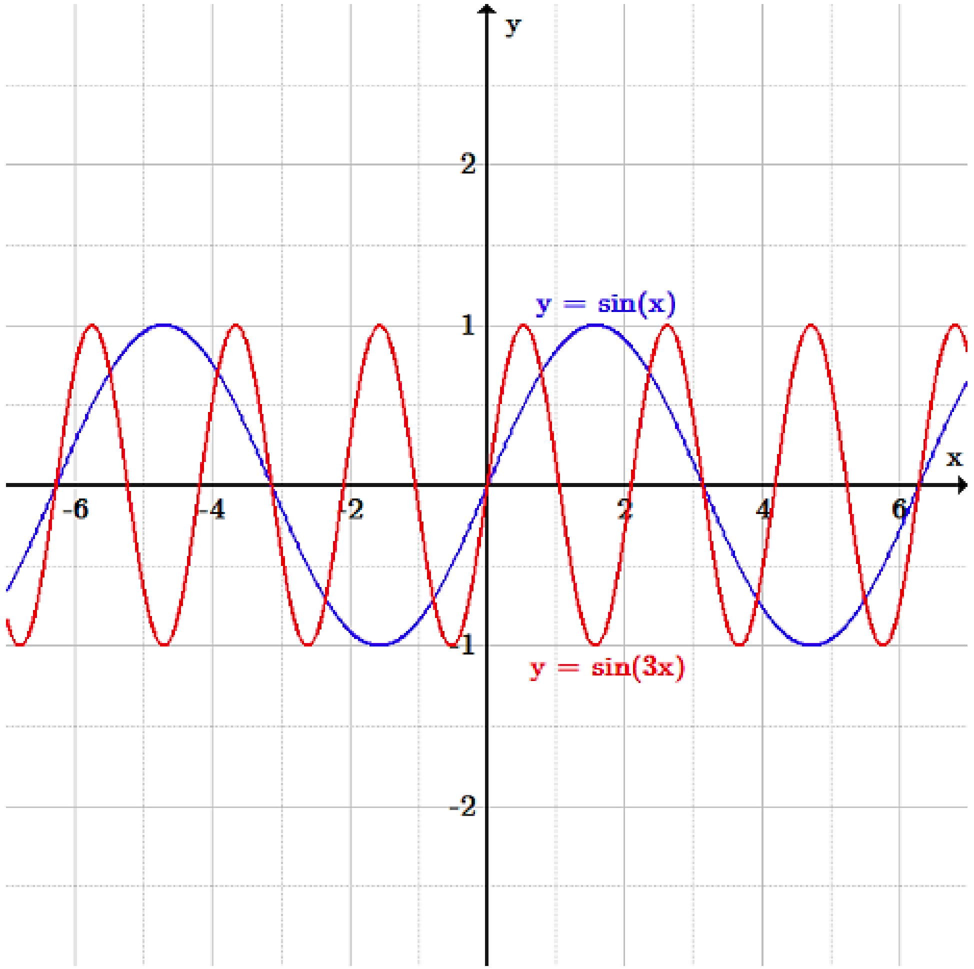
- If c is greater than one the function will undergo horizontal shrinking, and if c is less than one the function will undergo horizontal stretching.

4.7 Horizontal Scaling - Example

Consider Y = sin(x)

If we want to induce horizontal shrinking, the new function becomes:

Example: y = f(3x) = sin(3x)



Transformations of Functions - Summary

Transformation of $f(c > 0)$	Effect on the graph of f
$f(x) + c$	Vertical shift up c units
$f(x) - c$	Vertical shift down c units
$f(x + c)$	Shift left by c units
$f(x - c)$	Shift right by c units
$cf(x)$	Vertical stretch if $c > 1$; vertical compression if $0 < c < 1$
$f(cx)$	Horizontal stretch if $0 < c < 1$; horizontal compression if $c > 1$
$-f(x)$	Reflection about the x-axis
$f(-x)$	Reflection about the y-axis