Class: FY BSc

Subject : Calculus

Chapter: Unit 3 Chapter 5

Chapter Name: Definite & Indefinite integrals

Today's Agenda

1. Integration
   1. Introduction to integration
   2. What is an integral
   3. General formulas of integration
   4. Integration as the reverse of differentiation

2. Definite integrals

3. Indefinite integrals

4. Improper integrals

5. Properties of integral calculus

6. Methods of integration
   1. Integration by substitution
   2. Integration by parts
   3. Integration of some particular function

1 Integration

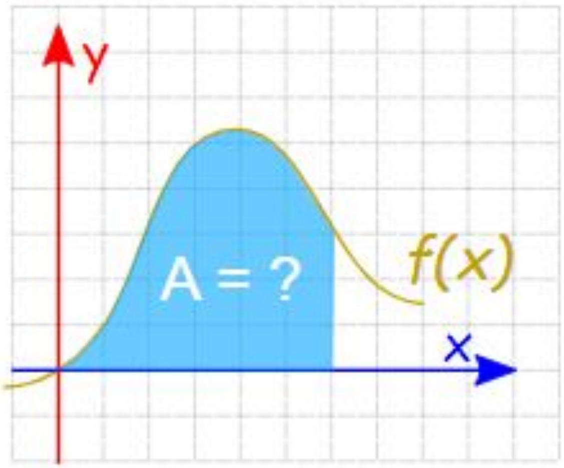
Integration is the process of finding the area of the region under the curve. This is done by drawing as many small rectangles covering up the area and summing up their areas. The sum approaches a limit that is equal to the region under the curve of a function.

The concept of integration has developed to solve the following types of problems:

1. To find the problem function, when its derivatives are given.
2. To find the area bounded by the graph of a function under certain constraints.

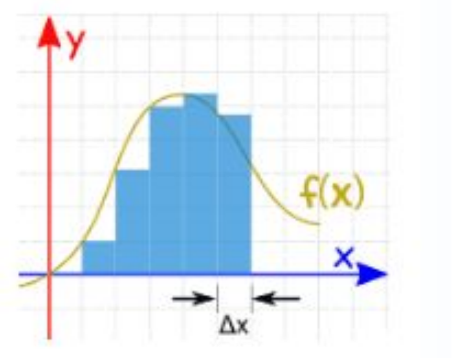
1.1 Introduction to Integration

Integration can be used to find areas, volumes, central points and many useful things. But it is easiest to start with finding the **area between a function and the x-axis** like this:

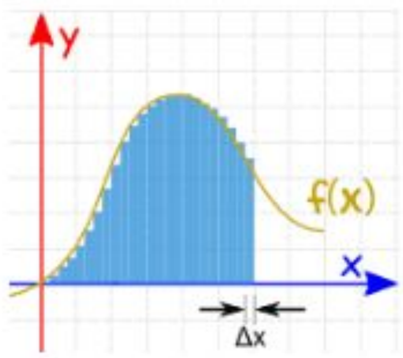


1.1 Introduction to Integration

1. We could calculate the function at a few points and **add up slices of width Δx** like this

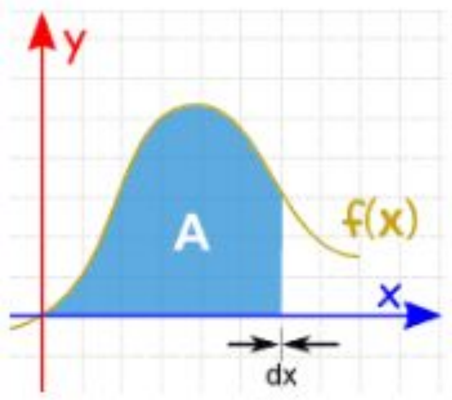


2. We can make **Δx** a lot smaller and **add up many small slices**



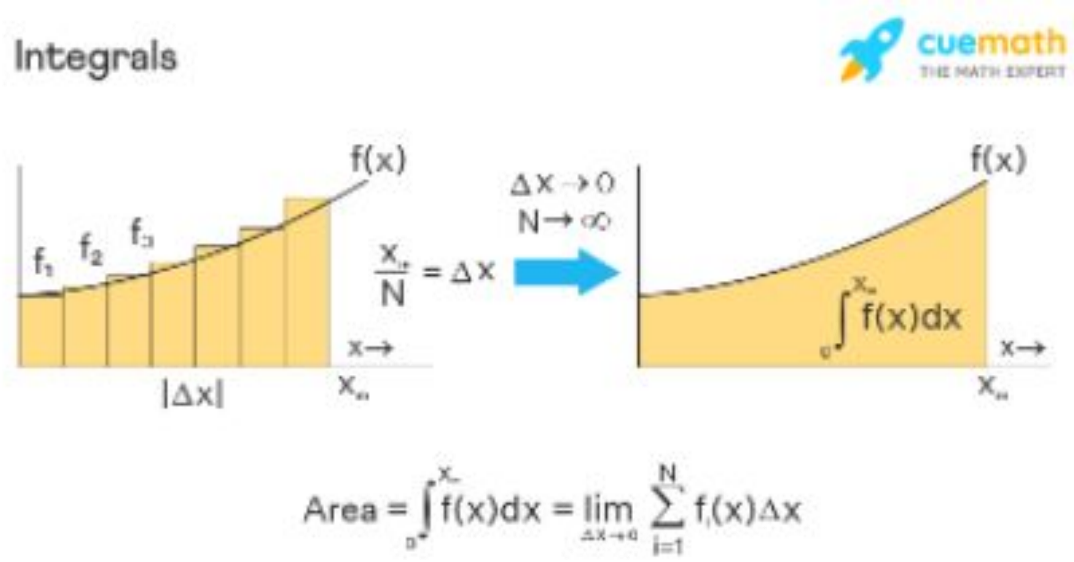
1.1 Introduction to Integration

And as the slices **approach zero in width**, the answer approaches the **true answer**.
We now write **dx** to mean the **Δx** slices are approaching zero in width.



1.2 What is an integral?

F(x) is called an antiderivative or Newton-Leibnitz integral or primitive of a function f(x) on an interval I. F'(x) = f(x), for every value of x in I.
Integral is the representation of the area of a region under a curve. We approximate the actual value of an integral by drawing rectangles. A definite integral of a function can be represented as the area of the region bounded by its graph of the given function between two points in the line. The area of a region is found by breaking it into thin vertical rectangles and applying the lower and the upper limits, the area of the region is summed up. We specify an integral of a function over an interval on which the integral is defined.



1.3 General Formulas of Integration

1. $\int 1 \, dx = x + C$

2. $\int a \, dx = ax + C$

3. $\int x^n \, dx = ((x^{n+1})/(n+1)) + C$; $n \neq 1$

4. $\int (1/x) \, dx = \ln |x| + C$

5. $\int e^x \, dx = e^x + C$

6. $\int a^x \, dx = (a^x/\ln a) + C$; $a>0$, $a \neq 1$

1.4 Integration as the reverse of differentiation

Differentiation is the process of finding the derivative of the functions and integration is the process of finding the antiderivative of a function. So, these processes are inverse of each other.

The symbolic representation of the antiderivative of a function (Integration) is:
$y = \int f(x)\, dx$
$\int f(x)\, dx = F(x) + c.$

The constant, c, is known as the constant of integration.

1.4 Example

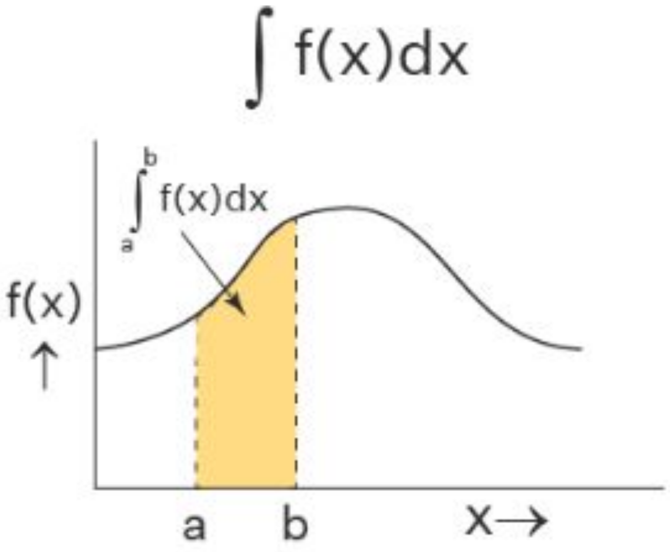
$y = 3x - x^2$
Differentiate w.r.t x
$dy/dx = 3 - 2x$

Taking the integral of dy/dx
$y = \int dy/dx$
$= \int 3-2x \, dx$
$= 3x - 2x^2 /2 + C$
$= 3x - x^2 + C$

Hence proved,
$y = \int dy/dx$

2 Definite Integrals

An integral that contains the upper and lower limits then it is a definite integral. On a real line, x is restricted to lie. Riemann Integral is the other name of the Definite Integral.



The area under the curve between points a and b:

$$\int_a^b f(x)dx$$

2 Example

$$\int_0^3 x^2 dx$$

$$= \left[\frac{x^3}{3}\right]_0^3$$

$$= \frac{(3)^3}{3} - \frac{(0)^3}{0}$$

$$= \frac{27}{3}$$

$$= 9$$

3 Indefinite Integrals

These are the integrals that do not have a pre-existing value of limits; thus making the final value of integral indefinite.

$$\int f(x)\,dx = F(x) + C$$

Where C is any constant and the function f(x) is called the integrand.

3 Example

Evaluate: $\int 4e^x \, dx$

Solution:
$\int 4e^x \, dx = 4e^x + C$

4 Improper Integrals

When we have definite integral with upper limit of ∞, or a lower limit of −∞.

We define the improper integral $\int_a^{\infty} f(x)\,dx$ as $\lim_{b\to\infty}\int_a^b f(x)\,dx$, provided the limit exists.

We define the improper integral $\int_{-\infty}^{b} f(x)\,dx$ as $\lim_{a\to-\infty}\int_a^b f(x)\,dx$, provided the limit exists.

5 Properties of Integral Calculus

- The derivative of an integral is the integrand itself.

  $\int f(x)\,dx = f(x) + C$

- Two indefinite integrals with the same derivative lead to the same family of curves and so they are equivalent.

  $\int [\, f(x)\,dx - g(x)\,dx] = 0$

- The integral of the sum or difference of a finite number of functions is equal to the sum or difference of the integrals of the individual functions.

  $\int [\, f(x)\,dx + g(x)\,dx] = \int f(x)\,dx + \int g(x)\,dx$

- The constant is taken outside the integral sign.

  $\int k\, f(x)\,dx = k \int f(x)\,dx$, where $k \in R$.(constant).

6 Methods of Integration

Sometimes, the inspection is not enough to find the integral of some functions. There are additional methods to reduce the function in the standard form to find its integral. Prominent methods are discussed below.

The methods of integration are:

- Integration by Substitution
- Integration using Partial Fractions
- Integration by Parts
- Integration of Some Particular Function

6.1 Integration by Substitution

Sometimes, it is really difficult to find the integration of a function, thus we can find the integration by introducing a new independent variable. This method is called Integration By Substitution.

The given form of integral function (say $\int f(x)$) can be transformed into another by changing the independent variable **x to t,**

Substituting $x = g(t)$ in the function $\int f(x)$, we get;

$dx/dt = g'(t)$
or $dx = g'(t).dt$

Thus, $\mathbf{I = \int f(x).dx = f(g(t)).g'(t).dt}$

6.1 Question

1. **Evaluate $\int x\sqrt{x+3}\,dx$.**

Solution:
set $u=x+3$
Differentiate w.r.t x
$du/dx = 1$
$du = dx$

But at this stage, we have:
$\int x\sqrt{x+3}\,dx = \int x\sqrt{u}\,du$.

We cannot evaluate an integral that has both an x and an u in it. We need to convert the x to an expression involving just u.
Since we set $u=x+3$, we can also state that $u-3=x$. Thus we can replace x in the integrand with $u-3$. It will also be helpful to rewrite $\sqrt{u}$ as $u^{1/2}$

6.1 Question

$$
\begin{aligned}
\int x\sqrt{x+3}\,dx &= \int (u-3)u^{\frac{1}{2}}\,du \\
&= \int \left(u^{\frac{3}{2}} - 3u^{\frac{1}{2}}\right) du \\
&= \frac{2}{5}u^{\frac{5}{2}} - 2u^{\frac{3}{2}} + C \\
&= \frac{2}{5}(x+3)^{\frac{5}{2}} - 2(x+3)^{\frac{3}{2}} + C.
\end{aligned}
$$

6.2 Integration by Parts

Integration by parts requires a special technique for integration of a function, where the integrand function is the multiple of two or more function.

Let us consider an integrand function to be f(x).g(x).

Mathematically, integration by parts can be represented as;

$$\int f(x).g(x).dx = f(x).\int g(x).dx - \int (f'(x).\int g(x).dx).dx$$

LIATE is a rule which helps to decide which term should you differentiate first and which term should you integrate first.

L– logarithm
I – Inverse
A– Algebraic
T – Trigonometric
E – Exponential

6.2 Question

1. Evaluate $\int \log x.dx$

Solution:

$\int \log x.1.dx$

Using the formula,

$\int f(x).g(x).dx = f(x).\int g(x).dx - \int (f'(x).\int g(x).dx).dx$

here, $f(x) = \log x$ and $g(x) = 1$

$$\begin{aligned}\int \log x.1.dx &= \log x.\int 1.dx - \int ((\log x)'.\int 1.dx).dx \\ &= \log x.x - \int (1/x\ .x).dx \\ &= x\log x - \int 1.dx \\ &= x\log x - x + C \\ &= x(\log x - 1) + C\end{aligned}$$

6.3 Integration of some particular functions

Many other standard integrals can be integrated using some important integration formulas.

S.No	Integral function	Integral value
1	$\int \frac{dx}{x^2-a^2}$	$\frac{1}{2a}\log\left\|\frac{x-a}{x+a}\right\| + C$
2	$\int \frac{dx}{a^2-x^2}$	$\frac{1}{2a}\log\left\|\frac{a+x}{a-x}\right\| + C$
3	$\int \frac{dx}{x^2+a^2}$	$\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
4	$\int \frac{dx}{\sqrt{x^2-a^2}}$	$\log\left\|x + \sqrt{x^2 - a^2}\right\| + C$
5	$\int \frac{dx}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\frac{x}{a}\right) + C$
6	$\int \frac{dx}{\sqrt{x^2+a^2}}$	$\log\left\|x + \sqrt{x^2 + a^2}\right\| + C$

6.3 Question

1. Find the Integral of the function. $\int \frac{dx}{\sqrt{7x^2-2x}}$

Solution:
The given function can be converted into the standard form

$$\int \frac{dx}{\sqrt{7x^2-2x}} = \int \frac{dx}{\sqrt{7}.\sqrt{x^2-\frac{2}{7}x}}$$

$$= \frac{1}{\sqrt{7}} \int \frac{dx}{\sqrt{\left(x-\frac{1}{7}\right)^2-\left(\frac{1}{7}\right)^2}}$$

(completing the squares)

Substituting
X– 1/7 = t

Differentiate w.r.t x
dx = dt

6.3 Question

$$\int \frac{dx}{\sqrt{7x^2-2x}} = \frac{1}{\sqrt{7}} \int \frac{dt}{\sqrt{t^2-\left(\frac{1}{7}\right)^2}}$$

$$= \frac{1}{\sqrt{7}} \log \left| t + \sqrt{t^2 - \left(\frac{1}{7}\right)^2} \right| + C$$

$$= \frac{1}{\sqrt{7}} \log \left| x - \frac{1}{7} + \sqrt{\left(x - \frac{1}{7}\right)^2 - \left(\frac{1}{7}\right)^2} \right| + C$$

$$= \frac{1}{\sqrt{7}} \log \left| x - \frac{1}{7} + \sqrt{x^2 - \frac{2}{7} x} \right| + C$$