INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Class: SY BSc

Subject : Calculus

Chapter: Unit 4 Chapter 1

Chapter Name: Differential equations

Today's Agenda

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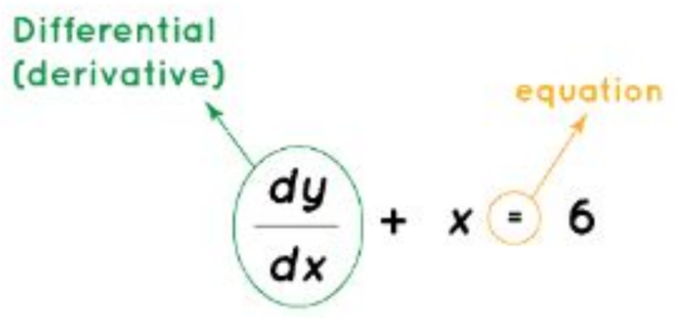
1 Differential equation

A **differential equation** is an equation which contains one or more terms and the derivatives of one variable (i.e., dependent variable) with respect to the other variable (i.e., independent variable)

$$dy/dx = f(x)$$

Here “x” is an independent variable and “y” is a dependent variable

A differential equation contains derivatives which are either partial derivatives or ordinary derivatives. The derivative represents a rate of change, and the differential equation describes a relationship between the quantity that is continuously varying with respect to the change in another quantity. There are a lot of differential equations formulas to find the solution of the derivatives.



1.1 Applications of Differential Equations

Differential equations have several applications in different fields such as applied mathematics, science, and engineering. Apart from the technical applications, they are also used in solving many real life problems.

1) Differential equations describe various exponential growths and decays.
2) They are also used to describe the change in return on investment over time.
3) They are used in the field of medical science for modelling cancer growth or the spread of disease in the body.
4) Movement of electricity can also be described with the help of it.
5) They help economists in finding optimum investment strategies.
6) The motion of waves or a pendulum can also be described using these equations.

1.2 Order of differential equation

The order of a differential equation is the highest order of the derivative appearing in the equation.

1.2 Order of differential equation

First Order Differential Equation

All the linear equations in the form of derivatives are in the first order. It has only the first derivative such as dy/dx, where x and y are the two variables and is represented as: $dy/dx = f(x, y) = y'$
Eg: $dy/dx = e^x$

Second-Order Differential Equation

The equation which includes second-order derivative is the second-order differential equation. It is represented as; $d/dx(dy/dx) = d^2y/dx^2 = f''(x) = y''$.
Eg: $(d^2y/dx^2)+ 2\,(dy/dx)+y$

Questions

What is the order of the differential equations:

1. $(d^2y/dx^2) + x(dy/dx) + y = 2\sin x$
2. $(d^4y/dx^4) + y = 0$
3. $(d^3y/dx^3) + x^2(d^2y/dx^2) = 0$

Solutions

Answer:

1. The order of the differential equation is 2
2. The order of the differential equation is 4
3. The order of the differential equation is 3

1.3 Degree of differential equation

The degree of the differential equation is the power of the highest order derivative, where the original equation is represented in the form of a polynomial equation in derivatives such as y',y", y"', and so on.

OR

The degree of the differential equation is the power of the highest ordered derivative present in the equation.

Note: If a differential equation is not expressible in terms of a polynomial equation having the highest order derivative as the leading term, then that degree of the differential equation is not defined.

Question

State the degree of differential equation

1. $$\left(\frac{\partial^3 y}{\partial x^3}\right)^2 + x^2\left(\frac{dy}{dx}\right)^3 + x^2 y = 0$$

2. $$\left(\frac{d^4 y}{dx^4}\right)^{\frac{1}{2}} = \left[1 + \left(\frac{d^2 y}{dx^2}\right)^2\right]^{\frac{1}{3}}$$

Solutions

Answer:

1. The degree of the differential equation is 2

2. The degree of the differential equation is 3

$$\left[\left(\frac{d^4y}{dx^4}\right)^{\frac{1}{2}}\right]^6 = \left[\left[1+\left(\frac{d^2y}{dx^2}\right)^2\right]^{\frac{1}{3}}\right]^6$$

$$\Rightarrow \left(\frac{d^4y}{dx^4}\right)^3 = \left[1+\left(\frac{d^2y}{dx^2}\right)^2\right]^2$$

2 Types of Differential Equations

Differential equations can be divided into several types namely

1. Ordinary Differential Equations
2. Partial Differential Equations
3. Linear Differential Equations
4. Non linear differential equations
5. Homogeneous Differential Equations
6. Non homogeneous Differential Equations

2.1 Ordinary Differential Equations

The “Ordinary Differential Equation” also known as ODE is an equation that contains only one independent variable and one or more of its derivatives with respect to the variable. Thus, the ordinary differential equation is represented as the relation having one independent variable x, the real dependent variable y, with some of its derivatives y’, y”, ….yn,…with respect to x.

The ordinary differential equation can be homogenous or non-homogenous

Example: $(d^2y/dx^2) + (dy/dx) = 3y \cos x$

The above differential equation example is an ordinary differential equation since it does not contain partial derivatives.

2.2 Homogeneous Differential Equations

A differential equation in which the degree of all the terms is the same is known as a homogenous differential equation. In general they can be represented as $P(x,y)dx + Q(x,y)dy = 0$, where $P(x,y)$ and $Q(x,y)$ are homogeneous functions of the same degree.

Examples of Homogenous Differential Equation:

$y + x(dy/dx) = 0$ is a homogenous differential equation of degree 1

$x^4 + y^4(dy/dx) = 0$ is a homogenous differential equation of degree 4

2.3 Non Homogeneous Differential Equations

A differential equation in which the degree of all the terms is not the same is known as a homogenous differential equation.

Example: $xy(dy/dx) + y^2 + 2x = 0$ is not a homogenous differential equation.

2.4 Linear Differential Equations

The differential equation of the form $(dy/dx) + Py = Q$ (Where P and Q are functions of x) is called a linear differential equation. $(dy/dx) + Py = Q$ (Where P, Q are constant or functions of y).
The general solution is $y \times (\text{I.F.}) = \int Q(\text{I.F.})dx + c$ where, I.F(integrating factor) = $e^{\int pdx}$

2.5 Partial Differential Equations

An equation involving only partial derivatives of one or more functions of two or more independent variables is called a partial differential equation also known as PDE.

A few examples are:

$\delta u/ dx + \delta/dy = 0$,

$\delta^2 u/\delta x^2 + \delta^2 u/\delta x^2 = 0$

3 Differential Equations solutions

A function that satisfies the given differential equation is called its solution.

The solution that contains as many arbitrary constants as the order of the differential equation is called a general solution.
The solution free from arbitrary constants is called a particular solution.

There exist two methods to find the solution of the differential equation.

1. Separation of variables
2. Integrating factor

3.1 Separation of variables

Separation of the variable is done when the differential equation can be written in the form of $dy/dx = f(y)g(x)$ where f is the function of y only and g is the function of x only. Taking an initial condition, rewrite this problem as $1/f(y)dy= g(x)dx$ and then integrate on both sides.

3.1 Example

$dy/dx) = x^2y + y$

Step 1: Divide the above differential equation by y. (We separate the variable)
$(1/y)(dy/dx) = (x^2 + 1)$
We consider y and x both as variables and rewrite this as
$(dy/y) = (x^2 + 1)dx$

Step 2: Now integrate L.H.S. with respect to y and with respect to x.
$\int(1/y)dx = \int(x^2 + 1)dx$

Step 3: After integrating, we get:
$\log y = (x^3/3) + x + c$
So, this is how the differential equation is solved.

3.2 Integrating factor

Integrating factor technique is used when the differential equation is of the form $dy/dx + p(x)y = q(x)$ where p and q are both the functions of x only.

First-order differential equation is of the form $y' + P(x)y = Q(x)$. where P and Q are both functions of x and the first derivative of y. The higher-order differential equation is an equation that contains derivatives of an unknown function which can be either a partial or ordinary derivative. It can be represented in any order.

3.2 Example

Solve the differential equation using the integrating factor: $(dy/dx) - (3y/x+1) = (x+1)^4$

Solution:
$(dy/dx) - (3y/x+1) = (x+1)^4$

First, find the integrating factor:
$\mu = e^{\int p(x)\, dx}$
$\mu = e^{\int(-3/x+1)\, dx}$
$\int(-3/\ x+1)dx = -3 \ln(x+1) = \ln(x+1)^{-3}$

Hence, we get
$\mu = e^{\ln(x+1)-3}$
$\mu = 1/\ (x+1)^3$

3.2 Example

Now, multiply the integrating factor on both the sides of the given differential eqaution:
$[1/ (x+1)^3] [dy/dx] – [3y/((x+1)^4)] = (x+1)$

Integrate both the sides, we get:
$[y/(x+1)^3] = [(1/2)x^2+x+c]$Here, c is a constant

Therefore, the general solution of the given differential eqaution is
$y = [(x+1)^3] [(1/2)x^2+x+c]$.