INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Class: FY BSc

Subject : Calculus

Chapter: Unit 4 Chapter 7

Chapter Name: Taylor and Maclaurin series

Today's Agenda

1. Taylor series
2. Maclaurin series
3. Common maclaurin series
4. Taylor series theorem
5. Taylor series proof

1 Taylor series

A Taylor series is a series expansion of a function about a point. A one-dimensional Taylor series is an expansion of a real function about a point is given by

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots\ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

A Taylor Series is an expansion of some function into an infinite sum of terms, where each term has a larger exponent like x, x^2, x^3, etc.

2 Maclaurin Series

A Maclaurin series is a Taylor series expansion of a function about 0

$$f(x) = f(0) + f'(0)(x) + \frac{f''(0)}{2!}(x)^2 + \ldots\ldots + \frac{f^{(n)}(0)}{n!}(x)^n$$

3 Common Maclaurin Series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \text{..........} + \frac{x^r}{r!}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \text{..........} + (-1)^{r-1}\frac{x^r}{r!}$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \text{..........} + \frac{n(n-1)(n-2)\text{.......}(n-r+1)}{r!}x^r$$

4 Taylor Series Theorem

Assume that if f(x) be a real or composite function, which is a differentiable function of a neighbourhood number that is also real or composite. Then, the Taylor series describes the following power series :

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots\ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

In terms of sigma notation, the Taylor series can be written as

$$\sum_{n=0}^{\infty} \frac{f^n(a)}{n}(x-a)^n$$

Where

$f^{(n)}$ (a) = n^{th} derivative of f

n! = factorial of n.

5 Taylor Series Proof

We know that the power series can be defined as

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

When x = 0,
f(x)= a_0

So, differentiate the given function, it becomes,
$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \ldots.$

Again, when you substitute x = 0, we get
$f'(0) = a_1$
So, differentiate it again, we get
$f''(x) = 2a_2 + 6a_3 x + 12a_4 x^2 + \ldots$
Now, substitute x=0 in second-order differentiation, we get
$f''(0) = 2a_2$

5 Taylor Series Proof

Therefore, [f”(0)/2!] = a_2
By generalising the equation, we get
$f^n(0) / n! = a_n$

Now substitute the values in the power series we get,

$$f(x) = f(0) + f'(0)(x) + \frac{f''(0)}{2!}(x)^2 + \ldots\ldots + \frac{f^{(n)}(0)}{n!}(x)^n$$

Generalise f in more general form, it becomes
$f(x) = b + b_1(x-a) + b_2(x-a)^2 + b_3(x-a)^3 + \ldots.$
Now, x = a, we get
$b_n = f^n(a) / n!$
Now, substitute b_n in a generalised form

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots\ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Hence, the Taylor series is proved.

5 Questions

1. Find the Taylor expansion of e^{-x} in the powers of $(x+4)$ up to and including the term in $(x+4)^3$

Solution

Answer 1:

Using the Taylor series ② with $f(x) = e^{-x}$ and $a = -4$,

$$f(x) = e^{-x} = f(-4) + f'(-4)(x+4) + \frac{f''(-4)}{2!}(x+4)^2 + \frac{f'''(-4)}{3!}(x+4)^3$$

$f(x) = e^{-x} \quad \Rightarrow f(-4) = e^4$

$f'(x) = -e^{-x} \quad \Rightarrow f'(-4) = -e^4$

$f''(x) = e^{-x} \quad \Rightarrow f''(-4) = e^4$

$f'''(x) = -e^{-x} \quad \Rightarrow f'''(-4) = -e^4$

Substituting the values in * gives

$$e^{-x} = e^4 - e^4(x+4) + \frac{e^4}{2!}(x+4)^2 - \frac{e^4}{3!}(x+4)^3 + \ldots$$

$$= e^4\left\{1 - (x+4) + \frac{1}{2}(x+4)^2 - \frac{1}{6}(x+4)^3 + \ldots\right\}$$