INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Subject : Statistical & Risk Modelling -2

Markov Chains

Markov property

Markov Property: The state of the system at time $t+1$ depends only on the state of the system at time t

$$\Pr\left[X_{t+1} = x_{t+1} \mid X_1 \cdots X_t = x_1 \cdots x_t\right] = \Pr\left[X_{t+1} = x_{t+1} \mid X_t = x_t\right]$$

$$\begin{tikzcd} X_1 \arrow[r, bend left] & X_2 \arrow[r, bend left] & X_3 \arrow[r, bend left] & X_4 \arrow[r, bend left] & X_5 \end{tikzcd}$$

$$\Pr\left[X_{t+1} = b \mid X_t = a\right] = p_{ab}$$

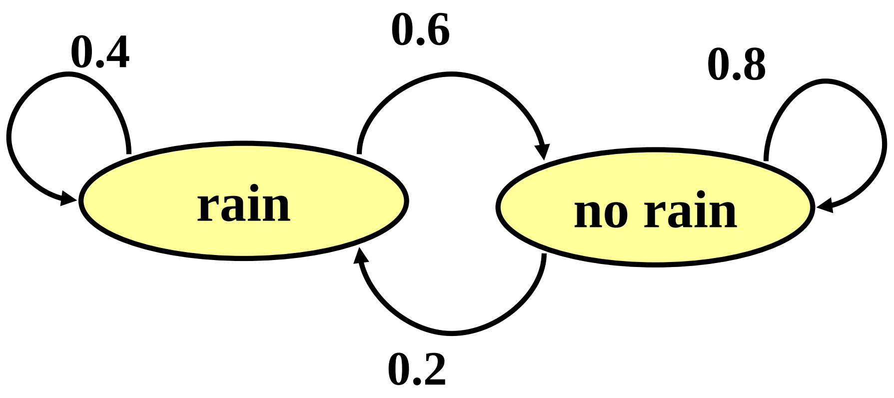
Bounded memory transition model

A Simple Markov Process

Weather:

- raining today → 40% rain tomorrow
  → 60% no rain tomorrow

- not raining today → 20% rain tomorrow
  → 80% no rain tomorrow



The transition matrix:

$$P = \begin{pmatrix} 0.4 & 0.6 \\ 0.2 & 0.8 \end{pmatrix}$$

Transition matrix features:

The transition matrix: $$P = \begin{pmatrix} 0.4 & 0.6 \\ 0.2 & 0.8 \end{pmatrix}$$

It is square, since all possible states must be used both as rows and as columns.

All entries are between 0 and 1, because all entries represent probabilities.

The sum of the entries in any row must be 1, since the numbers in the row give the probability of changing from the state at the left to one of the states indicated across the top.

Markov Chain - Definition

A process with a finite number of states (or outcomes, or events) in which the probability of being in a particular state at step n + 1 depends only on the state occupied at step n.

A sequence of trials of an experiment is a Markov chain if:

1. the outcome of each experiment is one of a set of discrete states;

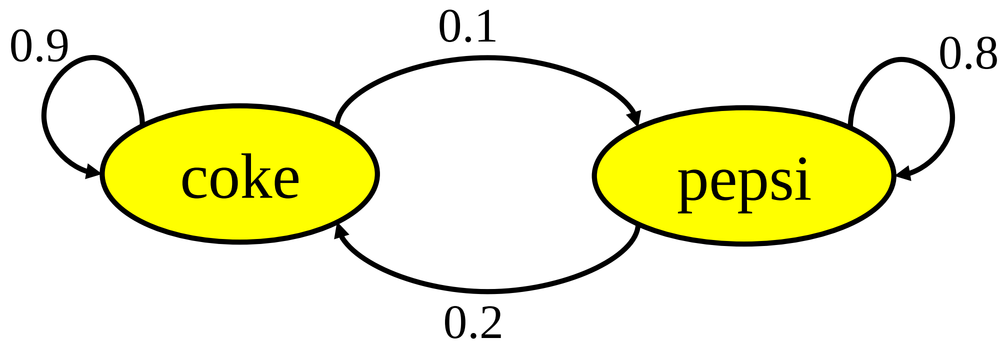
2. the outcome of an experiment depends only on the present state, and not on any past states.

Markov Process- Coke vs Pepsi Example

- Given that a person's last cola purchase was Coke, there is a 90% chance that his next cola purchase will also be Coke.
- If a person's last cola purchase was Pepsi, there is an 80% chance that his next cola purchase will also be Pepsi.

transition matrix:

$$P = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}$$



Markov Process- Coke vs Pepsi Example (Cont-2)

Given that a person is currently a Pepsi purchaser, what is the probability that he will purchase Coke two purchases from now?

Pr[Pepsi→?→Coke] =

Pr[Pepsi→Coke→Coke] + Pr[Pepsi→ Pepsi →Coke] =

0.2 * 0.9 + 0.8 * 0.2 = 0.34

$$P^2 = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix} \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix} = \begin{bmatrix} 0.83 & 0.17 \\ 0.34 & 0.66 \end{bmatrix}$$

Markov Process- Coke vs Pepsi Example (Cont-3)

Given that a person is currently a Coke purchaser, what is the probability that he will purchase Pepsi **three** purchases from now?

$$P^3 = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix} \begin{bmatrix} 0.83 & 0.17 \\ 0.34 & 0.66 \end{bmatrix} = \begin{bmatrix} 0.781 & 0.219 \\ 0.438 & 0.562 \end{bmatrix}$$

Markov Process- Coke vs Pepsi Example (Cont-4)

- Assume each person makes one cola purchase per week
- Suppose 60% of all people now drink Coke, and 40% drink Pepsi
- What fraction of people will be drinking Coke three weeks from now?

$$P = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix} \qquad P^3 = \begin{bmatrix} 0.781 & 0.219 \\ 0.438 & 0.562 \end{bmatrix}$$

$$\Pr[X_3 = \text{Coke}] = 0.6 * 0.781 + 0.4 * 0.438 = 0.6438$$

Q_i - the distribution in week i

$Q_0 = (0.6, 0.4)$ - initial distribution

$Q_3 = Q_0 * P^3 = (0.6438, 0.3562)$

Markov chains- Insurance Example:

An example: An insurance company classifies drivers as low-risk if they are accident-free for one year. Past records indicate that 98% of the drivers in the low-risk category (L) will remain in that category the next year, and 78% of the drivers who are not in the low-risk category (L') one year will be in the low-risk category the next year.

$$P = \begin{array}{c|cc} & L & L' \\ \hline L & 0.98 & 0.02 \\ L' & 0.78 & 0.22 \end{array}$$

If 90% of the drivers in the community are in the low-risk category this year, what is the probability that a driver chosen at random from the community will be in the low-risk category the next year? The year after next ? (answer 0.96, 0.972 from matrices)

$$S_0 = \begin{array}{cc} L & L' \\ 0.90 & 0.10 \end{array}$$

$$S_2 = S_1 P = S_o P \cdot P = S_0 P^2$$

$$S_1 = S_o P \longrightarrow$$

$$S_1 = \begin{array}{cc} L & L' \\ 0.96 & 0.04 \end{array}$$

$$S_2 = \begin{array}{cc} L & L' \\ 0.972 & 0.028 \end{array}$$

Special cases of Markov chains:

Regular Markov chains:

A Markov chain is a regular Markov chain if some power of the transition matrix has only positive entries. That is, if we define the (i; j) entry of P^n to be p^n_{ij} , then the Markov chain is regular if there is some n such that $p^n_{ij} > 0$ for all (i,j).

Absorbing Markov chains:

A state S_k of a Markov chain is called an absorbing state if, once the Markov chains enters the state, it remains there forever.

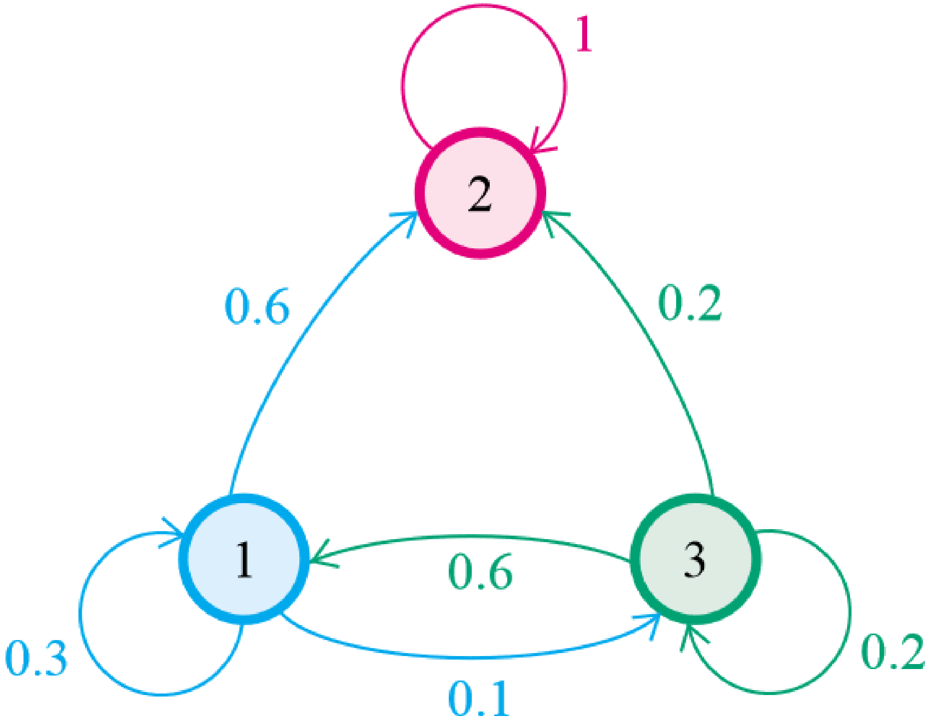
A Markov chain is called an absorbing chain if

- It has at least one absorbing state.
- For every state in the chain, the probability of reaching an absorbing state in a finite number of steps is nonzero.

Absorbing State:

State 2 is absorbing

Pii = 1 → P22 = 1



$$\begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0.3 & 0.6 & 0.1 \\ 0 & 1 & 0 \\ 0.6 & 0.2 & 0.2 \end{bmatrix} \end{array} = P.$$

Irreducible Markov Chain:

A Markov chain is irreducible if all the states communicate with each other, i.e., if there is only one communication class.
i and j communicate if they are accessible from each other. This is written $i \leftrightarrow j$.

Illustration of the irreducibility property. The chain on the left is not irreducible: from 3 or 4 we can't reach 1 or 2. The chain on the right (one edge has been added) is irreducible: each state can be reached from any other state.

Markov Chain - Periodicity

A state i is periodic if it is returned to after a time period > 1.

Formally, it is periodic if there exists an integer $k > 1$ where, for all j:

$$p_{ii}^{(n)} \begin{cases} \geq 0 & n = kj \\ = 0 & otherwise \end{cases}$$

Equivalently, a state is aperiodic if there is always a sufficiently large n that for all $m > n$:

$$p_{ii}^{(m)} > 0$$

Markov Chain – Periodicity Example

Illustration of the periodicity property. The chain on the left is 2-periodic: when leaving any state, it always takes a multiple of 2 steps to come back to it. The chain on the right is 3-periodic.

Thank You