INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Subject : Statistical & Risk Modelling -2

Examples – Markov Chains

Question-1

In analyzing switching by Business Class customers between airlines the following data has been obtained by British Airways (BA):

Next Flight/ Last Flight	BA	Competition
BA	0.85	0.15
Competition	0.10	0.9

- For example if the last flight by a Business Class customer was by BA the probability that their next flight is by BA is 0.85.
- Business Class customers make 2 flights a year on average.
- Currently BA have 30% of the Business Class market.

- What would you forecast BA's share of the Business Class market to be after two years?

Solution-1

- We have the initial system state s_1 given by s_1 = [0.30, 0.70] and the transition matrix P is given by

P =

0.85	0.15	*	0.85	0.15	=	0.7375	0.2625
0.1	0.9		0.1	0.9		0.175	0.825

where the square of transition matrix arises as Business Class customers make 2 flights a year on average.

- Hence after one year has elapsed the state of the system $s_2 = s_1P$ = [0.34375, 0.65625]
- After two years have elapsed the state of the system = $s_3 = s_2P$ = [0.368, 0.632] and note here that the elements of s_2 and s_3 add to one (as required).

- So after two years have elapsed BA's share of the Business Class market is 36.8%

Question -2

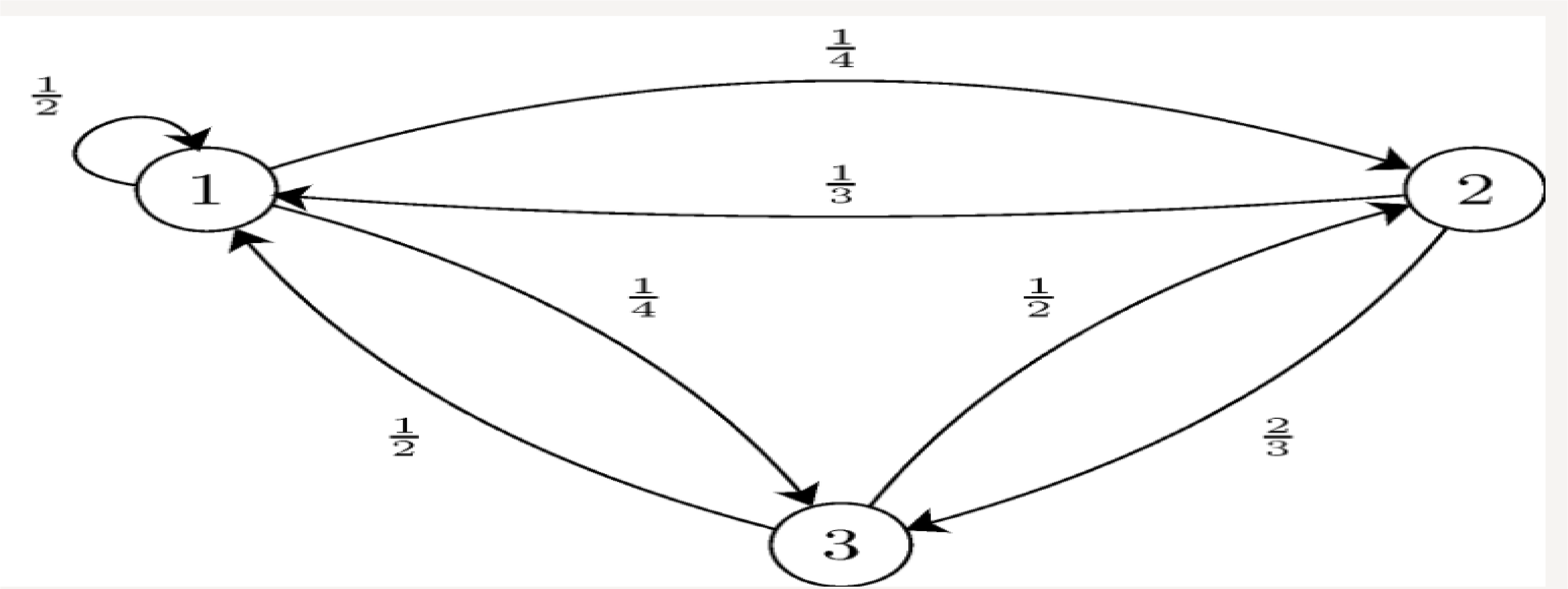
Consider the Markov chain with three states, $S = \{1, 2, 3\}$, that has the following transition matrix

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}.$$

a. Draw the state transition diagram for this chain.

b. If we know $P(X_1 = 1) = P(X_1 = 2) = \frac{1}{4}$, find $P(X_1 = 3, X_2 = 2, X_3 = 1)$.

Solution -2 (Transition Diagram)



Solution-2 (Continued)

b. First, we obtain

$$\begin{aligned} P(X_1 = 3) &= 1 - P(X_1 = 1) - P(X_1 = 2) \\ &= 1 - \frac{1}{4} - \frac{1}{4} \\ &= \frac{1}{2}. \end{aligned}$$

We can now write

$$\begin{aligned} P(X_1 = 3, X_2 = 2, X_3 = 1) &= P(X_1 = 3) \cdot p_{32} \cdot p_{21} \\ &= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{3} \\ &= \frac{1}{12}. \end{aligned}$$

Question -3

Consider the Markov chain in Figure 11.17. There are two recurrent classes, $R_1 = \{1, 2\}$, and $R_2 = \{5, 6, 7\}$. Assuming $X_0 = 3$, find the probability that the chain gets absorbed in R_1.

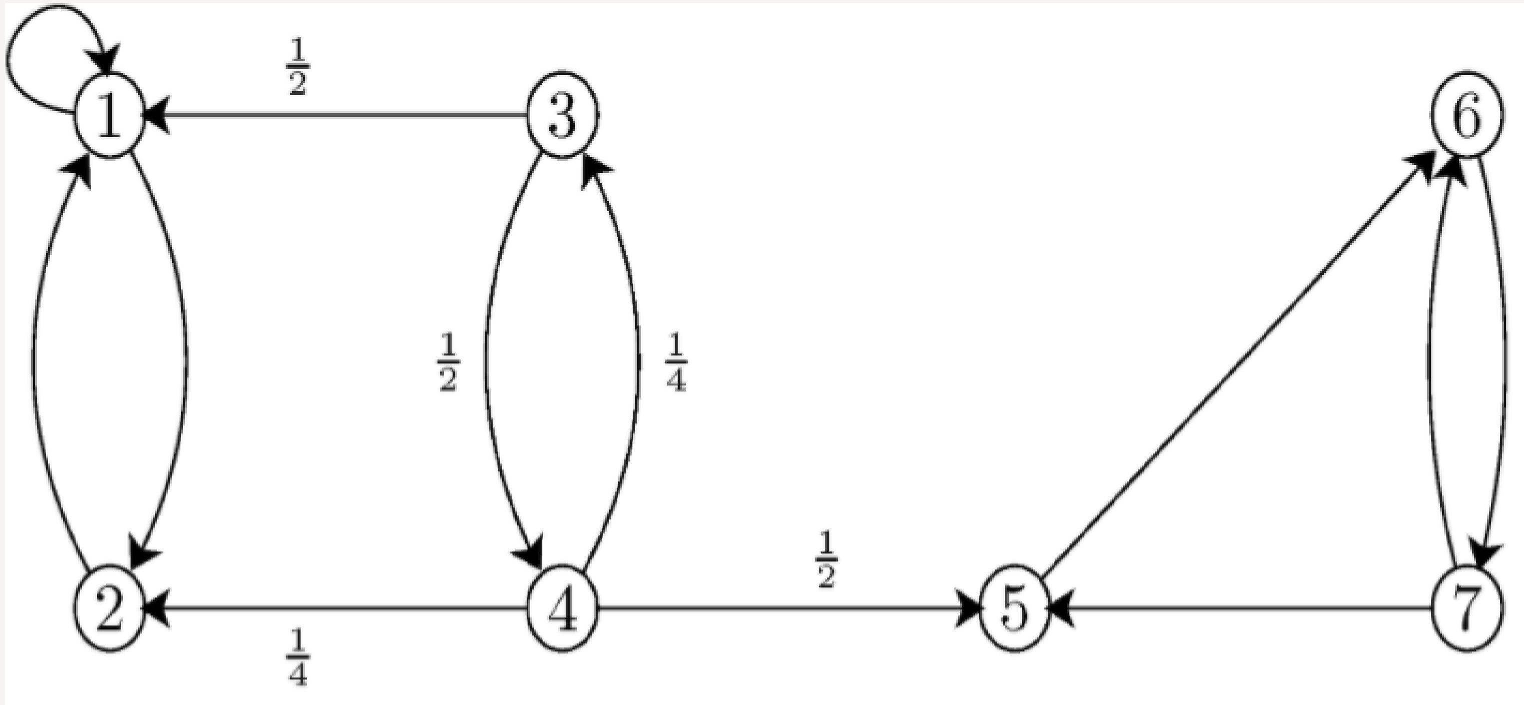


Figure 11.17 - A state transition diagram.

Solution-3

Here, we can replace each recurrent class with one absorbing state. The resulting state diagram is shown in Figure 11.18

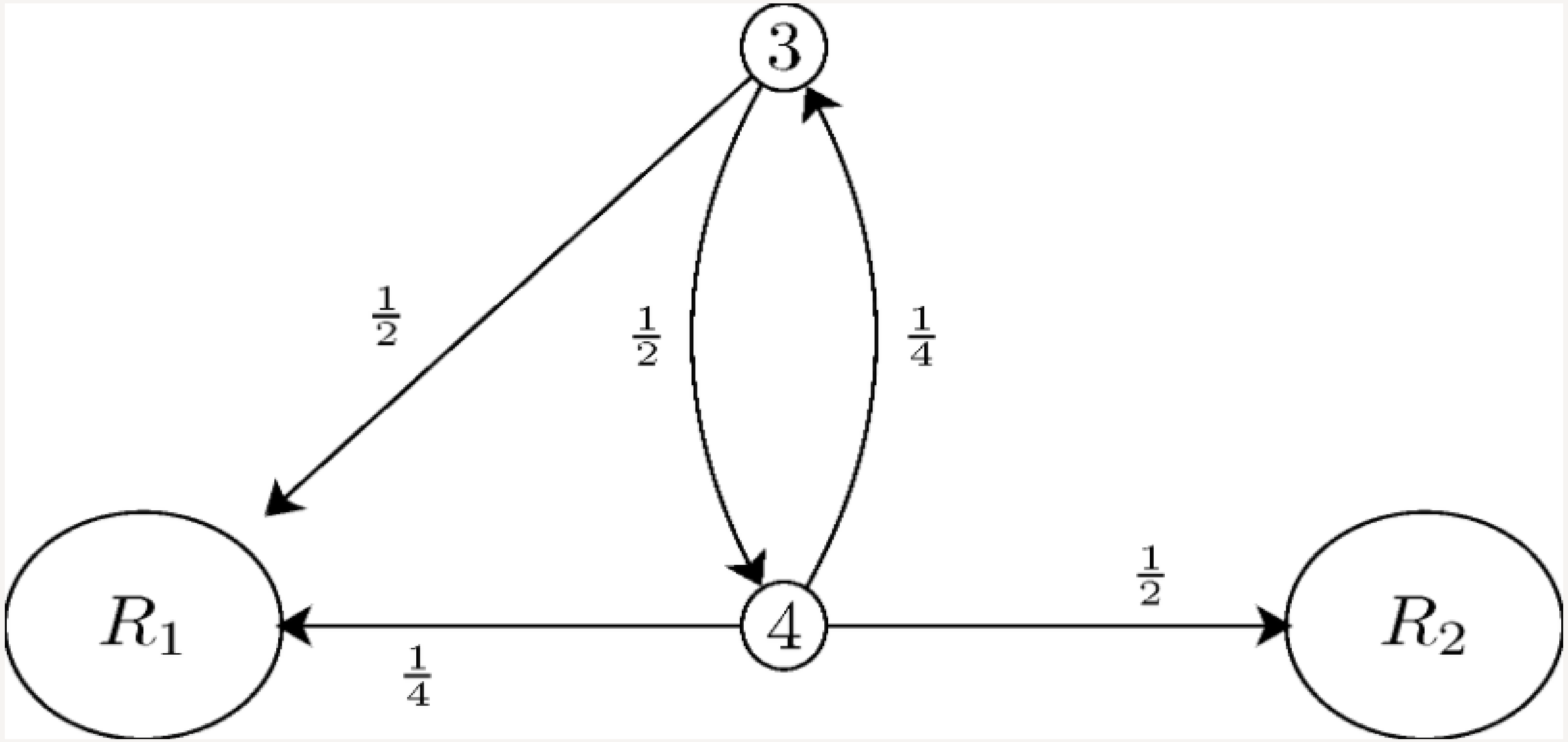


Figure 11.18 - The state transition diagram in which we have replaced each recurrent class with one absorbing state.

Solution-3 (Continued)

Now we can apply our standard methodology to find probability of absorption in state R_1. In particular, define

$$a_i = P(\text{absorption in } R_1 | X_0 = i), \quad \text{for all } i \in S.$$

By the above definition, we have $a_{R_1} = 1$, and $a_{R_2} = 0$. To find the unknown values of a_i's, we can use the following equations

$$a_i = \sum_k a_k p_{ik}, \quad \text{for } i \in S.$$

We obtain

$$\begin{aligned} a_3 &= \frac{1}{2} a_{R_1} + \frac{1}{2} a_4 \\ &= \frac{1}{2} + \frac{1}{2} a_4, \end{aligned}$$

$$\begin{aligned} a_4 &= \frac{1}{4} a_{R_1} + \frac{1}{4} a_3 + \frac{1}{2} a_{R_2} \\ &= \frac{1}{4} + \frac{1}{4} a_3. \end{aligned}$$

Solution -3 (Continued)

Solving the above equations, we obtain

$$a_3 = \frac{5}{7}, \quad a_4 = \frac{3}{7}.$$

Therefore, if $X_0 = 3$, the chain will end up in class R_1 with probability $a_3 = \frac{5}{7}$.

Stationary Distributions of Markov Chains

A stationary distribution of a Markov chain is a probability distribution that remains unchanged in the Markov chain as time progresses. Typically, it is represented as a row vector π whose entries are probabilities summing to 1, and given transition matrix $\mathbf{P}$, it satisfies

$$\pi = \pi\mathbf{P}.$$

Question -4

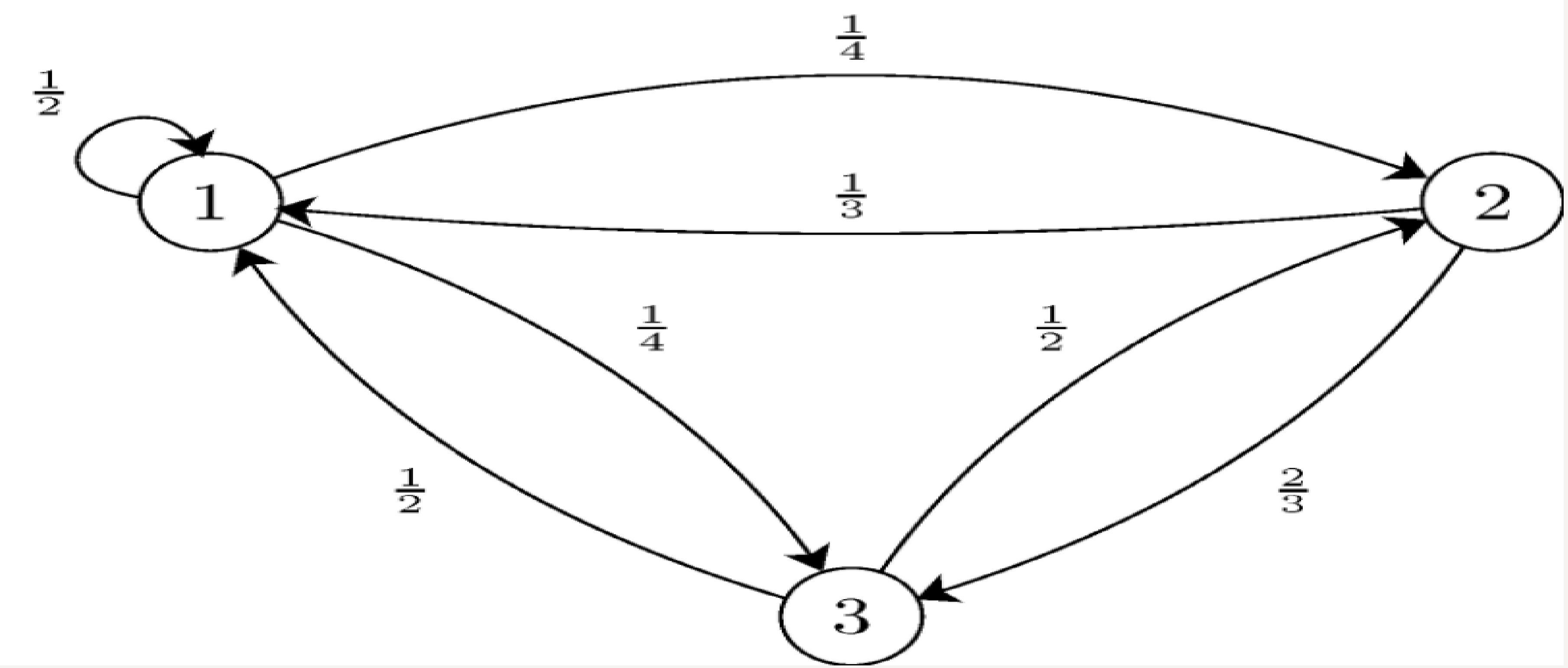


Figure 11.20 - A state transition diagram.

a. Is this chain irreducible?
b. Is this chain aperiodic?
c. Find the stationary distribution for this chain.

Solution -4

a. The chain is irreducible since we can go from any state to any other states in a finite number of steps.
b. The chain is aperiodic since there is a self-transition, i.e., $p_{11} > 0$.
c. To find the stationary distribution, we need to solve

$$\begin{aligned} \pi_1 &= \frac{1}{2}\pi_1 + \frac{1}{3}\pi_2 + \frac{1}{2}\pi_3, \\ \pi_2 &= \frac{1}{4}\pi_1 + \frac{1}{2}\pi_3, \\ \pi_3 &= \frac{1}{4}\pi_1 + \frac{2}{3}\pi_2, \\ \pi_1 + \pi_2 + \pi_3 &= 1. \end{aligned}$$

We find

$$\pi_1 \approx 0.457, \ \pi_2 \approx 0.257, \ \pi_3 \approx 0.286$$

Thank You