Lecture

INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Class: MSc

Subject : Portfolio Theory & Security Analysis

Subject Code:

Chapter: Unit 3 Chapter 1

Chapter Name: Modern Portfolio Theory

Agenda

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   2. Types of portfolio
   3. Risk Aversion

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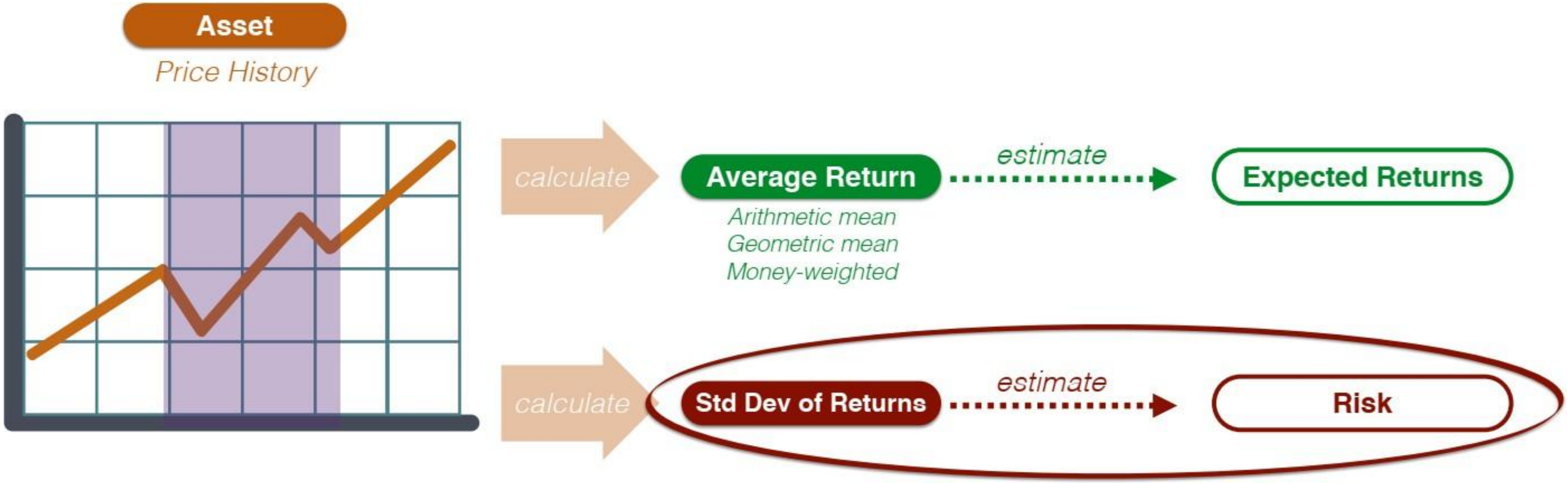
1 What is a portfolio?

A portfolio is a collection of financial investments like stocks, bonds, commodities, cash, and cash equivalents, including closed-end funds and exchange traded funds (ETFs).

1.1 Characteristics of a portfolio



1.2 Types of portfolio

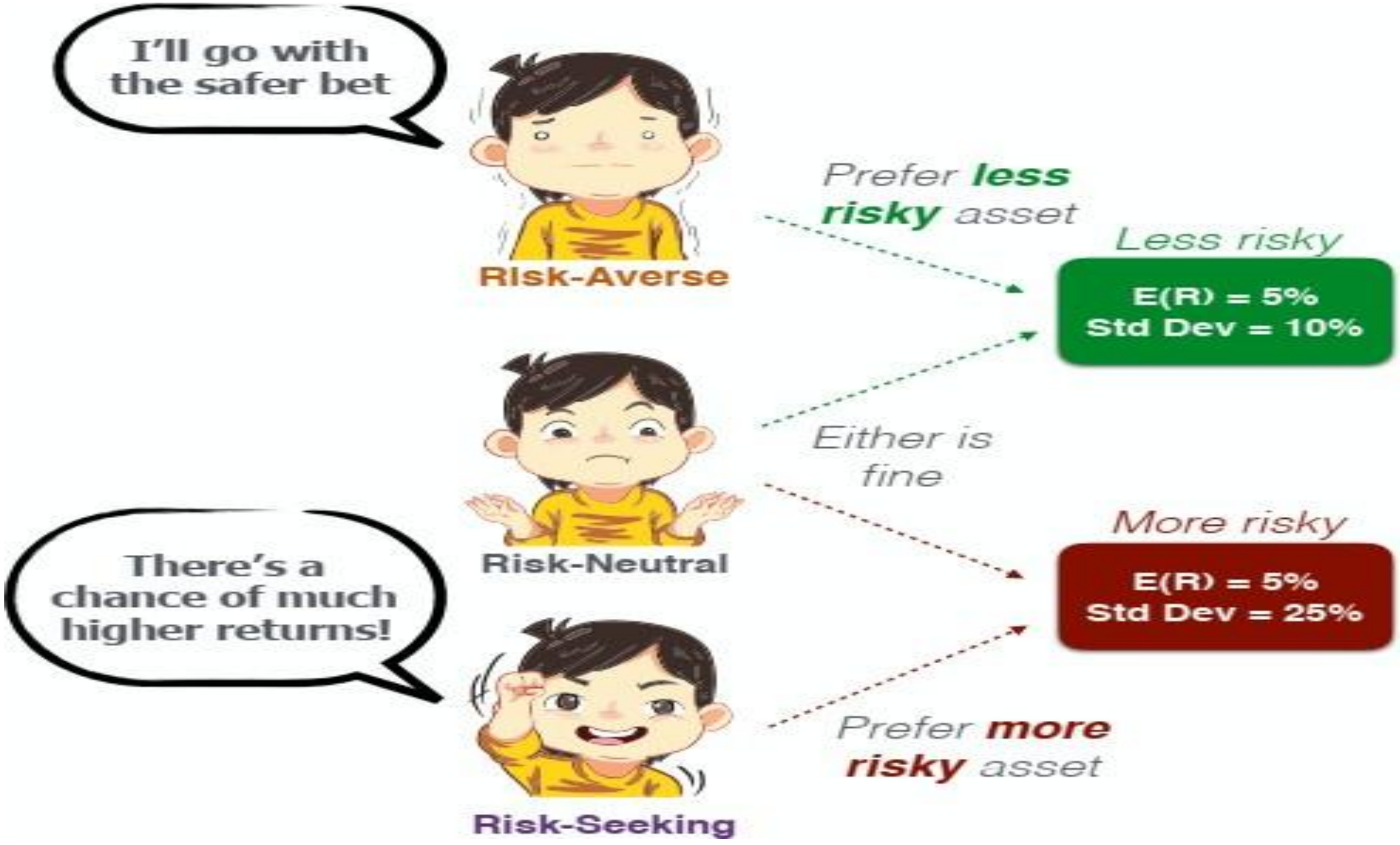
1. Inefficient Portfolio

A portfolio is inefficient if the investor can find another portfolio with the same (or higher) expected return and lower variance, or the same (or lower) variance and higher expected return. A portfolio is efficient if the investor cannot find a better one in the sense that it has either a higher expected return and the same (or lower) variance, or a lower variance and the same (or higher) expected return.
However, an investor may be able to rank efficient portfolios by using a utility function.

2. Efficient Portfolio

A portfolio which is not inefficient is an efficient portfolio.

1.3 Risk Aversion



2 Modern Portfolio Theory

- Mean-variance portfolio theory, sometimes called modern portfolio theory (MPT), specifies a method for an investor to construct a portfolio that gives the maximum return for a specified risk, or the minimum risk for a specified return.

- However, the theory relies on some strong and limiting assumptions about the properties of portfolios that are important to investors. For example, it uses variance to measure risk and so penalises gains as well as losses.

- In the form described here, the theory ignores the investor's liabilities, although it is possible to extend the analysis to include them.

- The application of the mean-variance framework to portfolio selection falls conceptually into two parts. First, the definition of the properties of the portfolios available to the investor – the opportunity set. Second, the determination of how the investor chooses one out of all the feasible portfolios in the opportunity set.

2.1 Investment Opportunity Set

- In specifying the opportunity set it is necessary to make some assumptions about how investors make decisions. Then the properties of portfolios can be specified in terms of relevant characteristics.

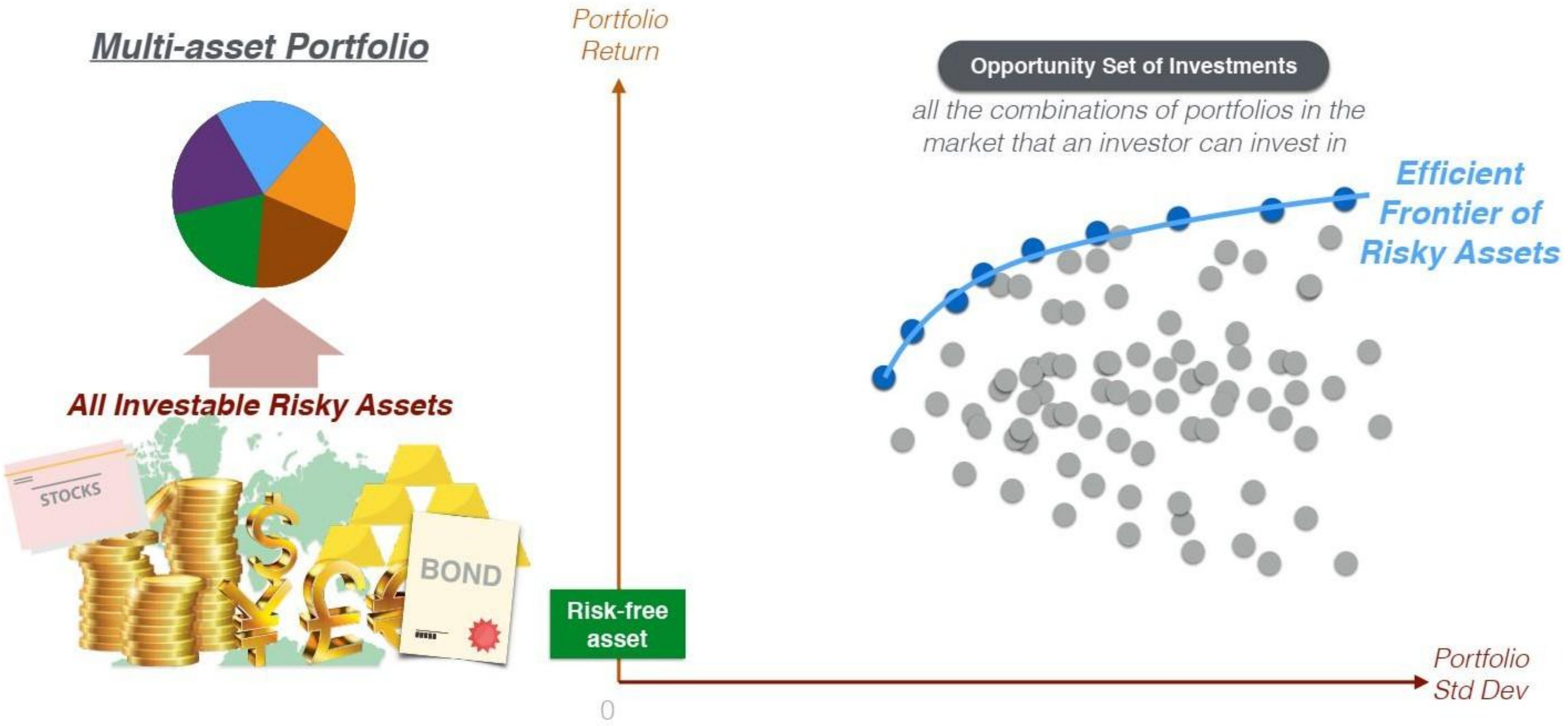
- It is assumed that investors select their portfolios on the basis of the expected return and the variance of that return over a single time horizon.

- Thus, all the relevant properties of a portfolio can be specified with just two numbers – the mean return and the variance of the return.

- The variance (or standard deviation) is known as the risk of the portfolio.

- To calculate the mean and variance of return for a portfolio it is necessary to know the expected return on each individual security, and also the variance/covariance matrix for the available universe of securities.

2.1 Investment Opportunity Set



2.2 Efficient portfolios

Two further assumptions about investor behaviour allow the definition of efficient portfolios.

The assumptions are:

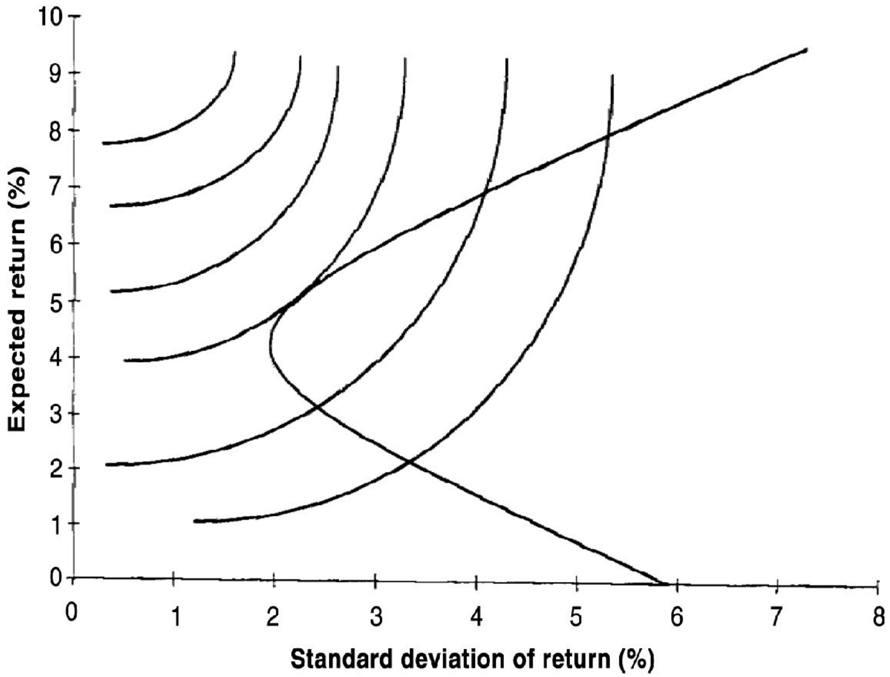
(i) Investors are never satiated. At a given level of risk, they will always prefer a portfolio with a higher return to one with a lower return.

(ii) Investors dislike risk. For a given level of return, they will always prefer a portfolio with lower variance to one with higher variance.

Once the set of efficient portfolios has been identified, all others can be ignored.

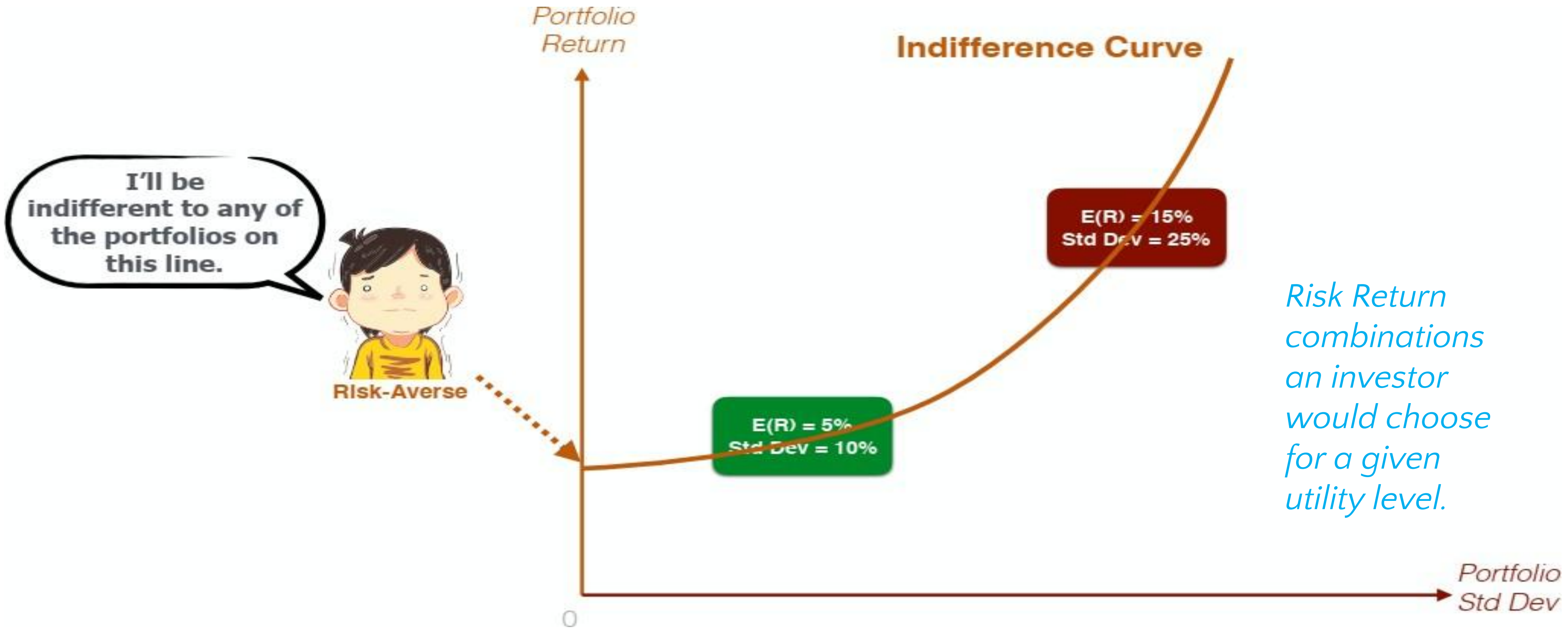
2.3 Choosing an efficient portfolio

- A series of indifference curves (curves which join all outcomes of equal utility) can be plotted in expected return – standard deviation space.
- Portfolios lying along a single curve all give the same value of expected utility and so the investor would be indifferent between them.
- Utility is maximised by choosing the portfolio on the efficient frontier at the point where the frontier is at a tangent to an indifference curve.



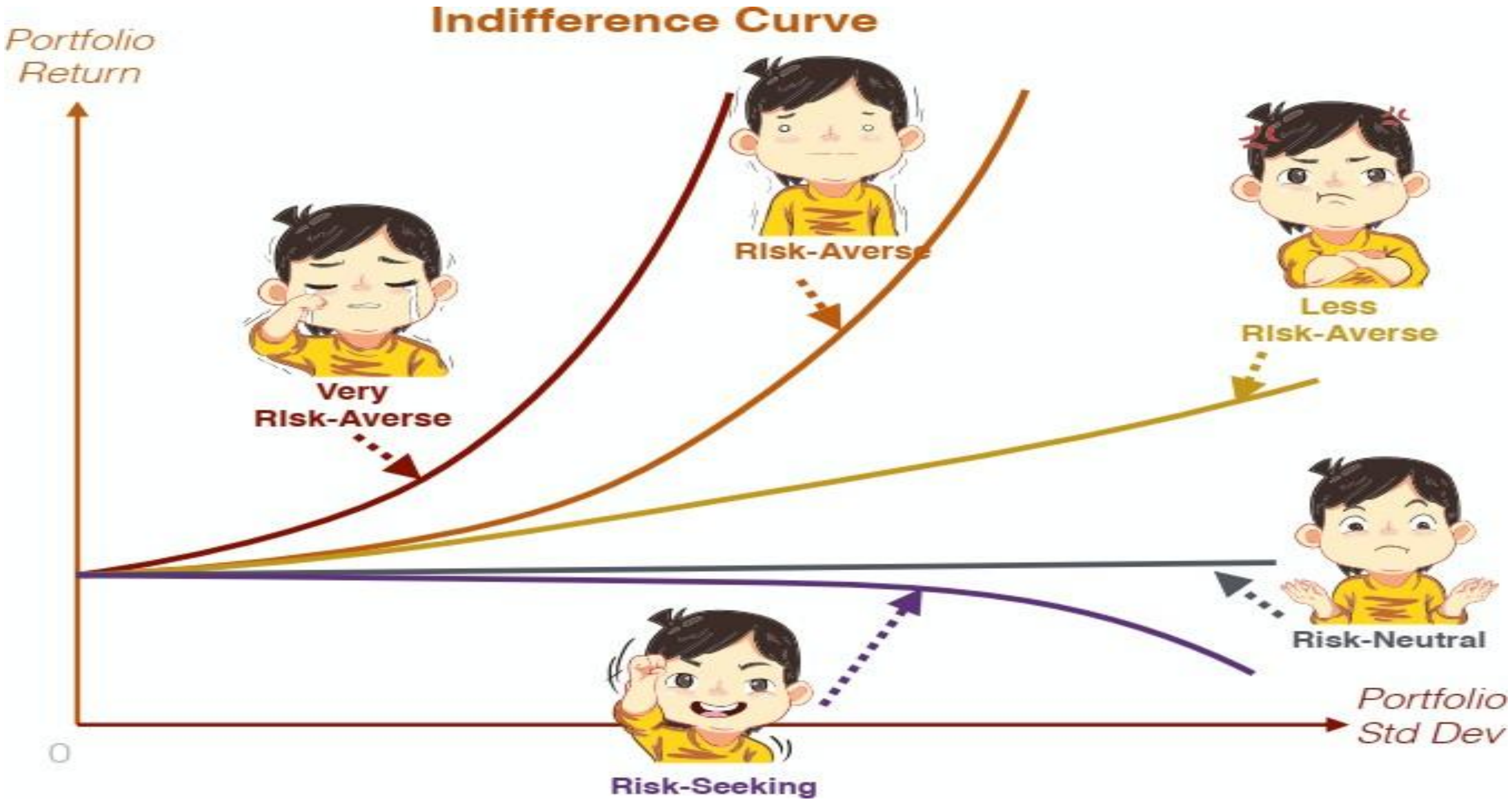
2.4 Indifference Curves

Indifference Curve is a term used in portfolio theory to describe investor demand for portfolios based on the trade-off between expected return and risk.

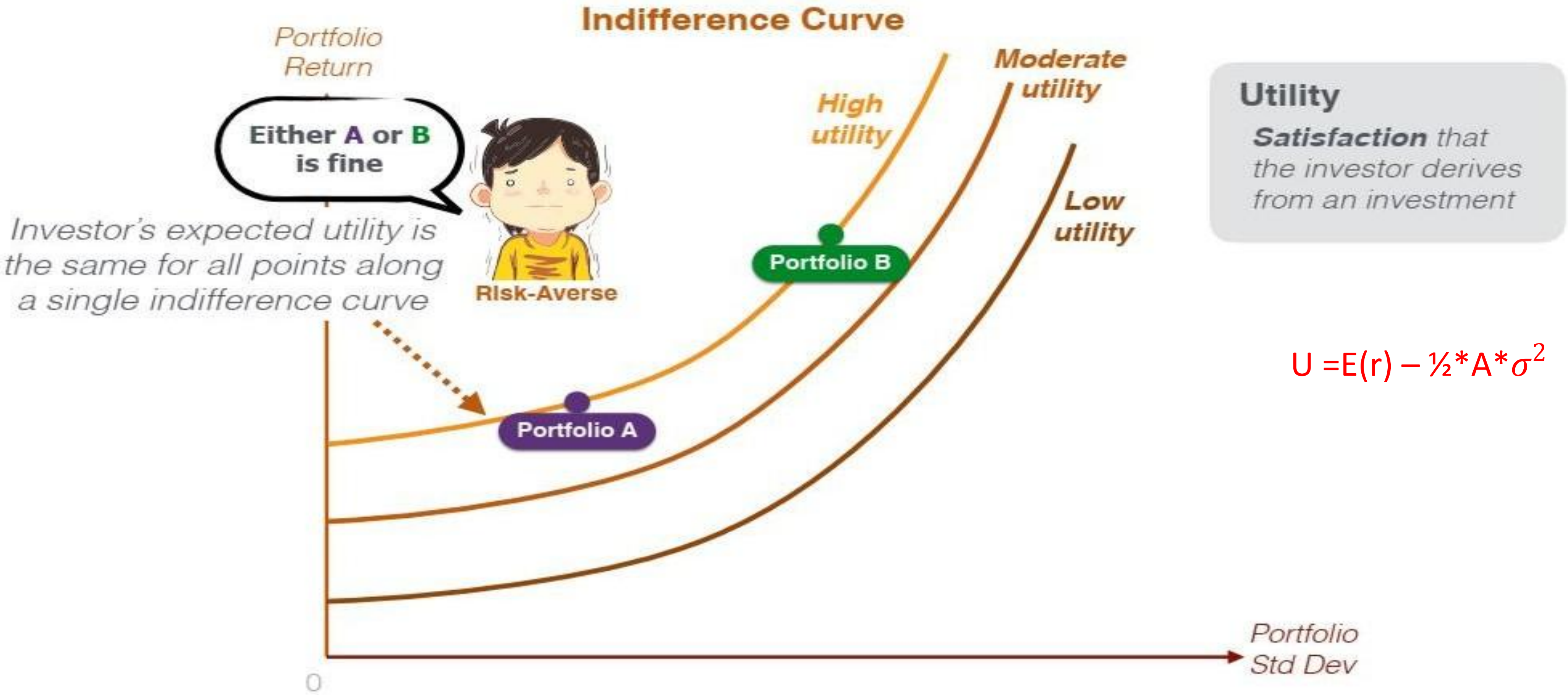


Risk Return combinations an investor would choose for a given utility level.

2.4 Indifference Curves for various investors



2.4 Indifference Curves for different levels of Utility



$U = E(r) - ½ * A * \sigma^2$

3 Aim of the Portfolio Theory

- Portfolio theory enables the investor to determine his optimal portfolio
- Optimal portfolio is the one which offers highest return for a given level of risk
- Mean-variance portfolio theory, sometimes called modern portfolio theory (MPT), specifies a method for an investor to construct a portfolio that gives the maximum return for a specified risk, or the minimum risk for a specified return.
- The theory relies on some strong and limiting assumptions

4 Assumptions of mean-variance portfolio theory

1. all expected returns, variances and covariances are known

2. investors make their decisions purely on the basis of expected return and variance

2. investors are non-satiated

2. investors are risk-averse

2. there is a fixed single-step time period

2. taxes or transaction costs are ignored

2. assets may be held in any amounts, *ie* short-selling is possible, we can have infinitely divisible holdings, and there are no maximum investment limits.

5 Two applications of MVPT

First

- First the definition of the properties of the portfolios available to the investor – the opportunity set. Here we are looking at the risk and return of the possible portfolios available to the investor.

Second

- Second, the determination of how the investor chooses one out of all the feasible portfolios in the opportunity set, *ie* the determination of the

6 Two Asset Portfolio: calculating weights

The Expected Return of a portfolio:

- To find an optimal portfolio, we need a method to define a portfolio and analyse its return.
- We can describe a portfolio by its portfolio weights, the fraction of the total investment in the portfolio held in each individual investment in the portfolio:
- $W_i = \frac{Value\ of\ investment}{total\ value\ of\ Portfolio}$
- These portfolio weights add up to 1(that is $\sum wi =)1$

6 Two Asset Portfolio: calculating weights

Given the portfolio weights, we can calculate the return on the portfolio. Suppose w_1 ,w_2 ……..,w_n are the portfolio weights of the n investments in a portfolio, and these investments have returns R_1 ,R_2 …. R_n . then the return on the portfolio R_p is the weighted average of the returns on the investments in the portfolio, where the weights correspond to portfolio weights

$$R_P = w_1R_1 + w_2R_2 + w_3R_3 + \cdots + w_n R_n = \sum_i w_iR_i$$

The return of a portfolio is straightforward to compute if we know the returns of the individual stocks and the portfolio weights.

6.1 The volatility of a two stock portfolio

What are the covariance and the correlation of a stock's return with itself?

Let R_i be the stock's return. From the definition of the covariance,

$$Cov(R_S, R_S) = E[(R_S - E[R_S])(R_S - E[R_S])] = E[(R_S - E[R_S])^2] = Var(R_S)$$

where the last equation follows from the definition of the variance. That is, the covariance of a stock with itself is simply its variance.

Then,

$$Corr(R_S, R_S) = \frac{Cov(R_S, R_S)}{SD(R_S)SD(R_S)} = \frac{Var(R_S)}{SD(R_S)^2} = 1$$

where the last equation follows from the definition of the standard deviation. That is, a stock's return is perfectly positively correlated with itself, as it always moves together with itself in perfect synchrony.

6.2 The Variance of a Two-Stock Portfolio

$$
\begin{aligned}
Var(R_P) &= Cov(R_P, R_P) \\
&= Cov(x_1 R_1 + x_2 R_2, x_1 R_1 + x_2 R_2) \\
&= x_1 x_1 Cov(R_1, R_1) + x_1 x_2 Cov(R_1, R_2) + x_2 x_1 Cov(R_2, R_1) + x_2 x_2 Cov(R_2, R_2)
\end{aligned}
$$

$$Var(R_P) = x^2 Var(R_1) + x^2 Var(R_2) + 2x_1 x_2 Cov(R_1, R_2)$$

6.2 The Variance of a Two-Stock Portfolio

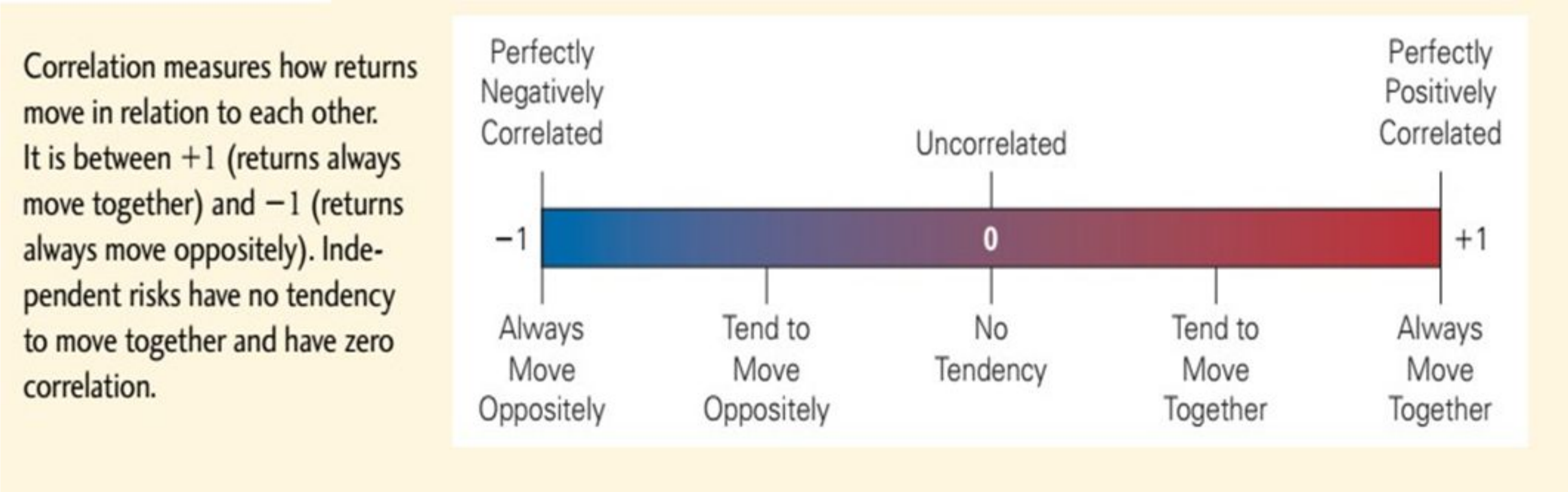
$$\rho_{1,2} = \frac{Cov_{1,2}}{\sigma_1 \sigma_2}$$

$$Cov_{1,2} = \rho_{1,2}\, \sigma_1 \sigma_2$$

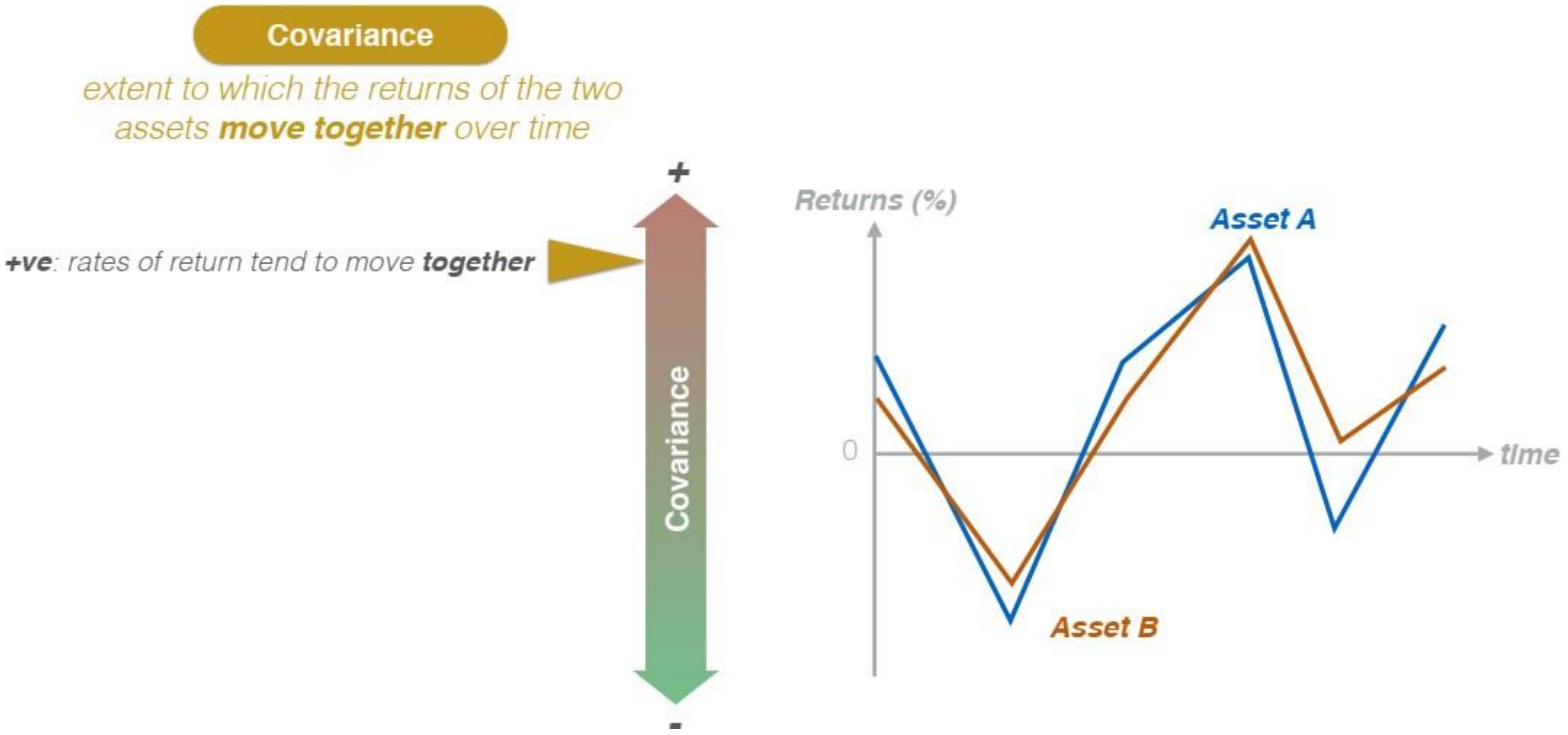
$$\text{Portfolio Variance} = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2Cov_{1,2}$$

$$\text{Portfolio Variance} = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho_{1,2}\sigma_1\sigma_2$$

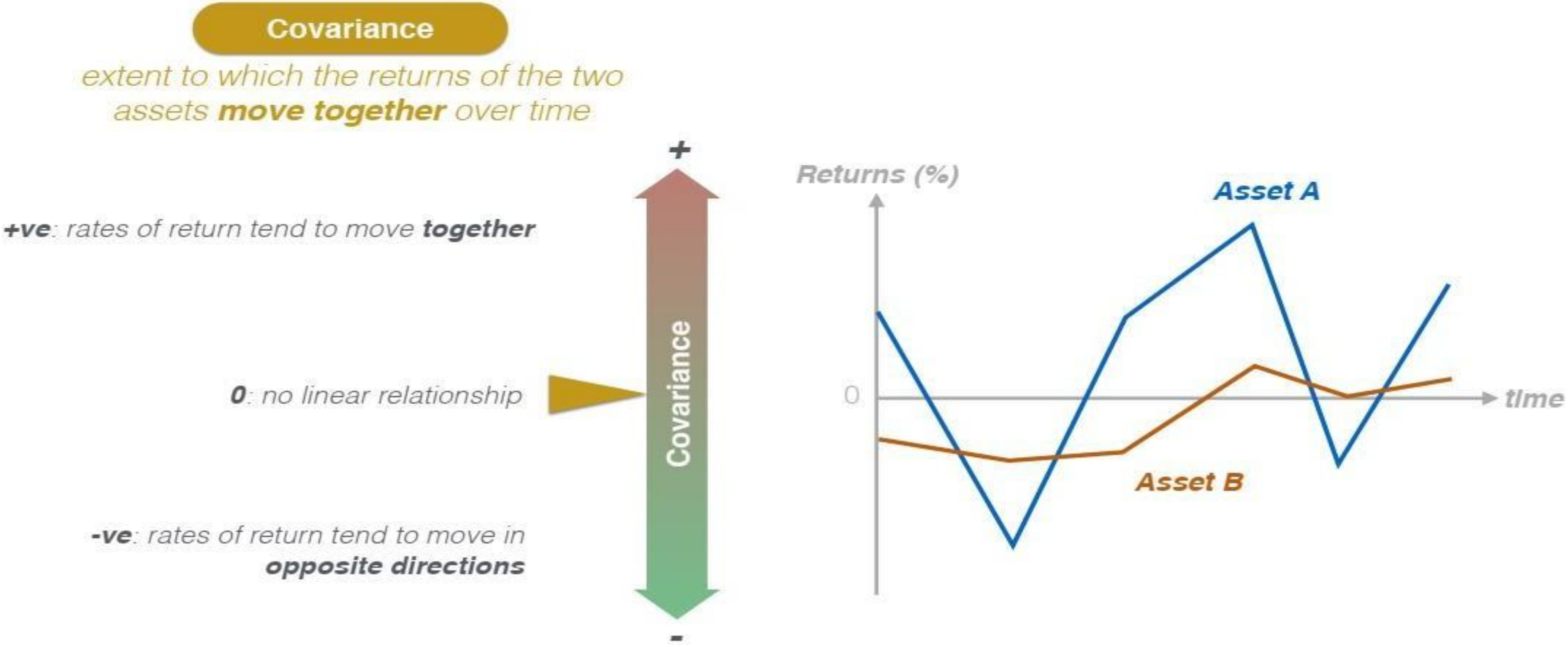
6.3 Correlation between two assets



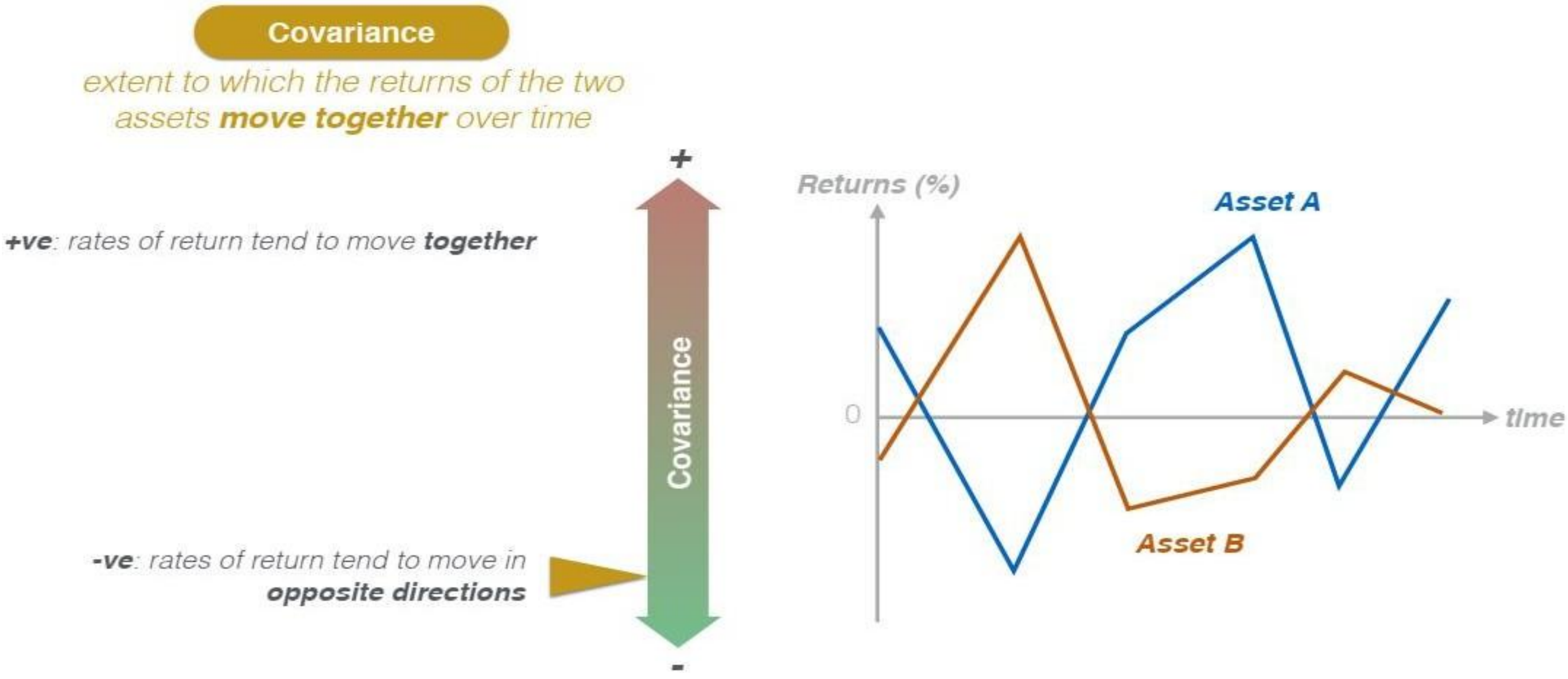
6.3 Positive correlation between two assets



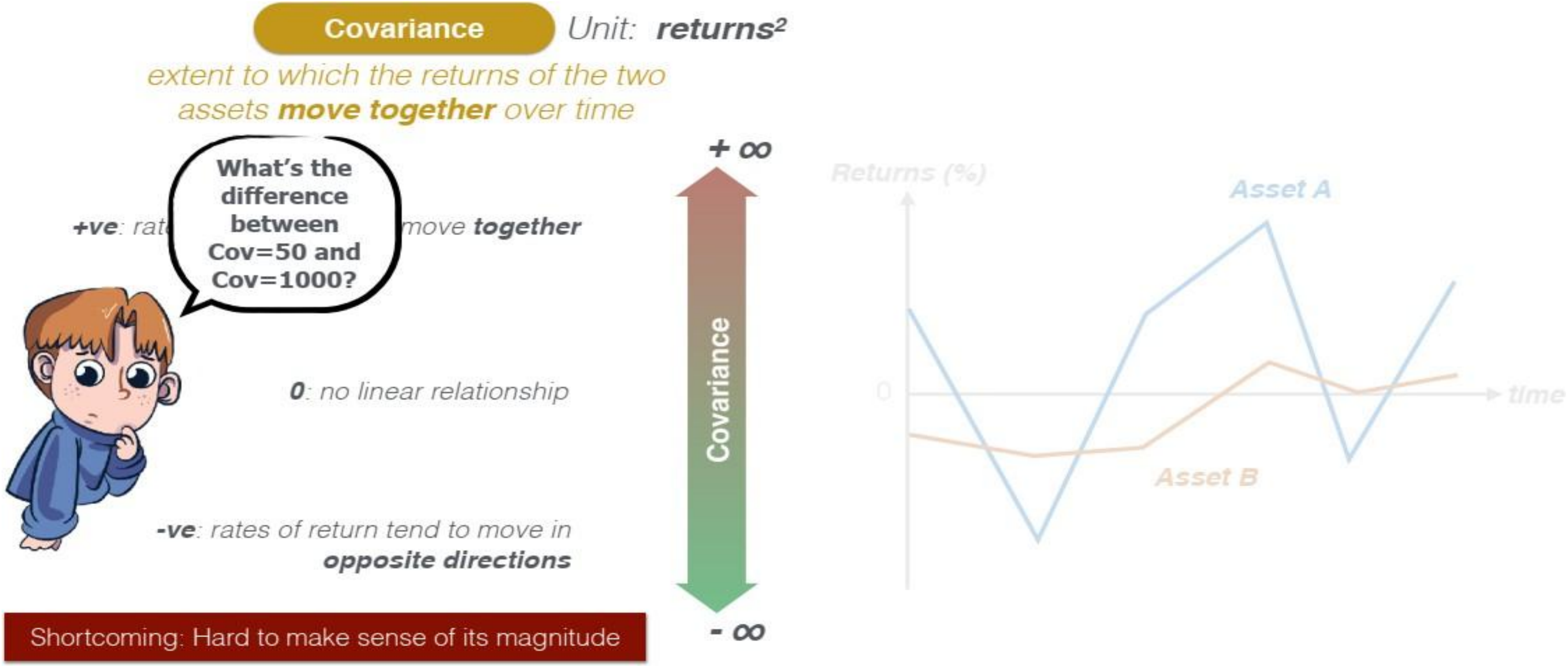
6.3 No correlation between two assets



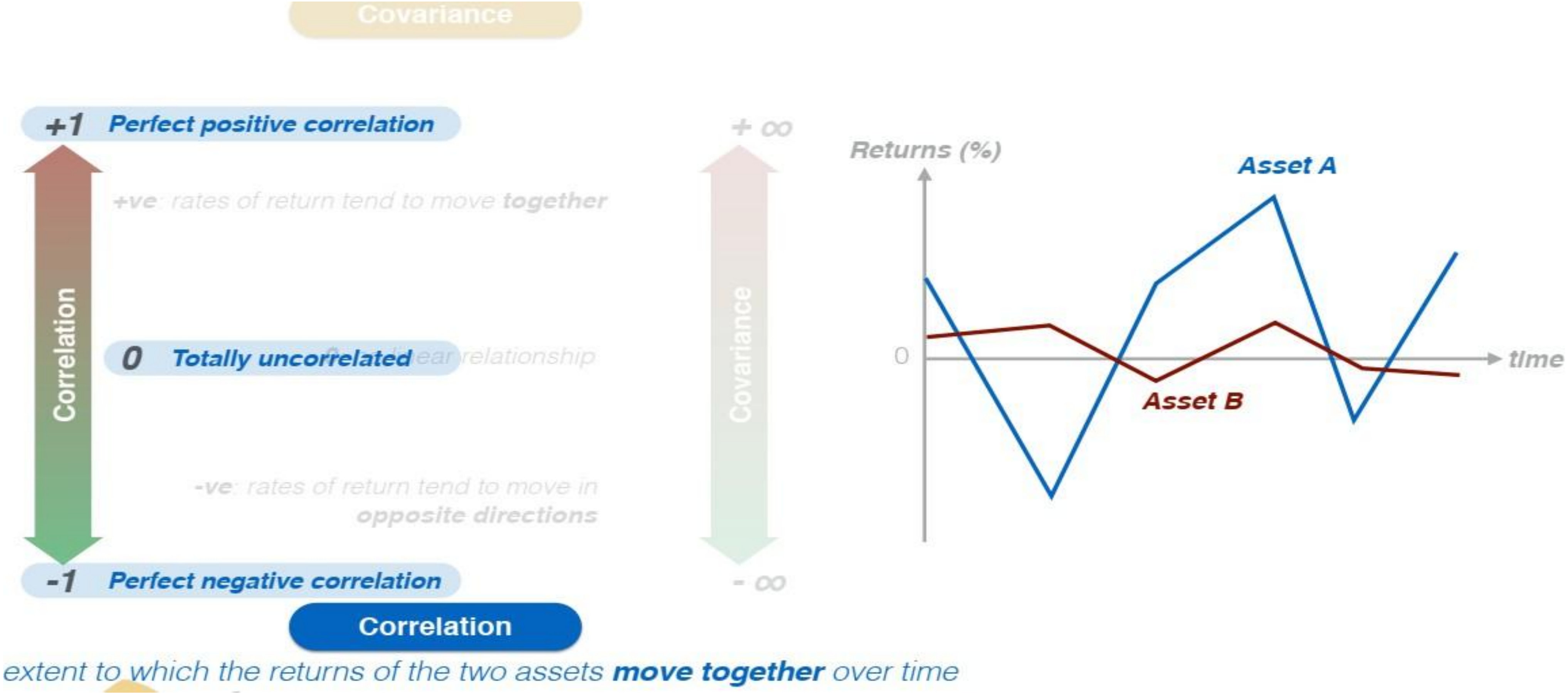
6.3 Negative correlation between two assets



6.4 Shortcoming of covariance



6.4 The Answer: Correlation coefficient



6.5 Minimum Variance Portfolio

Inorder to find the Minimum variance portfolio.

$$V_p = x_A^2 V_A + (1 - x_A)^2 V_B + 2x_A(1 - x_A)C_{AB}$$

$$V_p = f(x_A)$$

We need to find that x_A that minimizes our V_p

$$x_A = \frac{V_B - C_{AB}}{V_A - 2C_{AB} + V_B}$$

This is the proportion to be invested in Asset A which will maximize our V_p.

Check out the video on Minimum variance portfolio

6.5 Minimum Variance Portfolio

$$x_A = \frac{V_B - C_{AB}}{V_A - 2C_{AB} + V_B}$$

Prove the above result.

The variance of the portfolio return is:

$$V_P = x_A^2 V_A + x_B^2 V_B + 2x_A x_B C_{AB}$$

$$= x_A^2 V_A + (1 - x_A)^2 V_B + 2x_A(1 - x_A) C_{AB}$$

We want to choose the value for x_A that minimises the variance V_P. To do this, we differentiate and set to zero:

$$\frac{dV_P}{dx_A} = 2x_A V_A + 2(1 - x_A)V_B(-1) + 2(1 - x_A)C_{AB} - 2x_A C_{AB} = 0$$

$$\Leftrightarrow \quad 2x_A V_A - 2V_B + 2x_A V_B + 2C_{AB} - 4x_A C_{AB} = 0$$

$$\Leftrightarrow \quad x_A(V_A + V_B - 2C_{AB}) = V_B - C_{AB}$$

$$x_A = \frac{V_B - C_{AB}}{V_A - 2C_{AB} + V_B}$$

Question

CT8, September 2018, Q.4

An investor has £100 and is considering investing in two different stocks. The prices of both stocks are assumed to follow the lognormal model with the parameters below.

Stock	Current price	Drift μ	Volatility σ
A	5	5%	20%
B	5	8%	30%

(i) Calculate the expected value at time 3 of £100 invested in: (a) stock A (b) stock B.

(ii) Calculate the standard deviation at time 3 of £100 invested in: (a) stock A (b) stock B.

The investor decides to invest £50 in each stock.
(iii) Calculate the expected value of the investor's portfolio at time 3.

Question

The correlation of the two stocks is 0.3.
(iv) Calculate the standard deviation of the value of the investor's portfolio at time 3.

(v) Comment on the expected return and standard deviation of the portfolio compared to investing the whole £100 in one stock.

Solution

i)

a. $E[A_3] = A_0 \exp(\mu t + 0.5\sigma^2 t) = 100 \exp(0.05*3+0.5*0.2^2 *3) = £123.37$

b. $E[B_3] = B_0 \exp(\mu t+0.5\sigma^2 t) = 100 \exp(0.08*3+0.5*0.3^2 *3) = £145.50$

ii)

a. $SD[A_3] = \sqrt{(A_0^2 \exp(2\mu t+\sigma^2 t)(\exp(\sigma^2 t)-1))}$

$= \sqrt{(100^2 \exp(2*0.05*3+0.2^2 *3)(\exp(0.2^2 *3)-1))}$

$= £44.05$

b. $SD[B_3] = \sqrt{(B_0^2 \exp(2\mu t+\sigma^2 t)(\exp(\sigma^2 t)-1))}$

$= \sqrt{(100^2 \exp(2*0.08*3+0.3^2 *3)(\exp(0.3^2 *3)-1))}$

$= £81.01$

iii) $E[P_3] = 0.5E[A3] + 0.5E[B3] = £134.44$

iv) $V[P_3] = 0.5^2 V[A_3] + 0.5^2 V[B_3] + 2*\text{Correlation}*0.5*0.5*SD[A_3]*SD[B_3]$

$= 0.25*44.05^2 + 0.25*81.01^2 + 2*0.3*0.5*0.5*44.05*81.01$

$= 2{,}661.03$

$SD[P_3] = £51.59$

Solution

v) The expected return of the portfolio falls halfway between the expected return on each of the one-stock investment strategies.

But the standard deviation is well below halfway between the two one-stock strategies.
The price of risk for stock A is 23.37/44.05 = 0.53
The price of risk for stock B is 45.5/81.01 = 0.56
But the price of risk for the portfolio is 34.44/51.59 = 0.67

So the portfolio delivers a better expected return per unit of risk [1] This is because the assets are not fully correlated…
Which shows the benefit of diversification.

Question

CT8 April 2008, Q6

(i) Outline the assumptions used in modern portfolio theory regarding investor behaviour that are necessary to specify efficient portfolios.

(ii) An investor can construct a portfolio using only two assets X and Y. The statistical properties of the two assets are shown below:

	X	Y
Expected return	12%	8%
Variance of return	30%	15%

Correlation coefficient between assets X and Y = 0.5

Assuming that the investor cannot borrow to invest:

(a) Determine the composition of the portfolio which will give the investor the highest expected return.

(b) Calculate the composition of the portfolio which will give the investor the minimum variance.

Solution

(i) Investors select their portfolios on the basis of the expected return and the variance of the return over a single time horizon.

Investors are never satiated. At a given level of risk, they will always prefer a portfolio with a higher return to one with a lower return.

Investors dislike risk. For a given level of return they will always prefer a portfolio with lower variance to one with higher variance.

(ii)

(a) 100% in X – expected return of 12%

(b) Proportion in X $= (V_Y + C_{XY})/(V_X + V_Y + C_{XY})$

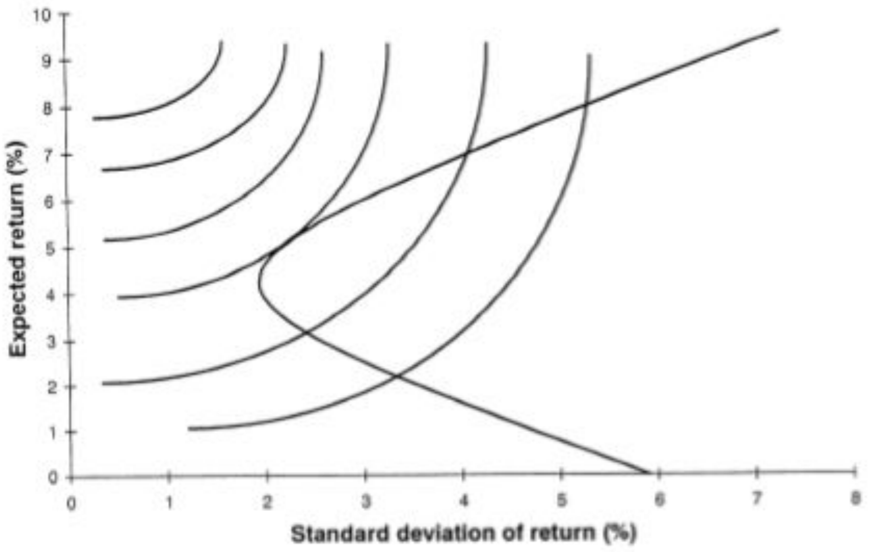
$= (15\%\% - 0.5 \times (30\%\% - 15\%\%)^{0.5})/(30\%\% + 15\%\% + 0.5 \times (30\%\% - 15\%\%)^{0.5})$

$= 18.47\%$

Solution

(iii) Plot indifference curves in return-standard deviation space.

Utility is maximised by choosing the portfolio on the efficient frontier where the frontier is at a tangent to the indifference curve.



7 Efficient Portfolios with two stocks

Consider a portfolio of Intel and Coca-Cola stock. Suppose an investor believes these stocks are uncorrelated and will perform as follows:

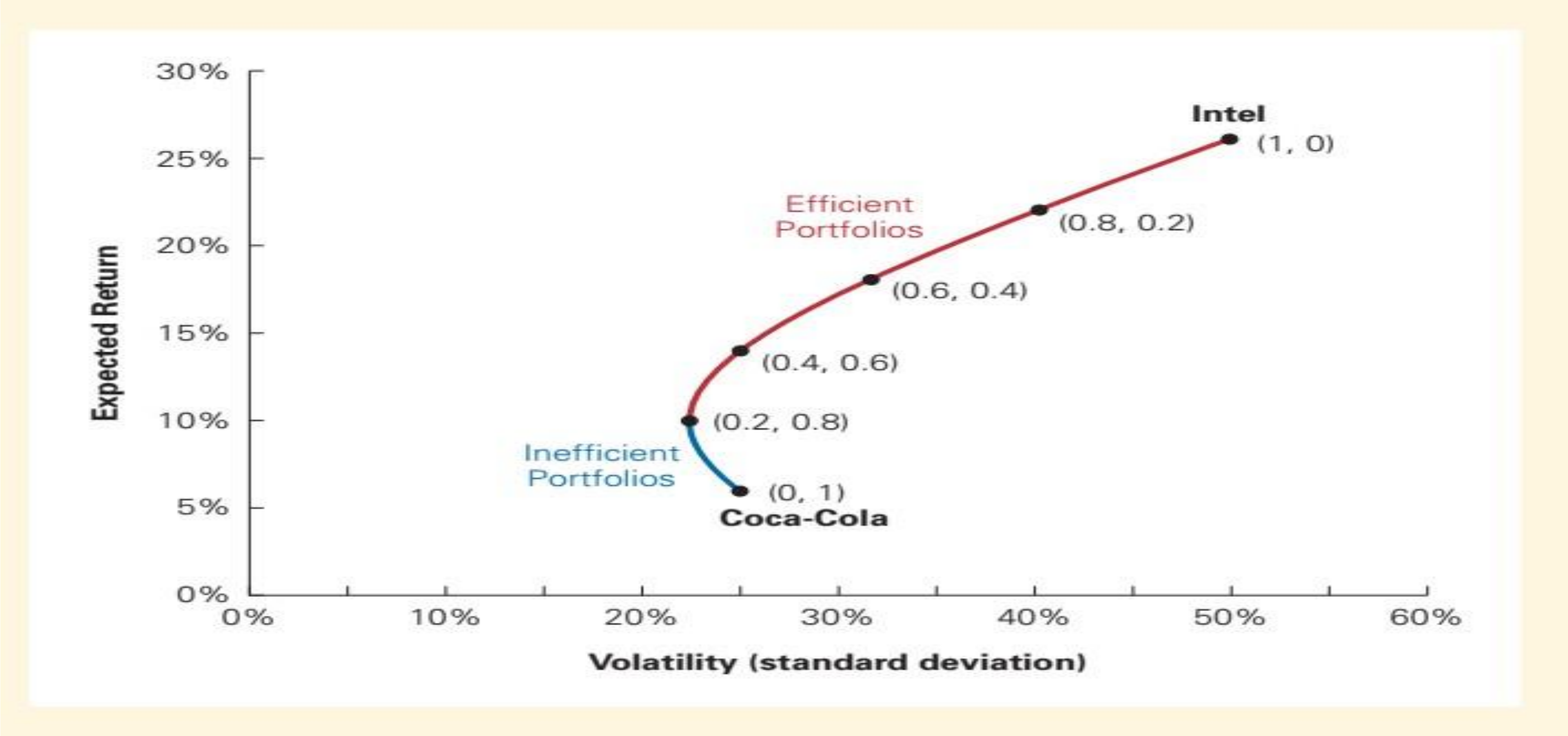
Stock	Expected Return	Volatility
Intel	26%	50%
Coca-Cola	6%	25%

How should the investor choose a portfolio of these two stocks? Are some portfolios preferable to others?

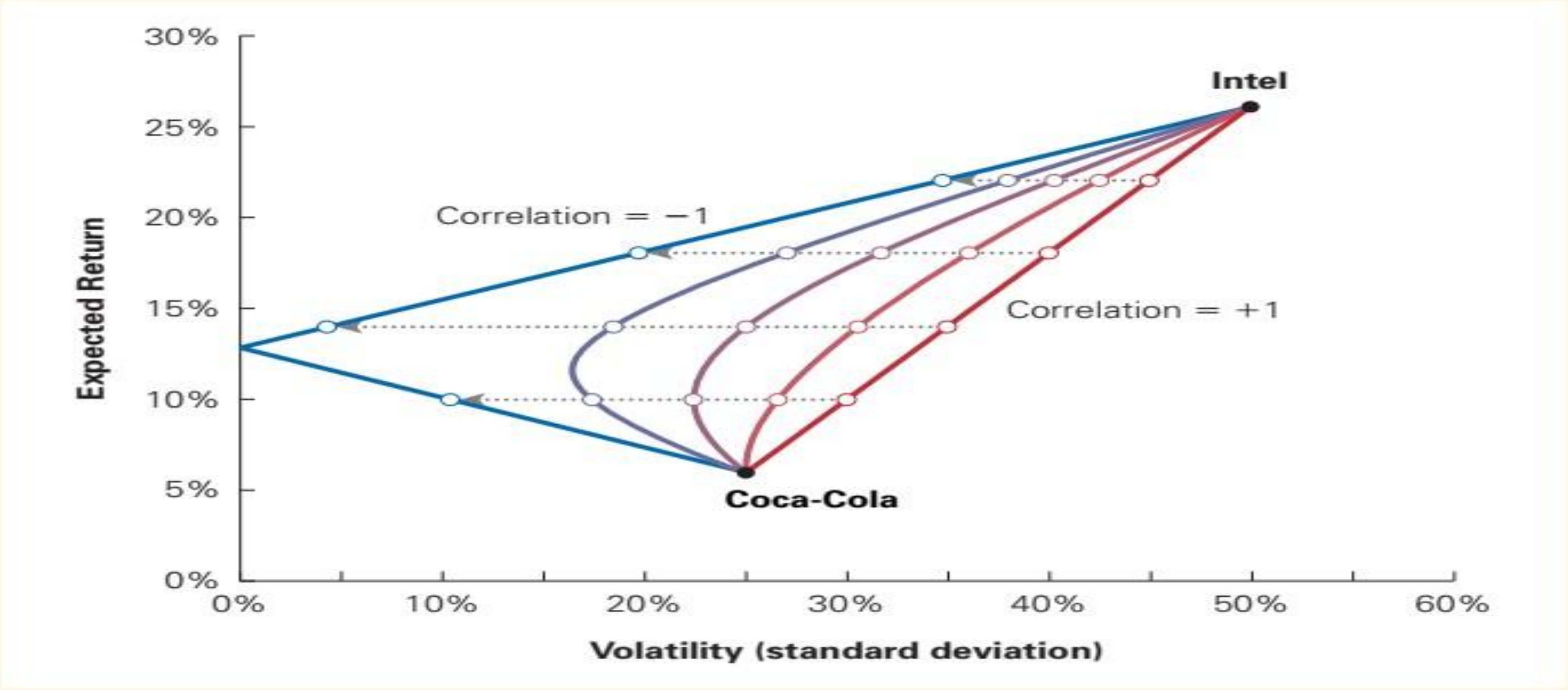
7 Efficient Portfolios with two stocks

Portfolio Weights		Expected Return (%)	Volatility (%)
x_I	x_C	$E[R_P]$	$SD[R_P]$
1.00	0.00	26.0	50.0
0.80	0.20	22.0	40.3
0.60	0.40	18.0	31.6
0.40	0.60	14.0	25.0
0.20	0.80	10.0	22.4
0.00	1.00	6.0	25.0

7 Efficient Portfolios with two



7 The Effect of Correlation on Efficient Frontier



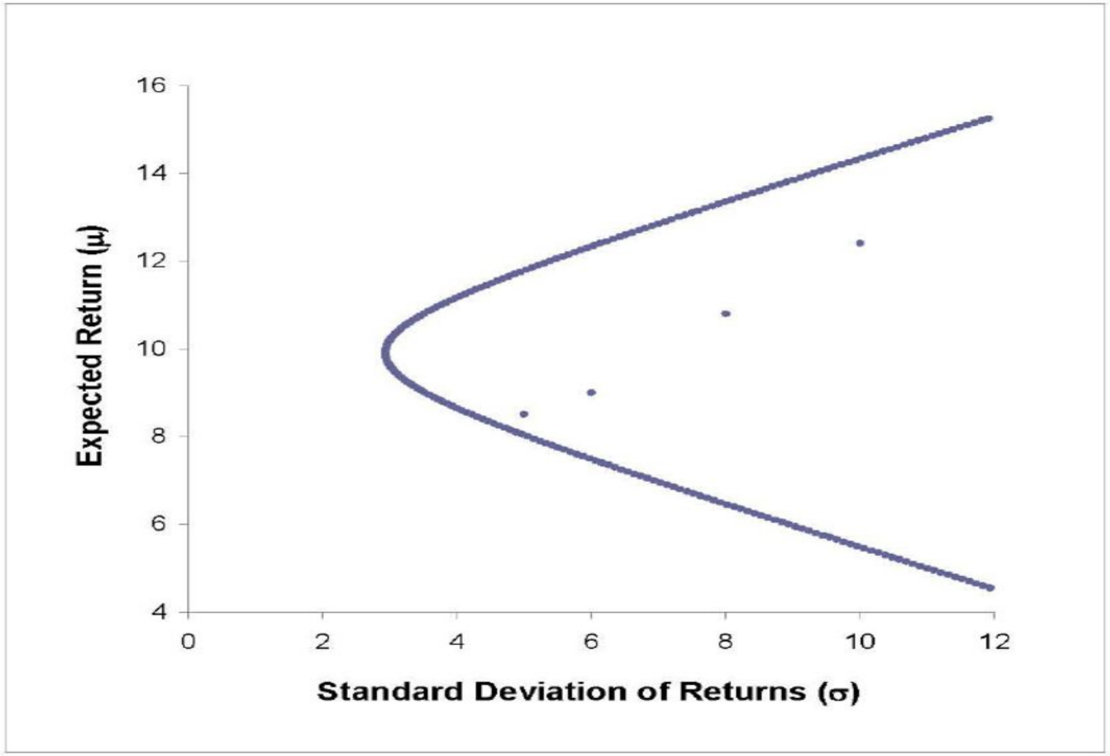
8 Volatility of a large portfolio

We can gain additional benefits of diversification by holding more than two stocks in our portfolio.

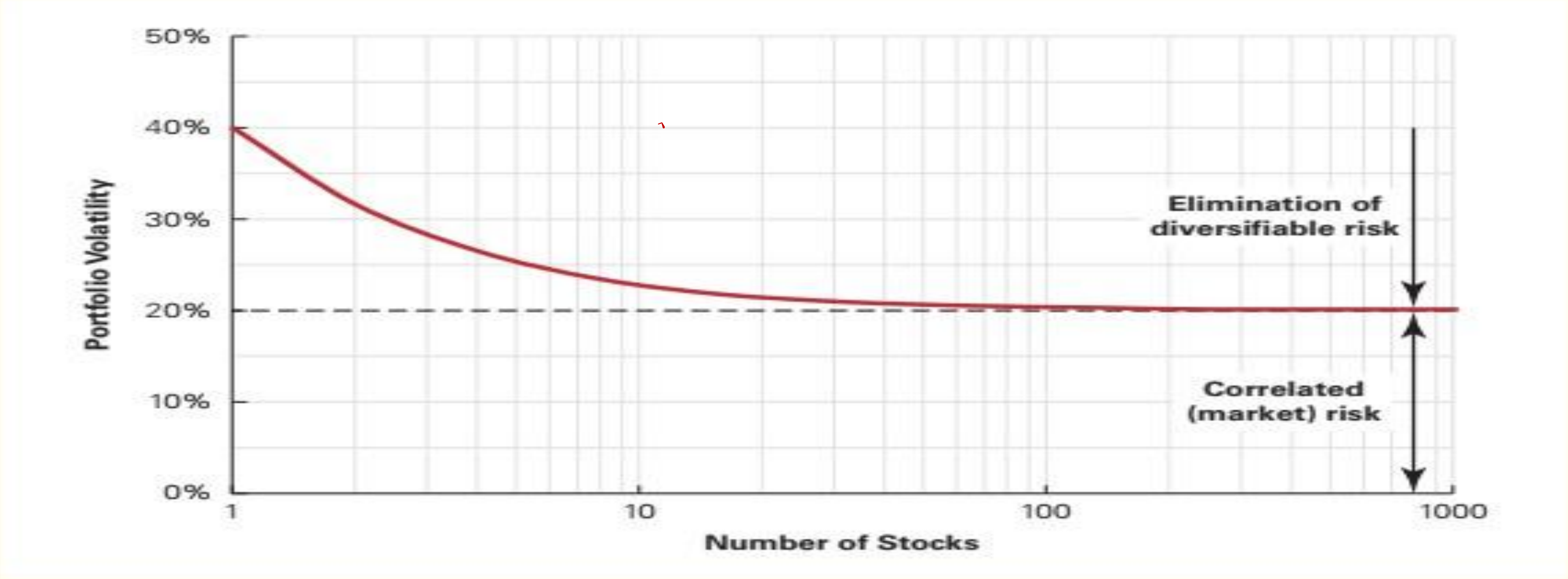
Large Portfolio Variance

9 Risk Return Space

The graph between expected return on the portfolio and the standard deviation of the portfolio is known as risk-return space or the E-sigma graph.



10 Effect of adding securities to a portfolio



The volatility declines as the number of stocks in the portfolio increases. Even in a very large portfolio, however, market risk remains.

10.1 N asset Portfolio

Portfolio Mean Return

Portfolio Mean, $E = E\ R_P$

Portfolio Variance

Portfolio Variance, $V = var\ R_P \quad = \sigma_i \sigma_j w_i w_j C_{ij}$

where $C_{ij} = \varrho_{ij} \sigma_i \sigma_j$

In order to calculate portfolio mean and variance we need:

- N means, one for each security
- N variances, one for each security
- N*(N–1)/2 covariances

So a total of N(N+3)/2 parameter values are required.

10.1 N asset Portfolio: components

Note that this means that with *N* different securities an investor must specify:

- N expected returns.
- N variances of return.
- N*(N-1)/2 covariances.

So, a total of N*(N+3)/2 parameter values are required.

10.2 N asset portfolio: variance/covariance matrix

Variance/ covariance matrix shows the covariance between each pair of the variables

For 3 variables, the matrix can be shown as:

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

Where:
(for i ≠ *j*)

c_{ij}: is the covariance between variables *i* and *j*

(for i = *j*)
c_{ii}: is the variance of variable i

Since, $c_{ij} = c_{ji}$, the matrix is symmetric about the leading diagonal.

Question

CT8, September 2017, Q.2

(i) Define in the context of mean-variance portfolio theory:
(a) an inefficient portfolio
(b) an efficient portfolio

(ii) State the two assumptions about investor behaviour that are needed for the existence of efficient portfolios.
An investment universe includes two assets, A and B, with expected return on asset i of ri and variance vi as set out below:

Asset	Expected return	Variance of return
A	$r_A = 0.05$	$V_A = 0.16$
B	$r_B = 0.07$	$V_B = 0.25$

The correlation of returns is $c_{AB} = -0.2$.

Question

In an efficient portfolio, let a be the proportion which is held in asset A.
(iii) Express the portfolio variance V in terms of a quadratic function in a, showing your workings.

Let R be the expected return on the portfolio.

(iv) Express the portfolio variance V in terms of a quadratic function in R, using your result from part (iii) and showing your workings. [Your expression should not include a.]

The expression in part (iv) represents the efficient frontier.
An investor uses a utility function that gives rise to an indifference curve $V = 16R - 200R^2$.

(v) Determine the two portfolios on the efficient frontier that also lie on the investor's indifference curve.

(vi) Comment on the implications for part (v) if short selling is not allowed in the market.

Solution

(i)
(a) A portfolio is inefficient if the investor can find another portfolio with the same expected return and lower variance, or the same variance and higher expected return.
(b) A portfolio is efficient if the investor cannot find a better one in the sense that it has both the same or higher expected return and the same or lower variance.

(ii) The assumptions are:
(a) Investors are never satiated. [At a given level of risk, they will always prefer a portfolio with a higher expected return to one with a lower return.]
(b) Investors dislike risk. [For a given level of return, they will always prefer a portfolio with lower expected variance to one with higher variance.]

(iii) $V = a^2 V_A + (1 - a)^2 V_B + 2a(1 - a)(V_A V_B)^{0.5} C_{AB}$
$= 0.16a^2 + 0.25 (1 - a)^2 - 2a(1 - a) (0.16 * 0.25)^{0.5} * 0.2$
$= 0.49a^2 - 0.58a + 0.25$

Solution

(iv) $R = aR_A + (1 - a)R_B$
$= -0.02a + 0.07$

So $R^2 = 0.0004a^2 - 0.0028a + 0.0049$
So $V = 1225R^2 - 142.5R + 4.2225$

(v) $1225R^2 - 142.5R + 4.2225 = 16R - 200R^2$
So $R = 0.0670$ or 0.0442
Hence $a = 0.1497$ or 1.2889

(vi) The second solution implies a proportion of −0.2889 invested in asset B so would not be allowed, hence only the first solution would remain.

Summary

- Indifference curve

  An indifference curve is a graphical representation of a combined products that gives similar kind of satisfaction to a consumer thereby making them indifferent. Every point on the indifference curve shows that an individual or a consumer is indifferent between the two products as it gives him the same kind of utility.

- Opportunity Set

  The expected return/standard deviation pairs of all *portfolios* that can be constructed from a given set of *assets*.

- Efficient portfolio

  A portfolio is *efficient* if the investor cannot find a better one in the sense that it has either a higher expected return and the same (or lower) variance or a lower variance and the same (or higher) expected return.

- Efficient frontier

  The *efficient frontier* is the set of optimal portfolios that offer the highest expected return for a defined level of risk or the lowest risk for a given level of expected return

Summary

In this portfolio, there are only 2 securities, A and B.

- w_A & w_B is the weight assigned to security A and B respectively
- Portfolio expected return, $E = w_A E_A + w_B E_B$
- Portfolio Variance, $V_{Port} = w_A^2 V_A + w_B^2 V_B + 2w_A w_B C_{AB}$
- Where $C_{AB} = \varrho_{AB} \sigma_A \sigma_B$
- E_A and E_B the expected return from security A and B respectively. Then the return on the portfolio,R_p is the weighted average of the returns on the investments in the portfolio, where the weights correspond to portfolio weights.
- V_A & V_B: variance of security A and B respectively.
- C_{AB}: covariance between security A and B.
- ϱ_{AB} correlation coefficient between security A and B.

Thank You