Lecture 1

INSTITUTE OF ACTUARIAL
& QUANTITATIVE STUDIES

Class: M.Sc. – Sem 2

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Chapter: Unit 3 Chapter 2

Chapter Name : Greeks

Today's Agenda

1. Factors affecting stock prices

2. The Greeks
   - 2.1 Delta
   - 2.2 Gamma
   - 2.3 Vega
   - 2.4 Theta
   - 2.5 Rho
   - 2.6 Lambda

3. The Relationship Between the Greeks

1. Factors affecting stock prices

There are six factors affecting the price of a stock option:

1. The current stock price at time 0, S_0

2. The strike price, K

3. The time to expiration, T -t

4. The volatility of the stock price,

5. The risk-free interest rate, r

6. The dividends that are expected to be paid.

We consider what happens to option prices when there is a change to one of these factors, with all the other factors remaining fixed. We understand this with the help of Greeks

The Greeks are a group of mathematical derivatives that can be used to help us to manage or understand the risks in our portfolio. We now introduce a set of six Greeks

2.1 Delta

Let $f(t, S_t)$ be the value/price at time t of a derivative when the price of the underlying asset at t is S_t.

Delta tells us the (approximate) change in the derivative price when the underlying asset price changes.

$$\Delta = \frac{df(t, s_t)}{dS_t}$$

Suppose C = value/ price of a call and P = value/ price of put.

$$\Delta_c = \frac{dC}{dS_t} \text{ and } \Delta_p = \frac{dP}{dS_t}$$

Δ_c is always positive and Δ_p is always negative.

The delta of the underlying asset is equal to the derivative of the underlying asset price with respect to itself. By definition, this must be equal to 1, *ie*:

$$\Delta_s = \frac{dS_t}{dS_t} = 1$$

2.1 Delta of the portfolio

A portfolio for which the weighted sum of the deltas of the individual assets is equal to zero is described as delta-hedged or delta-neutral.

We give delta of the portfolio by Δ_V.

When we consider delta hedging, we add up the deltas for the individual assets and derivatives (taking account, of course, of the number of units held of each). If this sum is zero and if the underlying asset prices follow a diffusion then the portfolio is instantaneously risk-free.

Instantaneously risk-free means that if we know the value of the portfolio at time t, then we can predict its value at time t + dt with complete certainty.

2.1

Delta of the portfolio - Example

Investor A has a portfolio with 2 long call options, 2 short put options and 3 units of the underlying asset. The delta of each call option is 0.5 and that of put option is -0.5.

Find the delta of the portfolio.

Solution:

$\Delta_{\mathbf{V}} = 2\ \Delta_{\mathbf{c}} - 2\ \Delta_{\mathbf{p}} + 3\ \Delta_{\mathbf{s}}$

$= 2\ (0.5)\ -\ 2(-0.5)\ +3\ (1)$

$= 1 - (-1) + 3$

$= 5$

2.1

Delta of the portfolio - Example

Hedge the above portfolio. Consider using call options for the same.

Solution:

Now in order to hedge the portfolio we need $\boldsymbol{\Delta_V}$ = 0. Let X be the number of calls required (buy or sell) to delta-hedge the portfolio. Therefore,

$\Delta_V = 2\,\Delta_{\mathbf{c}} - 2\,\Delta_{\mathbf{p}} + 3\,\Delta_s + X\,\Delta_{\mathbf{c}}$

$0 = 2\,(0.5) - 2(-0.5) + 3\,(1) + X\,(0.5)$

$-5 = X\,(0.5)$

$X = -10.$

Thus sell 10 calls in order to have a delta neutral portfolio.

Dynamic hedging and Static hedging

We can distinguish between dynamic hedging and static hedging.

- The process of simply constructing an initial portfolio with a total delta of zero, at time 0 say, and not rebalancing to reflect the subsequent changes in delta, is known as static delta hedging.
- However, as the share price St varies with time, so does: the price of the derivative and hence the delta.
- If it is intended that the sum of the deltas should remain close to zero (this is what is called delta hedging) then normally it will be necessary to rebalance the portfolio on a regular basis.
- The process of continuously rebalancing the portfolio in this way in order to maintain a constant total portfolio delta of zero is known as dynamic delta hedging.
- The extent of rebalancing depends primarily on gamma.

2.2 Gamma

Gamma is the rate of change of Δ with the price of the underlying asset.

It therefore measures the *curvature* or convexity of the relationship between the derivative price and the price of the underlying asset.

$$\Gamma = \frac{d^2 f\left(t, s_t\right)}{d{s_t}^2}$$

Gamma is always positive.

The gamma of the underlying asset, S_t, Γ = 0.

Suppose a portfolio is following a delta hedging strategy. If the portfolio has a high value of Γ then it will require more frequent rebalancing or larger trades than one with a low value of gamma.

Note that it may also be possible to construct a gamma-neutral portfolio – ie one with an overall gamma equal to zero.

Delta Gamma Neutral Portfolio

Investor A has a portfolio with 2 long call options, 2 long put options and 3 units of the underlying asset. The delta of each call option is 0.5 and that of put option is -0.5. Gamma of the call and put option is 100.

Consider adding x calls and y puts in order to ensure that Δ_V = 0 and γ_v = 0. We have two simultaneous equations as:

Δ_V = 2(0.5) + 2(-0.5) + 3(1) + x(0.5) + y(-0.5) = 0

Therefore, x(0.5) + y(-0.5) = -3

γ_v = 2(100) + 2(100) + 3(0) + x(100) + y(100) = 0

Therefore, x(100) + y(100) = -400

Solving the two simultaneous equations we have,

x = -5 and y = 1 → These lead to a delta gamma neutral portfolio.

2.3 Vega

This is the rate of change of the price of the derivative with respect to a change in the assumed level of volatility of S_t.

$$V = \frac{df\,(t, s_t\,)}{d\,\sigma}$$

Vega is positive for a call and a put option.

The value of a portfolio with a low value of vega will be relatively insensitive to changes in volatility. Put another way, it is less important to have an accurate estimate of σ if vega is low. Since σ is not directly observable, a low value of vega is important as a risk management tool.

Furthermore, it is recognised that σ can vary over time. Since many derivative pricing models assume that σ is constant through time, the resulting approximation will be better if vega is small.

2.4 Theta

Changes with respect to the time elapsed.

$$\theta = \frac{df(t, s_t)}{dt}$$

Theta is negative both for call and put options.

Since time is a variable which advances with certainty, it does not make sense to hedge against changes in t in the same way as we do for unexpected changes in the price of the underlying asset

2.5 Rho

Rho tells us about the sensitivity of the derivative price to changes in the risk-free rate of interest.

$$\rho = \frac{df\,(t, s_t)}{d\,r}$$

Rho is positive for a call option and negative for a put option.

The risk-free rate of interest can be determined with a reasonable degree of certainty, but it can vary by a small amount over the (usually) short term of a derivative contract.

As a result, a low value of ρ reduces risk relative to uncertainty in the risk-free rate of interest.

2.6 Lambda

$$\lambda = \frac{df(t, s_t)}{dq}$$

where q is the assumed, continuous dividend yield on the underlying security.

Lambda is negative for a call and positive for a put option.

3. Relationship between Greeks

Put-Call Parity relationship is given as (for same strike price and same maturity)

C + X e^{-rt} = S_0 + P.

We now differentiate the put-call parity with the various factors to understand the relationship between different greeks.

1. Differentiate wrt to $\boldsymbol{S_t}$

$\Delta_c + 0 = 1 + \Delta_p$

Therefore: $\Delta_c = 1 + \Delta_p$

2. Again differentiate wrt to $\boldsymbol{S_t}$ we have:

$\boldsymbol{\gamma_c = \gamma_p}$

3. Relationship between Greeks

3. Differentiate wrt to σ

$\sigma_c + 0 = 0 + \sigma_p$

Thus, $\boldsymbol{\sigma_c} = \boldsymbol{\sigma_p}$

4. Differentiate wrt to t

Put Call parity can be written at t as C(t) + X $e^{-r(T-t)}$ = S_t + P(t)

$\Theta_c + X\, \boldsymbol{e^{-r(T-t)}}.\, r = 0 + \Theta_p$

5. Differentiating wrt to r

$\boldsymbol{\rho_c} + X\, \boldsymbol{e^{-r(T-t)}}\, [-(T-t)] = 0 + \boldsymbol{\rho_p}$

Quick Recap

➢ Factors affecting option prices

1. The current stock price at time 0, S_0

2. The strike price, K

3. The time to expiration, T -t

4. The volatility of the stock price,

5. The risk-free interest rate, r

6. The dividends that are expected to be paid.

Quick Recap

- Delta

$$\Delta = \frac{d f(t, S_t)}{dS_t}$$

- Gamma

$$\Gamma = \frac{d^2 f(t, S_t)}{dS_t^{\,2}}$$

- Vega

$$\nu = \frac{d f(t, S_t)}{d\sigma}$$

- Theta

$$\theta = \frac{d f(t, S_t)}{dt}$$

- Rho

$$\rho = \frac{d f(t, S_t)}{dr}$$

- Lambda

$$\lambda = \frac{d f(t, S_t)}{dq}$$