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Subject: Financial Engineering 1

Chapter: Unit 2

Category: Practice Questions

1. Subject CT8 April 2013 Question 5

(i) State the five key features of a standard Brownian motion B_t.

Consider a stochastic differential equation

$$dX_t = Y_t dB_t + A_t\, dt \,,$$

Where A_t is a deterministic process & Y_t is a process adapted to the natural filtration of B_t

(ii) Write down Ito's lemma for $f(t, X_t)$, where f is a suitable function

(iii) Determine $df(t, X_t)$ Where $f(t, X_t) = e^{2tX_t}$.

2. Subject CT8 April 2013 Question 7

A non-dividend paying stock in an arbitrage free market has a current price of 150p. Over each of the next two years its price will either be multiplied by a factor of 1.2 or divided by 1.2. The continuously compounded risk free rate is 1% p.a. The value of an option on the stock is 50p.

Denote by P_{uu} the value of payoff if both stock price moves are up, P_{ud} the value of payoff if one move is up and one is down (this is the same whichever order the price moves) and P_{dd} the value of payoff if both stock price moves are down. The price of the stock is to be modelled using a binomial tree approach with annual time steps.

i. Derive, and simplify an equation for P_{uu} in terms of P_{ud} and P_{dd}.
ii. Calculate using your answer to part (i) or otherwise the range of values that P_{uu} could take.
iii. Determine the value of the option in each of the two cases below, assuming that P_{uu} takes its maximum possible value:
   a) If the first stock price move is up.
   b) If the first stock price move is down.

3. Subject CT8 April 2015 Question 5

Let (X_t; $t \geq 0$) be a stochastic process satisfying $dX_t = \mu_t dt + \sigma_t \, dW_t$ where W_t is a standard Brownian motion.

Let $f(t,x)$ be a function, twice partially differentiable with respect to x, once with respect to t.

i. State the stochastic differential equation for $f(t, X_t)$.

Let $dX_t = \lambda X_t dt + \sigma dW_t$.

ii. Solve this differential equation, by considering $X_t = U_t e^{\lambda t}$ or otherwise.

4. Subject CT8 April 2016 Question 8

Consider a three-period binomial tree model for the stock price process S_t. Let $S_0 = 100$ and let the price rise by 10% or fall by 5% at each time step.

Assume also that the risk-free rate is 4% per time period, continuously compounded.

(i) (a) State the conditions under which the market is arbitrage free.
(b) Verify that there is no arbitrage in the given market.

(ii) Calculate the price of a European call option on this stock, with maturity at the end of the third period and a strike price of 103.

A special option, called a European "Paylater" call option, has the following payoff at maturity T:

$$(S_T - K - c) \text{ if } S_T > K$$

and zero otherwise.

K is the strike price and c is the premium paid for the option.

The premium is paid at maturity, and is only paid if the option expires in-the-money.

Further, the option premium is set such that the value of the option at time $t = 0$ is zero. Assume that $K = 103$ and the maturity of the contract is at time $t = 3$.

iii) Determine the premium c of this contract.

5. Subject CT8 April 2017 Question 5

Consider a three-period binomial tree model for a stock price process S_t , under which the stock price either rises by 18% or falls by 15% each month. No dividends are payable.

The continuously compounded risk-free rate is 0.25% per month.
Let $S_0 = \$ 85$
Consider a European put option on this stock, with maturity in three months (i.e. at time $t = 3$) and strike price $90.

(i) Calculate the price of this put option at time $t = 0$.

(ii) Calculate the risk-neutral probability that the put option expires out-of-the- money.

(iii) Assess whether the probability calculated in part (ii) would be higher or lower under the real-world probability measure. [No further calculation is required.]

6. Subject CT8 April 2008 Question 2

State the stochastic differential equation for geometric Brownian motion and its solution. (No proof is required.)

7. Subject CT8 April 2008 Question 3

Consider a two-period Binomial model of a stock whose current price $S0 = 100$. Suppose that:

- over each of the next two periods, the stock price can either move up by 10% or move down by 10%
- the continuously compounded risk-free rate is r = 8% per period

(i) Show that there is no arbitrage in the market.

(ii) Calculate the price of a one-year European call option with a strike price $K = 100$.

8. Subject CT8 April 2009 Question 10

Consider a three-period binomial model for a stock with the following parameters: $u = 1.2$, $d = 0.9$ and $S_0 = 60$. Assume that the discretely compounded risk-free rate of interest is r = 11% per period.

(i) (a) Verify that there is no arbitrage in the market.
(b) Construct the binomial tree.

(ii) Calculate the price of a standard European call option with maturity date in three periods and strike price $K = 60$.

A new "knock-in" option is introduced which has the following characteristics:

If the value of the stock crosses the level 80 during the whole life of the option, the contract holder has the right to obtain the difference between the value of the stock at maturity (in three periods) and 60.

(iii) Calculate the price of this new option.

9. Subject CT8 April 2010 Question 3

Consider a two-period binomial model for a non-dividend paying stock whose current price is $S_0 = 100$. Assume that:

- over each six-month period, the stock price can either move up by a factor $u = 1.2$ or down by a factor $d = 0.8$
- the continuously compounded risk-free rate is $r = 5\%$ per six-month period

(i) (a) Prove that there is no arbitrage in the market.

(b) Construct the binomial tree.

(ii) Calculate the price of a standard European call option written on the stock S with strike price K = 100 and maturity one year.

Consider a special type of call option with strike price K=100 and maturity one year. The underlying asset for this special option is the average price of the stock over one year, calculated as the average of the prices at times 0, 0.5 and 1 measured in years.

(iii) Calculate the initial price of this call option assuming it can be exercised only at time 1.

10. Subject CT8 April 2012 Question 3

A non-dividend paying stock has a current price of $S_0 = 150$p & trades in market which is arbitrage free and has a constant effective risk-free rate of interest r. After one year the price of the stock could increase to 280p, or decrease to 120p. Over the following year the price could increase from 280p either to 420p or to 322p. If the stock price had decreased to 120p, then over the following year it could increase to 168p or decrease to 112p.

i. Determine the range of values that the annual risk-free rate of interest could take.

Assume that r takes the value 20% p.a.

ii. Calculate the price at time 0 of a non-standard derivative which pays off $(S_2 - 100)^2$ at the end of two years.

11. Subject CT8 September 2008 Question 5

State the defining properties of a standard Brownian motion.

12. Subject CT8 September 2008 Question 7

Consider a one-period Binomial model of a stock whose current price is $S_0 = 40$ suppose that:

- over a single period, the stock price can either move up to 60 or down to 30
- the continuously compounded risk-free rate is $r = 5\%$ per period

(i) Show that there is no arbitrage in the market.

(ii) Calculate the price of a European call option with maturity date in one period & Strike Price K=45 using any of the method

a. by constructing a risk-neutral portfolio; and
b. by constructing a replicating portfolio

13. Subject CT8 September 2009 Question 5

A derivative security entitles the holder to a payment, at time *T*, of $max_{0 \le t \le T}$ St, where St is the price at time *t* of a security.

Assume that *S* satisfies $St = S0\exp(\sigma Bt + (r - 1/2\ \sigma^2)t)$ under the risk neutral measure, where *B* is a standard Brownian motion and *r* is the risk-free rate of interest.

(i) Derive the probability density of $max_{0 \le s \le t}$ $Bs + \mu s$. (Hint: use the formula in section 7.2 of the Formulae and Tables for Actuarial Examinations).

(ii) Determine an expression for pt, the fair price of the derivative security at time *t*. You need not evaluate the resulting integral.

14. Subject CT8 September 2010 Question 6

Under the real-world measure P, W is a standard Brownian motion and the price of a stock, S is given by $S_t = S_0 \exp(\sigma W_t + (\mu - ½\ \sigma^2)t)$. The continuously compounded risk-free rate of interest is r and a zero coupon bond with maturity T has price $B_t = e^{-r(T-t)}$. Suppose that in the market any contract which pays f(S_t) at time T is valued at:

$p_t = E[e^{-r(T-t)}\ f(S_t)\Delta_T | F_t]$

Where $\Delta_T = \exp(mW_t - \frac{1}{2} m^2 t)$ for $t \le T$ for some real number m.

i) a) Prove using Ito's formula that, Δ_T is a martingale.

b) Show that $E[\exp(mW_t)] = \exp(\frac{1}{2} m^2 t)$

ii) a) Derive an expression for p_0 when f(x) = x.

b) Show that there is an arbitrage in the market unless m = (r – μ)/σ.

15. Subject CT8 September 2010 Question 9

Consider a two-period binomial model for a non-dividend paying stock whose current price is $S_0 = 100$. Assume that:

- over each of the next six-month periods, the stock price can either move up by a factor $u = 1.2$ or down by a factor $d = 0.8$
- the continuously compounded risk-free rate is $r = 6\%$ per period

(i) (a) Prove that there is no arbitrage in the market.
(b) Construct the binomial tree for the model.

(ii) Calculate the price of a standard European call option written on the stock S with strike price $K = 100$ and maturity one year.

Consider a special European call option with strike price K=100 & maturity one year the owner of option has a right to exercise her option at the end of the year only if the price goes above the level L=130 during or at the end of the year

(iii) (a) Calculate the initial price of this call option.

(b) Comment on the relationship between the price of the special call option and the option in (ii).

16. Subject CT8 September 2013 Question 5

The share price in Santa Insurance Co, S_t, is currently 97p and can be modelled by the stochastic differential equation:

$$dS_t = 0.4S_t\,dt + 0.5S_t\,dB_t$$

where B_t is a standard Brownian motion

(i) (a) Determine $d\log S_t$, using Ito's Lemma.

(b) Calculate the expectation and variance of the Santa Insurance Co share price in two years' time.

UNIT 2

PRACTICE QUESTIONS

The share price in Rudolf financial services plc. R_t is also currently at 97p & can be modelled by the stochastic differential equation.

$$dR_t = -0.4R_t\, dt + 0.5dB_t$$

Let $U_t = e^{0.4t} R_t$

(ii) (a) Calculate dU_t.

(b) Calculate the expectation and variance of the Rudolf Financial Services plc share price in two years' time.

17. Subject CT8 September 2014 Question 3

Let $(Z_t : t \geq 0)$ be a standard Brownian motion.

(i) Calculate the probability of the event that $Z1 > 0$ and $Z2 < 0$.

Hint: Write $Z1 = W$, $Z2 = W + X$, where W and X are both independent, identically distributed $N(0,1)$ random variables.

(ii) State the model for geometric Brownian motion.

(iii) Explain why the standard Brownian motion is less suitable than the geometric Brownian motion as a model of stock prices.

18. Subject CT8 September 2014 Question 4

A non-dividend paying stock currently trades at $65. Every two years the stock price either increases by a multiplicative factor 1.3 or decreases by a multiplicative factor 0.8. The effective risk-free rate is 4% p.a.

Calculate the price of an American put option written on the stock with strike price $70 and maturity four years, using a two-period binomial model.

19. Subject CT8 September 2015 Question 5

An actuary plans to retire in five years' time, and hopes to celebrate retirement with a round-the-world cruise. The cruise will cost €20,000. The actuary chooses to save for the cruise by buying non-dividend paying shares with price S_t governed by the Stochastic Differential Equation:

$$dS_t = S_t(\mu dt + \sigma dZ_t)$$

where:

- Z_t is a standard Brownian motion.
- $\mu = 10\%$.
- $\sigma = 20\%$.
- t is the time from now measured in years; and
- $S_0 = 1$.

The instantaneous, constant, continuously compounded risk-free rate of interest is 4% p.a.

Derive the distribution of S_t.

20. Subject CT8 September 2017 Question 4

Consider a one-period binomial tree model for the stock price process S_t.Let S_0 = \$100 and assume that in three months' time the stock price is either \$125 or \$105. No dividends are payable on this stock.

Assume also that the continuously compounded risk-free rate is 5% per annum.

(i) Verify that this market is not arbitrage-free by considering the relationship between the risk-free rate and the stock price movements.

(ii) (a) Identify a portfolio which would generate an arbitrage profit.

(b) Calculate this profit.

Now assume that the continuously compounded risk-free rate is 20% per annum. Consider a European put option on this stock, expiring in three months' time and with strike price K = \$120.

(iii) Calculate the current price of the put option.

21. Subject CT8 April 2018 Question 4

Mr and Mrs Jones both wish to buy stocks in Widgets Inc. They don't have enough money right now, so they are considering buying either forwards or options on the stocks, both with a term of 4 years.

The stock price at time 0 is £10 with standard deviation of 12% per annum. The stock does not pay any dividend. The continuously compounded risk-free rate of interest is 5% per annum.

(i) Calculate the 4 year forward price on one stock.

(ii) Calculate the price at time 0 of a 4 year call option on one stock with a strike price of £12.21.

Mrs Jones enters into one forward contract, while Mr Jones buys one call option. At time 4 the stock is worth £12.

(iii) Calculate the accumulated profit or loss at time 4 for Mrs Jones.

(iv) Calculate the accumulated profit or loss at time 4 for Mr Jones.

(v) Explain why Mr Jones makes a loss despite having an option that does not force him to buy the stock.

(vi) Calculate the range of stock prices at time 4 which would leave Mr Jones better off than Mrs Jones.

22. Subject CM2 September 2019 Question 6

Let pt denote the value at time t of a European put option on a non-dividend paying share St with maturity at time T and a strike price K. The risk-free rate of interest is r.

(i) Derive the lower bound for pt in terms of St and K.

(ii) Explain how the lower bound would change if pt were an American put option.

The put option pt has the following characteristics:

- Strike price = £100
- Time to expiry 6 months. The risk-free rate of interest is 4% per annum.

(iii) Calculate an upper bound for the value of the option pt.

(iv) Explain the conditions necessary for the option price to approach the upper bound in part (iii).

23. Subject CT8 September 2018 Question 6

Consider a call option ct and a put option pt written on a non-dividend paying stock St .

(i) Prove the put-call parity relationship by constructing two portfolios that produce the same value at maturity.

A stock market includes four options set out below. All the options are for a term of 10 years and relate to a single non-dividend paying stock, currently priced at \$5. The continuously compounded risk-free rate is 3% per annum.

	Type	*Strike price*	*Option price*
Option A	European Call	\$8	\$0.32
Option B	European Put	\$8	?
Option C	European Put	\$10	?
Option D	American Put	\$10	?

(ii) Calculate the price of Option B.

(iii) Determine lower and upper bounds for the price of option C.

(iv) Determine lower and upper bounds for the price of option D.