Lecture 7

INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Class: FY MSc

Subject : Financial Mathematics

Subject Code: PPSAS102

Chapter: Unit 4 Chapter 7

Chapter Name: Loan Schedules

Today's Agenda

1. Introduction
    1. How loans work?

1. Calculation of Capital Outstanding
    1. Prospective loan calculation
    2. Retrospective loan calculation
    3. Calculating capital & interest elements
    4. The loan schedules
    5. Instalments payable more frequently than annually

2. Flat rates & APR


LOAN
APPLICATION
APPROVED
Personal Information
Name (Last)
PUBLIC
Address (Mailing Address)
12345 MAIN STREET
E-Mail Address
Services needed
UNDER REVIEW
(Middle Initial)
Home Telephone
Other Telephone
Zip
999999
UNDER REVIEW
SUBJECT
REVIEW
Yes No

1 Introduction

Loan - A very common transaction involving compound interest is a loan that is repaid by regular instalments, at a fixed rate of interest, for a predetermined term

The Instalments are divided in two parts:

- *Principal*
- *Interest*

How this works?

Each instalment payment is first used to pay any interest due since the last payment, and then to reduce the amount of capital outstanding.

1.1 How loans work?

Learn How Loans Work Before You Borrow

https://www.thebalance.com/how-loans-work-315449

Illustration

Bank A lends a loan of amount Rs. 2000 to an investor B. The loan is to be repaid in three years by equal annual payments at the end of each year. The bank will charge an effective rate of interest of 8% per annum.

Suppose that the equal annual payment is $ W.

The equation of value for the transaction gives:

2000 = W $a_{\overline{3}|}$@8%

Thus W = $776.067027

These three payments cover both the interest due and the loan amount of $2000.

Interest and Capital Elements

For the illustration we considered, let’s now look at the separation of the repayment amount into interest and capital element.

Year	Repayment Amount	Interest Element	Capital Element	Capital Outstanding
1	776.06	= 2000 x 0.08 = 160	= 776.06 – 160 = 616.06	= 2000 – 616.06 = 1383.94
2	776.06	= (2000 – 616.06) x 0.08 =110.7152	= 776.06 – 110.7152 = 665.3448	= 1383.94 – 665.34 = 718.59
3	776.06	= (2000 – 616.06 – 665.3448) x 0.08 = 57.487	= 776.06 – 57.487 = 718.573	= 718.59 – 718.573 = 0

This is how we generally represent a Loan Schedule.

Important Note

- One important point is that each repayment must pay first for interest due on the outstanding capital. The balance is then used to repay some of the capital outstanding.

- Each payment therefore comprises both interest and capital repayment. It may be necessary to identify the separate elements of the payments –for example if the tax treatment of interest and capital differs.

- Notice also that, where repayments are level, the interest component of the repayment instalments will decrease as capital is repaid, with the consequence that the capital payment will increase.

2 Calculation of Capital Outstanding

There are two ways to calculate the capital outstanding immediately after a repayment has been made:

(a) by **calculating the accumulated value** of the original loan less the accumulated value **of the repayments** made to date – called **the retrospective method.**

(b) by **calculating the present value of future repayments – called the prospective method**.

The Equation of Value for the loan at time 0 is :

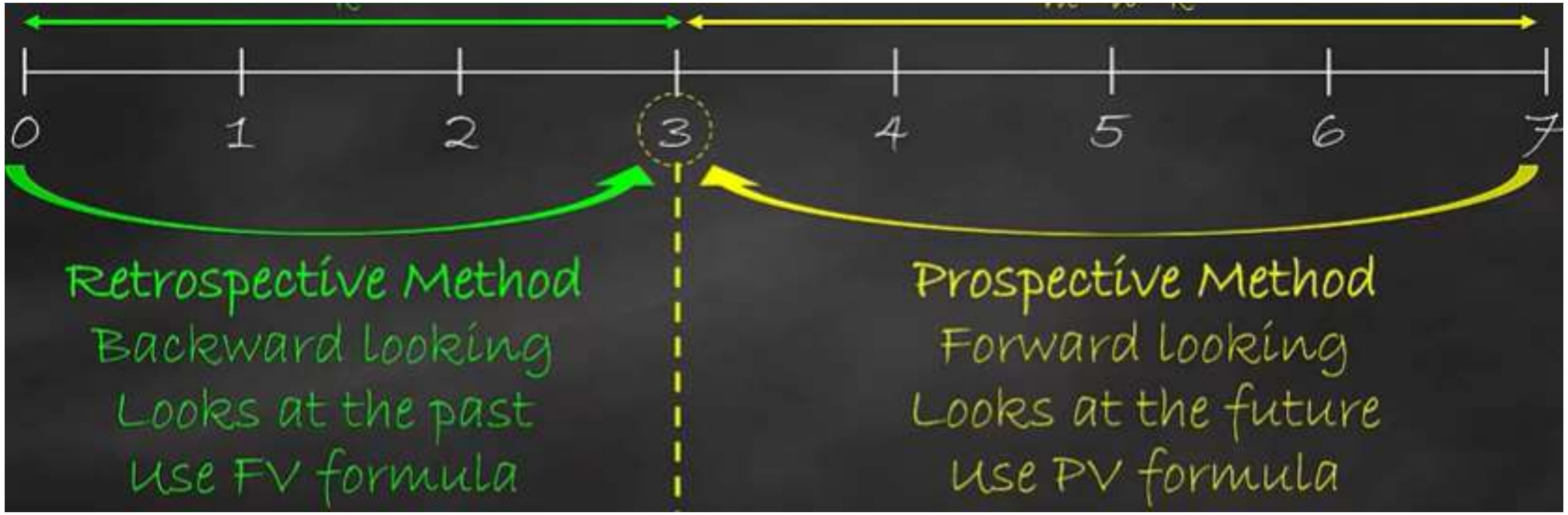
$$\boldsymbol{L_0 = X_1\, v + X_2\, v^2 + \ldots\ldots. + X_n\, v^n}$$

2

Calculation of Capital Outstanding

Loan Amortisation - Retrospective vs Prospective Method

https://www.youtube.com/watch?v=2tUT1PluTvU



2.1 Prospective Loan Calculation

The loan outstanding at time t is the present value (discounted value) at time t of the future payment installments.

Condition - The present value must be calculated at a repayment date.

The Equation of Value that calculates the loan outstanding at time t is ,

$$\boldsymbol{L_t = X_{t+1}\, v + X_{t+2}\, v^2 + \ldots\ldots. + X_n\, v^{n-t}}$$

Where

L_t - is the loan outstanding at time t

X - is the amount of installment (Can be level or not)

Question

A loan is being repaid with 10 payments of $2000 followed by10 payments of $1000 at the end of each half-year. If the nominal rate of interest convertible semiannually is 10%, find the outstanding loan balance immediately after five payments have been made.

Use the prospective method for calculation.

Solution

The rate of interest is 5% per half-year. Prospectively, the outstanding loan balance is

$$B_5^p = 1000\left(a_{\overline{15|}} + a_{\overline{5|}}\right) = 1000(10.37966 + 4.32948) = \$14,709$$

to the nearest dollar.

2.2 Retrospective Loan Calculation

The loan outstanding at time t is the accumulated value at time t of
the original loan less the accumulated value at time t of the repayments to date.

If we have level regular instalments, then the Equation of Value that calculates the loan outstanding at time t is ,

$$\begin{aligned} L_t &= L_0 (1+i)^t - X[(1+i)^{t-1} + (1+i)^{t-2} + \ldots.. + (1+i) + 1] \\ &= L_0 (1+i)^t - X s_{\overline{t|}} \end{aligned}$$

Question

A loan is being repaid with 10 payments of $2000 followed by10 payments of $1000 at the end of each half-year. If the nominal rate of interest convertible semiannually is 10%, find the outstanding loan balance immediately after five payments have been made.

Use the retrospective method for calculation.

2.3 Calculating Interest & Capital Element

Given the outstanding capital at any time, we can calculate the interest and capital element of any instalment.

<u>For a single payment</u>:

We first calculate the loan outstanding immediately after the previous installment at t-1 , $\boldsymbol{L_{t-1}}$.

Then interest element can be calculated by multiplying by the effective interest rate, i.e. $\boldsymbol{i.L_{t-1}}$.

The capital repaid can be found using $\boldsymbol{X_t}$ - $\boldsymbol{i.L_{t-1}}$ **or** $\boldsymbol{L_{t-1}}$ - $\boldsymbol{L_t}$.

2.3 Calculating Interest & Capital Element

Given the outstanding capital at any time, we can calculate the interest and capital element of any instalment.

For a series of payment:

- Calculate the capital repaid by subtracting the loan outstanding after the payments from the loan outstanding before the payments

- Then calculate the interest paid by subtracting the capital repaid from the total of the payments made.

2.4 The Loan Schedule

We set out the loan outstanding and the capital and interest part of each payment in a table. This type of table is called a loan schedule.

Year $r \to r+1$	Loan outstanding at r	Instalment at $r+1$	Interest due at $r+1$	Capital repaid at $r+1$	Loan outstanding at $r+1$
$0 \to 1$	L_0	X_1	iL_0	$X_1 - iL_0$	$L_1 = L_0 - (X_1 - iL_0)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$t \to t+1$	L_t	X_{t+1}	iL_t	$X_{t+1} - iL_t$	$L_{t+1} = L_t - (X_{t+1} - iL_t)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-1 \to n$	L_{n-1}	X_n	iL_{n-1}	$X_n - iL_{n-1}$	0

Question

A loan is to be repaid by a series of instalments payable annually in arrear for 15 years. The first instalment is £1,200 and payments increase thereafter by £250 per annum.

Repayments are calculated using a rate of interest of 6% per annum effective.

Determine:
(i) the amount of the loan. [3]

(ii) the capital outstanding immediately after the 9th instalment has been made. [2]

(iii) the capital and interest components of the final instalment. [2]

[Total 7]

Solution

(i) Loan $= 950\, a_{\overline{15|}} + 250(Ia)_{\overline{15|}}$ at 6%

$$= 950 \times 9.7122 + 250 \times 67.2668$$

$$= £26{,}043.29$$

Solution

(ii) Capital outstanding after 9 payments:

$$3200\, a_{\overline{6}|} + 250(Ia)_{\overline{6}|} = 3200 \times 4.9173 + 250 \times 16.3767 = £19,829.54$$

[2]

(iii) Capital outstanding after 14 payments $= 4700v$ at 6%

$= £4,433.96$

$=$ Capital in final payment

$\Rightarrow$ Interest in final payment $= 4700 - 4433.96$

$= £266.04$

[2]

(above uses factors from Formulae and Tables Book – exact answers are £26,043.34 for (i) and £19,829.61 for (ii))

2.5 Instalment payable more frequently than annually

Most loans will be repaid in quarterly, monthly or weekly instalments. Care needs to be taken in calculating the interest due at any instalment date.

For the case where the interest is expressed as an effective annual rate, with repayment instalments payable $p$thly, we have the equation of value for the loan, given repayments of X_t a time t = 1/p, 2/p, 3/p, ……….. , n is

$$L_0 = X_{1/p}\, v^{1/p} + X_{2/p}\, v^{2/p} + \ldots. + X_n\, v^n$$

2.5 Formula

If the loan is repaid by level instalments of amount *X* payable *p*thly, expressions for the loan outstanding at time *t* simplify to:

Prospectively -> L_t = pX $a_{\overline{n-t|}}^{(p)}$

Retrospectively -> L_t = L_0 $(1+i)^t$ - pX $s_{\overline{t|}}^{(p)}$

3 Flat Rates & APR

- Flat rate of interest is defined as the total amount of interest paid over the whole transaction, per unit of initial loan, per year of the loan.
- Flat rate ignores the details of gradual repayment of capital over the term of the loan.
- It is used to compare loans of equal term.
- Flat rate is lower than the true effective rate of interest charged on the loan.

It is calculated as:

$$\text{Flat rate} = \frac{\textit{total interest}}{\textit{original loan} \times \textit{term in years}} = \frac{\textit{total repayment} - \textit{original loan}}{\textit{original loan} \times \textit{term in years}}$$

3 Flat Rates & APR

APR, which is defined as the effective annual rate of interest, rounded to the nearer 1/10th of 1%.

For calculating APR, we could start by calculating the flat rate, and then double it to provide a first guess for the APR.

The APR is approximately:

$$\text{APR} \approx \frac{\textit{average annual interest}}{\textit{average loan amount}} = \frac{\textit{average annual interest}}{\frac{1}{2} \times \textit{original loan}} = 2 \times \frac{\textit{average annual interest}}{\textit{original loan}}$$

which is twice the flat rate.

Question

A company has borrowed £800,000 from a bank. The loan is to be repaid by level instalments, payable annually in arrear for 10 years from the date the loan is made.
The annual repayments are calculated at an effective rate of interest of 8% per annum.

(i) Calculate the amount of the level annual payment and the total amount of interest which will be paid over the 10 year term. [3]

(ii) At the beginning of the eighth year, immediately after the seventh payment has been made, the company asks for the term of the loan to be extended by two years. The bank agrees to do this on condition that the rate of interest is increased to an effective rate of 12% per annum for the remainder of the term and that payments are made quarterly in arrear.

(a) Calculate the amount of the new quarterly payment.
(b) Calculate the capital and interest components of the first quarterly instalment of the revised loan repayments. [6]

[Total 9]

Solution

(i) $800,000 = P\, a_{\overline{10}|}^{8\%} = P \times 6.7101$

$\Rightarrow P = 119,223.26$

Total amount of interest = 10 × 119,223.26 – 800,000

= £392,232.60

Solution

(ii) (a) Capital o/s at start of 8th year

$$= 119{,}223.26\ a_{\overline{3|}}^{8\%} = 119{,}223.26 * 2.5771 = 307{,}250.26$$

Let new payment be P' per annum, then

$$P' a_{\overline{5|}\,12\%}^{(4)} = P' * 1.043938 * 3.6048 = 307{,}250.26$$

$$\Rightarrow P' = 81{,}646.28$$

$$\Rightarrow \text{q'ly payment} = 20{,}411.57$$

Solution

(b) Capital o/s after 7 years = 307,250.26

$$\Rightarrow \text{Interest in 1st q'ly payment } = 30,7250.26 * \left((1.12)^{\frac{1}{4}} - 1 \right) = 8,829.56$$

$$\Rightarrow \text{capital component} = 20{,}411.57 - 8{,}829.56 = 11{,}582.01$$

Question

A loan of £3,000 is to be repaid by a level annuity-certain, payable annually in arrears for 25 years and calculated on the basis of an interest rate of 12% per annum. Find

(i) The annual repayment;

(ii) The capital repayment and interest paid at the end of (1) the 10th year and (2) the final year;

(iii) After which repayment the outstanding loan will first be less than £1,800; and

(iv) For which repayment the capital content will first exceed the interest content.

Solution

(i) Let the annual repayment be X. We have

$$Xa_{\overline{25}|} = 3{,}000 \text{ at } 12\%$$

So

$$X = \frac{300}{a_{\overline{25}|}} = £382.50$$

(ii) (1) The loan outstanding just after the payment at time 9 years is

$$382.50a_{\overline{16}|} = 2{,}667.56$$

and just after the payment at time 10 years it is

$$382.50a_{\overline{15}|} = 2{,}605.17$$

Hence, the capital repaid at the end of the tenth year is

$$2{,}667.56 - 2{,}605.17 = £62.39$$

and the interest paid at this time is

$$382.50 - 62.39 = £320.11$$

(2) The capital outstanding just after the payment at time 24 years is

$$382.50a_{\overline{1}|} = £341.52$$

The capital repaid at the end of the 25th year is therefore £341.52, and the interest paid at this time is

$$382.50 - 341.52 = £40.98$$

JAQS

Solution

(iii) The loan outstanding after the tth repayment is $382.50a_{\overline{25-t|}}$. This first falls below 1,800 when $a_{\overline{25-t|}}$ (at 12%) first falls below 4.7059. By the compound interest tables, the smallest value of t for which $a_{\overline{25-t|}} < 4.7059$ is 18, so the answer is the 18th repayment.

(iv) Using the loan schedule, the capital content exceeds the interest content of the tth installment when

$$1 - v^{26-t} < v^{26-t}$$

i.e., when

$$v^{26-t} > 0.5$$

This first occurs when $t = 20$, i.e., for the 20th payment.

Question

Previous question continued

Immediately after making the 15th repayment, the borrower requests that the term of the loan be extended by 6 years, the annual repayment being reduced appropriately. Assuming that the lender agrees to the request and carries out his calculations on the original interest basis, find the amount of the revised annual repayment.

Solution

The loan outstanding just after the 15th annual payment has been made is

$$382.50a_{\overline{10}|} \text{ at } 12\% = 2{,}161.20$$

Let Y be the revised annual payment. The equation of value is

$$Ya_{\overline{16}|} = 2{,}161.20 \text{ at } 12\%$$

which gives $Y = £309.89$.