Lecture

INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Class: FY MSc

Subject : Probability and Statistics

Subject Code: PPSAS101

Chapter: Unit 1 Chp 4

Chapter Name: Random Variables

Topics to be covered

1. Geometric Series
2. Random Variable
3. Types of Random Variables
4. Probability Functions
5. Properties of Probability Functions
6. Required Properties of Probability Distribution for Discrete Random Variables
7. Probability Density Function
8. Calculating Probability
9. Warning
10. Cumulative Probability Distribution
11. Cumulative Distribution Function
12. Properties of a Distribution Function

Continued ...

13. Properties of CDF
14. Expectation of a Random Variable
15. Expectation of a Function
16. Variance
17. Expectation of a linear function
18. Variance of a linear function
19. Skewness
20. Coefficient of Skewness
21. Median of a Random Variable
22. Percentile
23. Mode
24. Monte Carlo Simulation

Continued …

25. Inverse Transform Method for continuous distributions
26. Disadvantage of ITM
27. Inverse Transform Method for discrete distributions

1 Geometric Series

A geometric series $\sum_k a_k$ is a series for which the ratio of each two consecutive terms a_{k+1}/a_k is a constant function of the summation index k.

- A geometric series is a series of the form $a, ar, ar^2, ar^3 \ldots, ar^n$. The sum of series for $r \neq 1$ is given by:

$$\boldsymbol{a + ar + ar^2 + \cdots + ar^n = a\left(\frac{1 - r^{n+1}}{1 - r}\right)}$$

- The number r is called the ratio or common ratio. If $|r| < 1$, we can sum the infinite geometric series:

$$\boldsymbol{a + ar + ar^2 + \cdots + ar^n + \cdots = a\left(\frac{1}{1 - r}\right)}$$

2 Random Variable

A random variable is a numerical quantity whose value depends on chance.

- We typically write a random variable in a capital letter such as **X**.

Example:

- If we flip a coin and observe which side is up, the sample space is {H, T}. If we assign H=1 and T=0, our random variable is:

$$X = \begin{cases} 1 & \text{with probability of } 0.5 \\ 0 & \text{with probability of } 0.5 \end{cases}$$

- Random Variables are denoted by upper case letters **(X).**
- Individual outcomes for a RV are denoted by lower case letters **(x).**

3 Types of Random Variables

Random Variables

Discrete

A random variable is a **discrete random variable** if it can take on no more than a countable number of values.

Continuous

A random variable is a **continuous random variable** if it can take any value in an interval.

4 Probability Function

- **Probability Distribution:** Table, Graph, or Formula that describes values a random variable can take on, and its corresponding probability (discrete RV) or density (continuous RV).

- **Discrete Probability Distribution:** Assigns probabilities (masses) to the individual outcomes, denoted by:

$$P(x) = P(X=x)$$

- **Continuous Probability Distribution:** Assigns density at individual points, probability of ranges can be obtained by integrating density function, denoted by: $f(x)$

- Cumulative Distribution Function: $F(X) = P(X \leq x)$

5 Properties of probability functions

- If X is a discrete random variable that takes on the values $a_1, a_2, \dots$, then:

$$P(X = a) \geq 0 \; for \; all \; x$$
$$\sum_{all\ x} P(X = x) = 1$$

- The probability function always takes on a value greater than or equal to zero, and always sums to one.

6

Required Properties of Probability Distribution for Discrete Random Variables

- Let X be a discrete random variable with **probability distribution, P(x).** Then,

i. $0 \le P(x) \le 1$ for any value x, and
ii. the individual probabilities sum to 1, that is,

$$\sum_{x} P(x) = 1$$

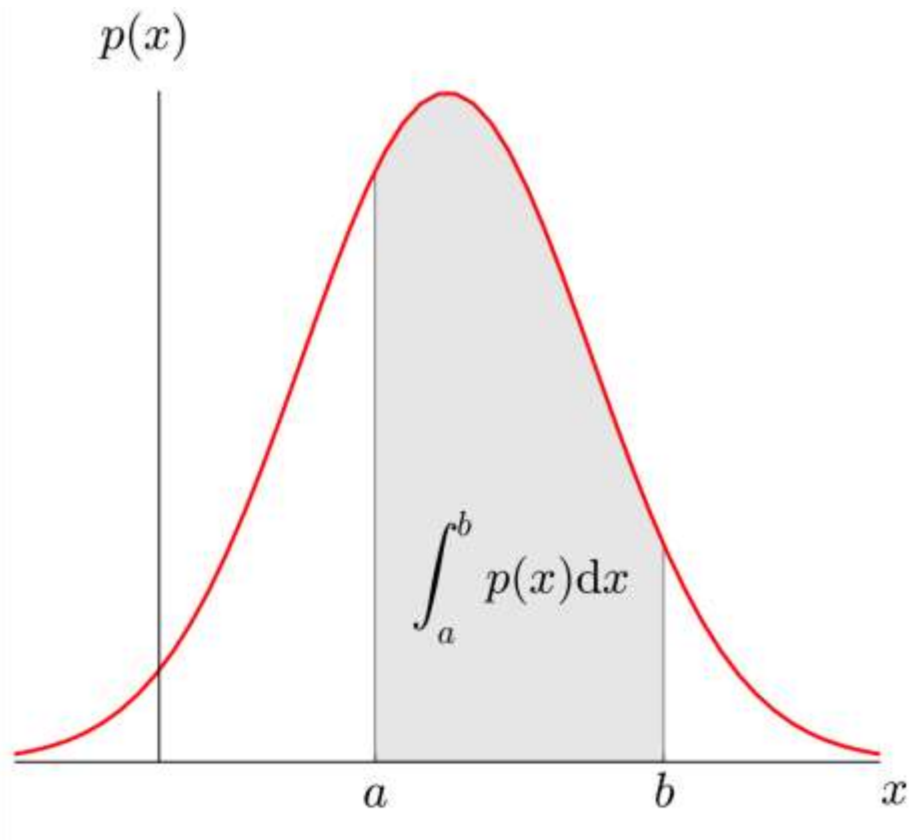
where the notation indicates summation over all possible values of x

7 Probability Density Function

- A probability density function, is a function that assigns probabilities to a continuous random variable.

$$f_X(x) \geq 0$$

$$\int_{\text{lower bound}}^{\text{upper bound}} f(x)dx = 1 \text{ (The area under the graph is 1)}$$



8 Calculating Probability

- Probability in terms of probability density function (p.d.f):

$$\boldsymbol{P(x_1 < X < x_2) = \int_{x_1}^{x_2} f(x)dx}$$

$$\boldsymbol{P(a < X < b) = P(a \le X < b) = P(a \le X \le b) = P(a < X \le b) = \int_{a}^{b} f(x)dx}$$

8 Calculating Probability

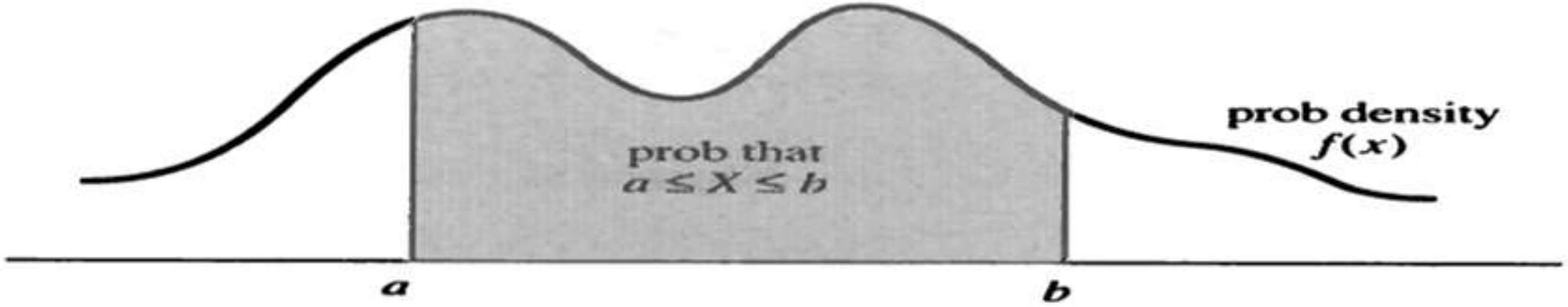
- Suppose X is a random variable and we have a function f(x) which is integrated as follows to get probabilities of events:

$$P(a \le X \le b) = \int_a^b f(x)dx$$

$$P(X \le b) = \int_{-\infty}^b f(x)dx$$

$$P(a \le X) = \int_a^{\infty} f(x)dx$$

- And so on. Then *f(x)* is called the probability *density function* of X, and the random variable X is called *continuous*.
- The set of points where $f(x) \neq 0$ is called the universe. You can think of probabilities as areas under the $f(x)$ graph:



9

Warning

- The values of the density function are not probabilities. The units on f(x) are probability per unit length. It's f(x) times dx that is a probability, namely,

$$f(x)dx = P(X \approx x)$$

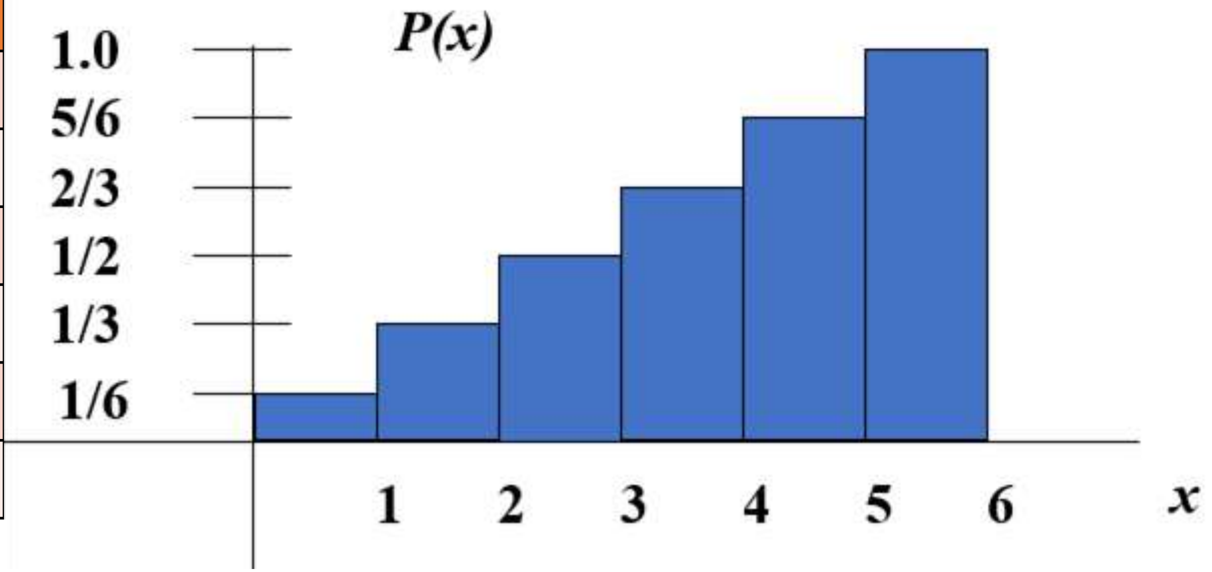
10 Cumulative Probability Distribution

- The cumulative probability distribution, $F(x_0)$, of a random variable X, represents the probability that X does not exceed the value x_0, as a function of x_0. That is,

$$\mathbf{F(x_0) = P(X \leq x_0)}$$

where the function is evaluated at all values of x_0

x	P(X=x)	F(x)
1	1/6	1/6
2	1/6	2/6
3	1/6	3/6
4	1/6	4/6
5	1/6	5/6
6	1/6	6/6



11 Cumulative Distribution Function

Discrete Random Variable

- The cumulative distribution function for a discrete random variable is defined to be:

$$F_X(x) = P(X \leq x) = \sum_{all\ x} P(X = x)$$

Continuous Random Variable

- The cumulative distribution function for a continuous random variable is defined to be:

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(t)dt$$

11 Cumulative Distribution Function

- If a random variable is discrete, we say PMF (probability mass function); if a random variable is continuous, we say PDF (probability density function).

- Whether a random variable is discrete or continuous, we always say CDF (cumulative probability function).

12

Properties of a Distribution Function

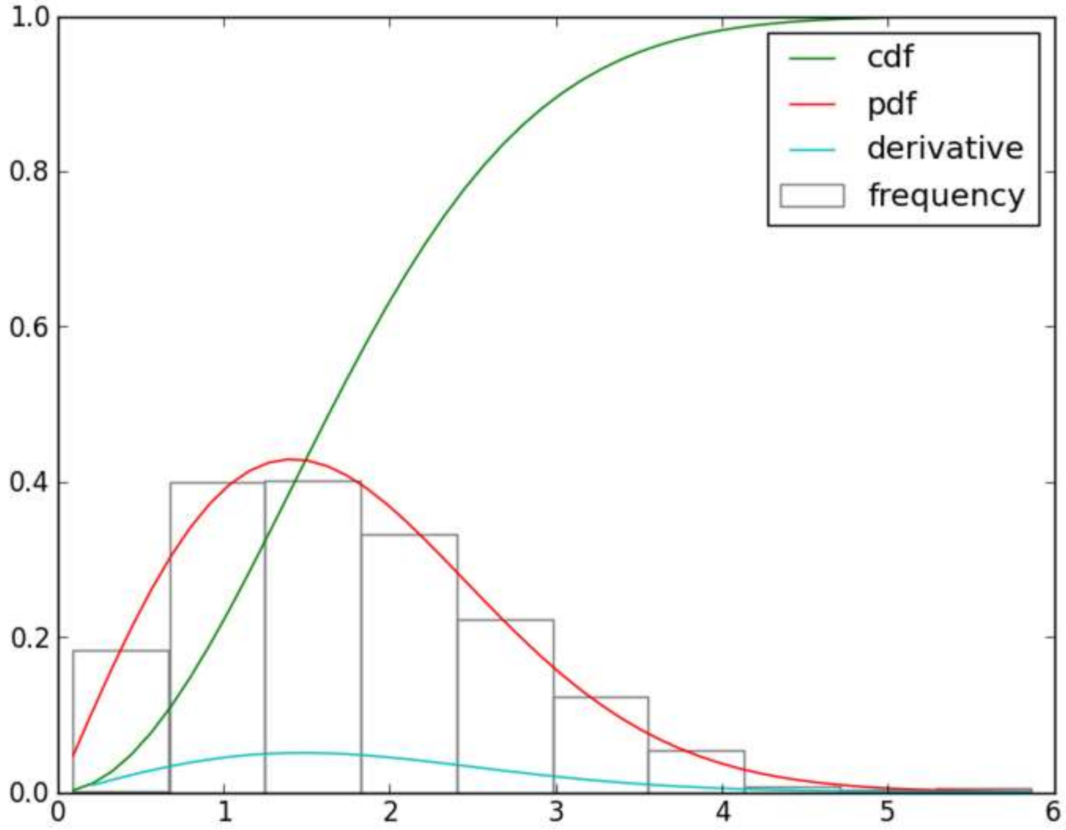
- Let X be a random variable with distribution function F(x). $F(x) = P(X \leq x)$

- *F(x)* collects cumulative probability so its graph starts at height 0 and rises to a height of 1. In other words:

$$\boldsymbol{F\ is\ non-decreasing}.$$

(F can increase or stay level but not go down)

$$\boldsymbol{F(-\infty) = 0, F(\infty) = 1}$$



13 Properties of CDF

- If X is discrete and takes integer values, the PMF and CDF can be obtained from each other by summing or differencing:

$$F(k) = \sum_{i=-\infty}^{k} p_x(i) \text{ ... this is the definition of F(k).}$$
$$\boldsymbol{p_x(k) = P(X \leq k) - P(X \leq k-1) = F(k) - F(k-1)}$$

- If X is continuous , the PDF and CDF can be obtained from each other by integration or differentiation:

$$\boldsymbol{F(x) = \int_{-\infty}^{x} f(t)dt, \qquad f(x) = \frac{d}{dx}F(x)}$$

- By definition, $F(x) = P(X \leq x) = P(-\infty \leq X \leq x) = \int_{-\infty}^{x} f(t)dt$. Taking the derivative at both sides of $F(x) = \int_{-\infty}^{x} f(t)dt$ gives is $f(x) = \frac{d}{dx}F(x)$

14 Expectation of a Random Variable

Discrete Random Variable

- The mean (or expectation) for a discrete random variable is defined to be:

$$\boldsymbol{E}(\boldsymbol{X}) = \sum_{\boldsymbol{all\ x}} \boldsymbol{x} * \boldsymbol{P}(\boldsymbol{X} = \boldsymbol{x})$$

- The expectation or expected value or mean of a random variable X is a weighted average of the values of X, where each value x is weighted by the probability of its occurrence.

Continuous Random Variable

- The mean (or expectation) for a continuous random variable is defined to be:

$$E(x) = \int_{-\infty}^{\infty} x f_X(x) dx$$

15 Expectation of a Function

Discrete Random Variable

- The mean (or expectation) of g(X) for a discrete random variable is defined to be:

$$E(X) = \sum_{all\ x} g(x) * P(X = x)$$

Continuous Random Variable

- The mean (or expectation) of g(X) for a continuous random variable is defined to be:

$$E(x) = \int_{-\infty}^{\infty} g(x) f_X(x) dx$$

16 Variance

- Let X be a random variable with finite mean **μ = E(X).** The variance of X, denoted by Var(X), is defined as follows:

$$\boldsymbol{Var(X) = E[(X - \mu)^2] = E[X^2] - [E(X)]^2}$$

- If X has infinite mean or if the mean of X does not exist, we say that Var(X) does not exist. The standard deviation of X is the non-negative square root of Var(X) if the variance exists.

- The square root of variance, typically denoted by σ, is called **standard deviation**.

Variance

Discrete Random Variable

- The variance for a discrete random variable is defined to be:

$$var(X) = \sum_{all\ x} x^2 P(X = x) - \left(x \sum_{all\ x} P(X = x) \right)^2$$

Continuous Random Variable

- The variance for a continuous random variable is defined to be:

$$var(x) = \int_{-\infty}^{\infty} x^2 f_X(x) dx - \left(\int_{-\infty}^{\infty} x f_X(x) dx \right)^2$$

17 Expectation of a linear function

$$E(aX + b) = aE(X) + b$$

18 Variance of a linear function

$$var(aX + b) = a^2 var[X]$$

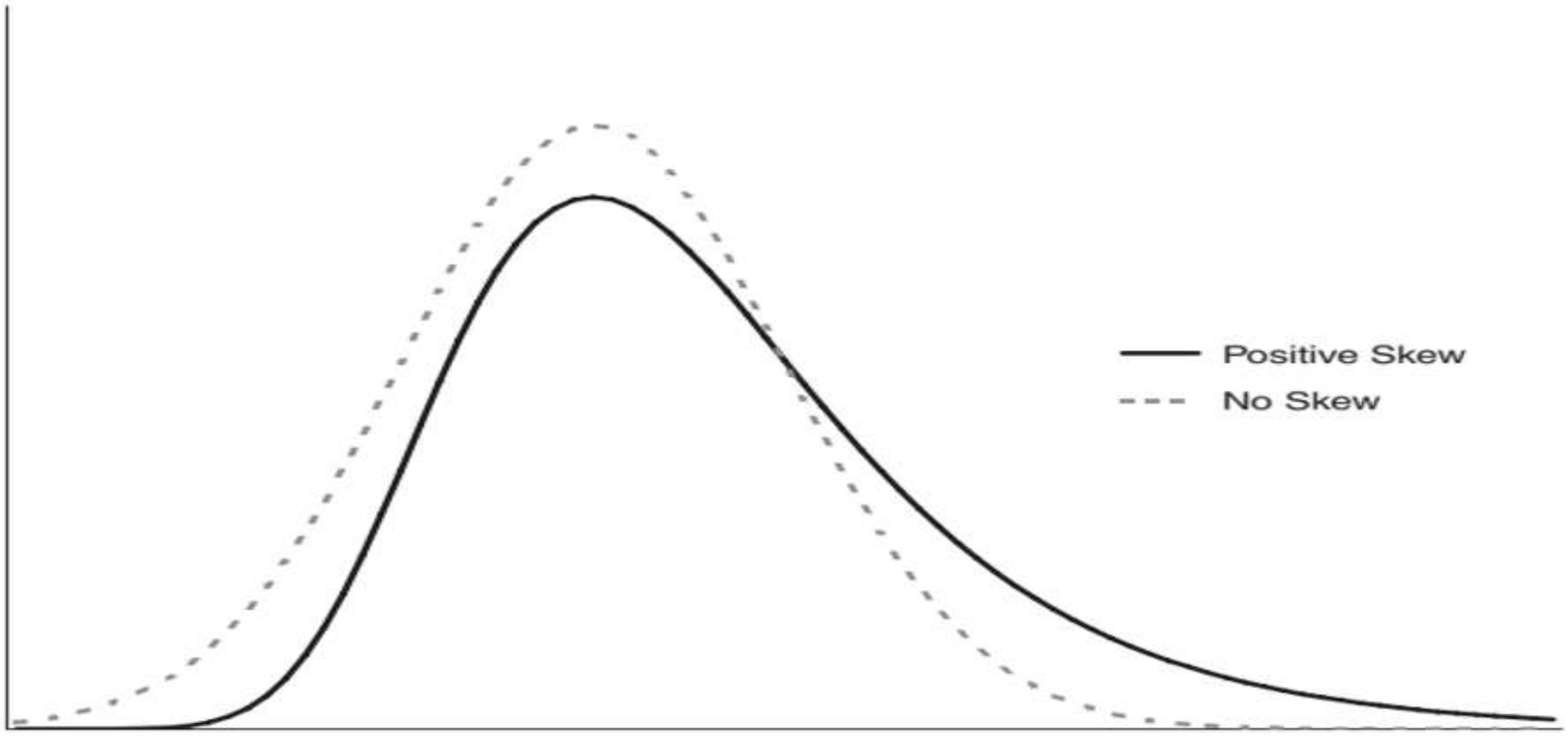
19 Skewness

- The third central moment is the skewness. The formula for skewness is:

$$E[(X-\mu)^3] = E[X^3] - 3\mu E[X^2] + 2\mu^3$$

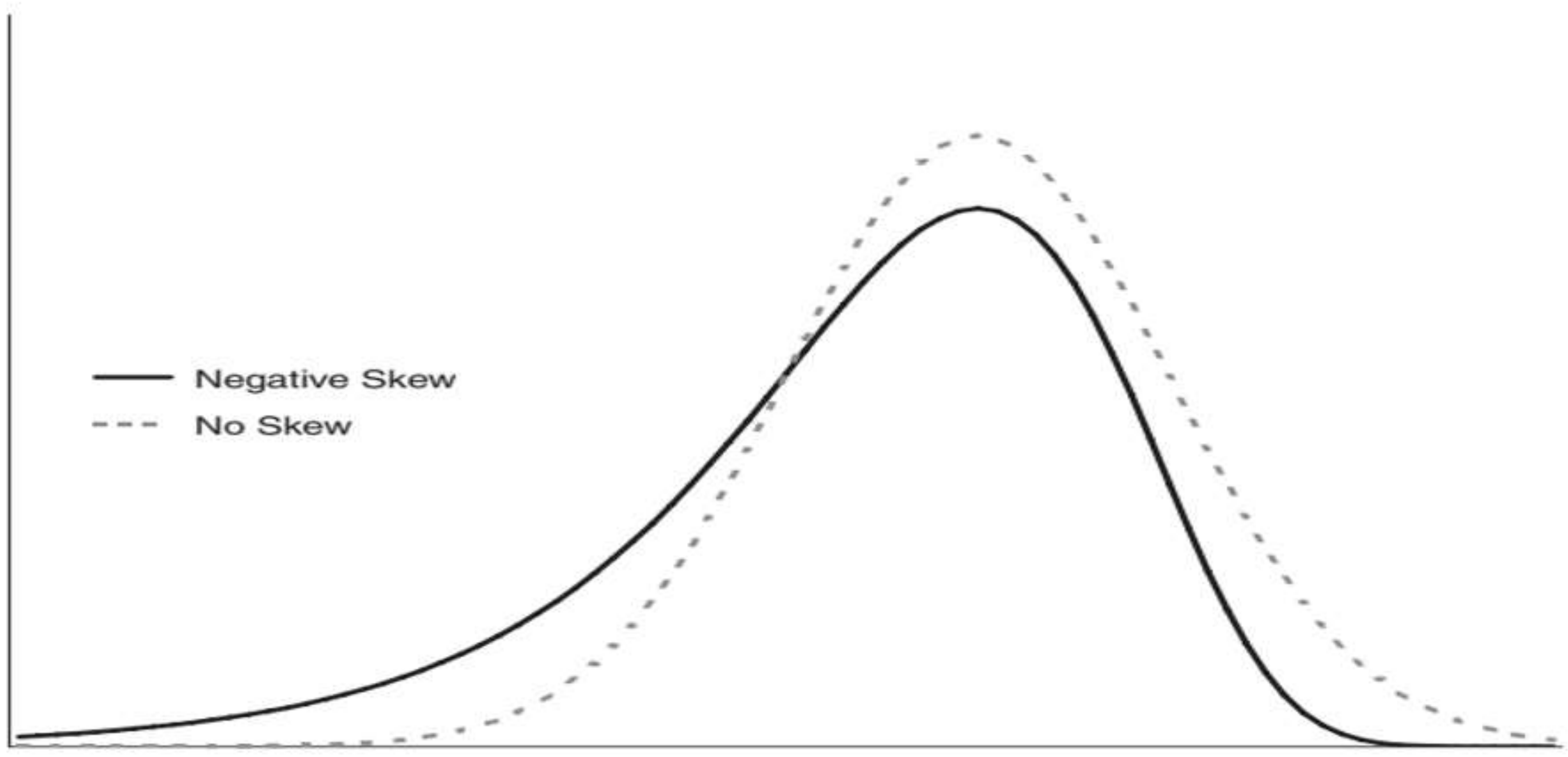
Where $\mu = E[X]$

19 Positive Skew



19

Negative Skew



Coefficient of Skewness

- The formula for coefficient of skewness is:

$$\frac{\boldsymbol{E}\left[(\boldsymbol{X}-\boldsymbol{\mu})^{3}\right]}{(\boldsymbol{var}[\boldsymbol{X}])^{1.5}}$$

- The coefficient of skewness is dimensionless, so it makes it easier to compare different distributions.

21

Median of a Random Variable

- The median of a random variable X is the value m such that:

$$P(X < m) \leq 0.5 \leq P(X \leq m)$$

- In particular, if X is a continuous random variable, the median m is defined by the equation:

$$F_X(m) = 0.5$$

22 Percentile

- **Percentile.** If your CAT score is in the 90^{th} percentile, then 90% of the people who took the same test scored below you or got the same score as you did; only 10% of the people scored better than you.

- For a random variable X , the p^{th} percentile, means that:

$$\boldsymbol{P(X \leq p^{th}\, percentile) = p\% \Leftrightarrow F(p^{th}\, percentile) = p\% \Leftrightarrow P(X > p^{th}\, percentile) = (1-p)\%}$$

- Median = Middle = 50^{th} percentile

23

Mode

- **Mode** = Most Often = Most Observed
- For a **discrete rv**: Mode is the value where the function **P(X=x)** has max. value
- For a **continuous rv**: Mode is the value where pdf, i.e. **f(x),** is maximum (Use maxima-minima techniques to find out the maximum value of f(X)).

Monte Carlo Simulation

- With the advent of high-speed personal computers Monte Carlo simulations have become one of the most valuable tools of the actuarial profession. This is because the vast majority of the practically important problems are not amenable to analytical solution.

- We outline one basic simulation technique that can be used to simulate values from most of the standard distributions. This is known as the **inverse transform method**. It can be applied to both continuous and discrete distributions.

Inverse Transform Method for continuous distributions

- The method works by first generating a random number from a uniform distribution on the interval (0,1) . We then use the cumulative distribution function of the distribution we are trying to simulate to obtain a random value from that distribution.

- First we generate a random number, U , from the U(0,1) distribution. We can use this to simulate a random variate X with PDF f (x) by using the CDF, F(x).
- Let U be the probability that X takes on a value less than or equal to x ,ie:

$U = P(X \leq x) = F(x)$. Then x can be derived as:

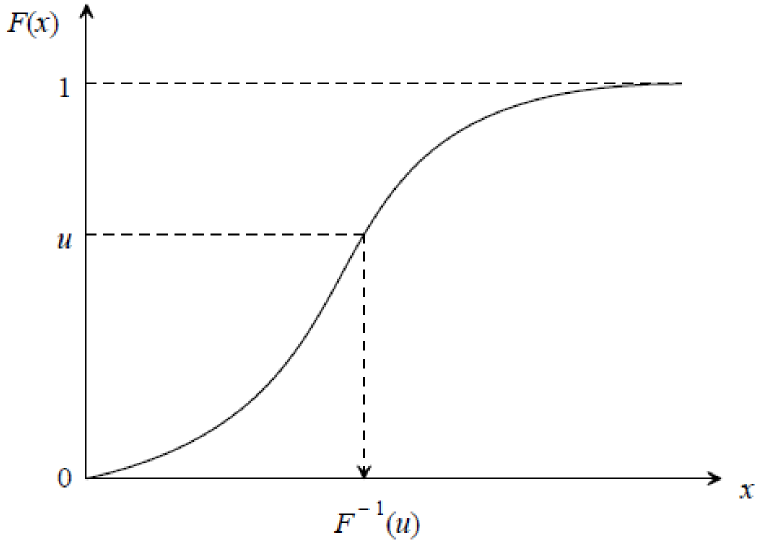
$$x = F^{-1}(u)$$

- Hence, the following two-step algorithm is used to generate a random variate x from a continuous distribution with CDF F(x) :

i. generate a random number u from U(0, 1),
ii. return $x = F^{-1}(u)$.

Inverse Transform Method for continuous distributions

- We can represent this on a diagram as follows. We have a random value, u , between 0 and 1. Recall that the cumulative distribution, F(x) , increases from 0 to 1 as x increases:



- If we set u =F(x) we can obtain a random value, x , by inverting the cumulative distribution, $x = F^{-1}(u)$. Hence this method is called the inverse transform method.
- This method requires that our distribution has a cumulative distribution function, F(x) , in the first place. This rules out the gamma, normal, lognormal and beta distributions.
- Formally, we can prove that the random variable $X = F^{-1}(U)$ has the CDF F(x) , as follows:

$$\boldsymbol{P(X \le x) = P\left[F^{-1}(U) \le x\right] = P[U \le F(x)] = F(x)}$$

Disadvantage of ITM

- The main disadvantage of the inverse transform method is the necessity to have an explicit expression for the inverse of the distribution function F(x) . For instance, to generate a random variate from the standard normal distribution using the inverse transform method we need the inverse of the distribution function

$$\boldsymbol{F(X) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt}$$

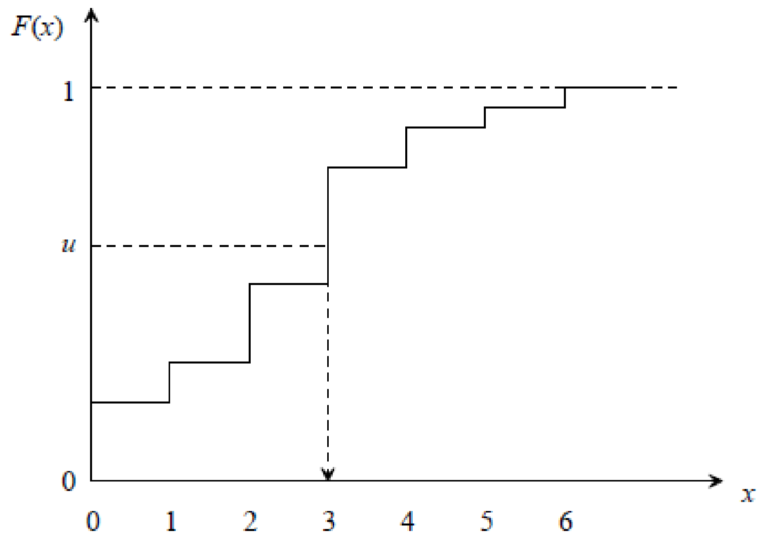
- However, no explicit solution to the equation u □ F(x) can be found in this case.

Inverse Transform Method for discrete distributions

- We cannot invert algebraically the distribution function of a discrete random variable, as it is a step function. The distribution function, F(x) , is the sum of the probabilities so far, eg:

$$\boldsymbol{F(5) = P(X \le 5) = P(X = 0) + P(X = 1) + \cdots + P(X = 5)}$$

- Given a random value, u , from U(0,1) we can read off the x value from the distribution function graph as follows:



- From the graph, we can see that in this particular case our value of u lies between F(2) and F(3) .
- This gives x = 3 as our simulated value.

27 Inverse Transform Method for discrete distributions

- So in general, if our value u lies between $F(x_{j-1})$ and $F(x_j)$ then our simulated value is x_j . If the value of u corresponds exactly to the position of a step, then by convention we use the lower of the x values, ie the point corresponding to the left hand end of the step.

- Let X be a discrete random variable which can take only the values $x_1, x_2, \ldots, x_N$ where $x_1 < x_2 < \cdots < x_N$.
- The first step is to generate a random number, U , from the U(0,1) distribution. We can use this to simulate a random variate X with PDF f (x) by using the CDF, F(x).
- Let U be the probability that X takes on a value less than or equal to x. Then $X = x_j$ if:

$$\boldsymbol{F(x_{j-1}) < U \leq F(x_j)}$$

ie $\boldsymbol{P(X = x_1) + P(X = x_2) + \cdots + P(X = x_{j-1}) < U \leq P(X = x_1) + P(X = x_2) + \cdots + P(X = x_j)}$

Inverse Transform Method for discrete distributions

- Note that for $x < x_1$, we have F(x)=0.
- Hence, the following three-step algorithm is used to generate a random variate x from a discrete distribution with CDF F(x) :

i. generate a random number u from U(0,1) .
ii. find the positive integer i such that $F(x_{i-1}) < u \le F(x_i)$.
iii. return $x = x_i$.

- We can see that the algorithm can return only variates x from the range $\{x_1, x_2, .., x_N\}$ and that the probability that a particular value x = x_i is returned is given by:

$$\boldsymbol{P(value\ returned\ is\ x_i) = P[F(x_{i-1}) < U \le F(x_i)] = F(x_i) - F(x_{i-1}) = P(X = x_i)}$$

- We can use a similar approach for the binomial distribution.

Thank You