INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Subject: Financial Engineering 1

Chapter:

Category: Assignment Solutions

1.

i) Arbitrage opportunity is a situation where we can make a certain profit with no risk. This is sometimes described as a free lunch.

An arbitrage opportunity means that:

(a) we can start at time 0 with a portfolio that has a net value of zero (implying that we are long in some assets and short in others). This is usually called a zero-cost portfolio.

(b) at some future time T:

• the probability of a loss is 0

• the probability that we make a strictly positive profit is greater than 0.

If such an opportunity existed then we could multiply up this portfolio as much as we wanted to make as large a profit as we desired.

ii) Law of one price

The Law of one price states that any two portfolios that behave in exactly the same way must have the same price. If this were not true, we could buy the 'cheap' one and sell the 'expensive' one to make an arbitrage (risk- free) profit.

iii)

a) Using Put- Call parity, the value of put option should be:

$$p_t = c_t + K\exp(-r(T-t)) - S_t\exp(-q(T-t))$$

$$= 30 + 120\exp(-.05^*.25) - 125\exp(-.15^*.25)$$
$$= 28.11$$

b) Arbitrage profit

If the put options are only Rs. 23 then they are cheap. If things are cheap then we buy them.

So looking at the put-call parity relationship, we “buy the cheap side and sell the expensive side”,
ie we buy put options and shares and sell call options and cash.
For example:

- sell 1 call option Rs. 30

- buy 1 put option (Rs. 23)

- buy 1 share (Rs.125)

- sell (borrow) cash Rs.118

This is a zero-cost portfolio and, because put-call parity does not hold, we know it will make an arbitrage profit. We can check as follows:

In 3 months' time, repaying the cash will cost us:
118exp(0.05*3 /12)= Rs. 119.48

We also receive dividends d on the share.

1. If the share price is above 120 in 3 months' time then the other party will exercise their call option and we will have to give them the share. They will pay 120 for it and our profit is:

$120 - 119.48 + d = 0.52 + d$

(The put option is useless to us)

2. If the share price is below 120 in 3 months' time then we will exercise our put option and sell it for 120. Our profit is:

$120 - 119.48 + d = 0.52 + d$

(The call option is useless to the other party and will expire worthless)

2.

Let $dX_t = A_t\, dt + B_t\, dZ_t$,

Where, $A_t = \alpha\, \mu\, (T-t)$, $B_t = \sigma \sqrt{(T-t)}$ Eq 1

$dF = \frac{\partial f}{\partial x} B_t\, dZ_t + \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} A_t + \frac{1}{2} \frac{\partial 2 f}{\partial x 2} B_t \right) dt$ *(Ito's lemma)*

$= - f B_t\, dZ_t + f \frac{\partial m}{\partial t} (T-t) dt - f A_t\, dt + \frac{1}{2} f B_t^2\, dt$

[Since $\frac{\partial f}{\partial x} = - e^{(m(T-t)-x)}$ and $\frac{\partial f}{\partial t} = \frac{\partial m}{\partial t} (T - t) * e^{(m(T-t)-x)}$ (using chain rule) and $\frac{\partial 2 f}{\partial x 2} = e^{(m(T-t)-x)}$]

$dF = f \left(\frac{\partial m}{\partial t} (T-t) - A_t + \frac{1}{2} B_t^2 \right) dt - f B_t\, dZ_t$

For f to be a martingale, $\frac{\partial m}{\partial t} (T-t) - A_t + \frac{1}{2} B_t^2 = 0$

Thus, $\frac{\partial m}{\partial t} (T-t) = A_t - \frac{1}{2} B_t^2$

Substituting Eq 1 above gives $\frac{\partial m}{\partial t} = \alpha\, \mu - \frac{1}{2} \sigma^2$

3.

i) Given

$F(t,x) = e^{-t}x^2$

$\frac{df}{dt} = - e^{-t}x^2 \quad = -f$

$\frac{df}{dx} = 2e^{-t}x$

$\frac{df}{dx^2} = 2e^{-t}$

$Y_t = f(t, X_t)$

Applying Ito's lemma

$dY_t = \frac{df}{dt}\, dt + \frac{df}{dx}\, dX_t + ½ \frac{d^2 f}{dx^2}\, \sigma^2 X_t^2\, dt$

$= -fdt + 2e^{-t} X_t dX_t + e^{-t}\sigma^2 X_t^2\, dt$

$= -Y_t dt + 2e^{-t} X_t^2 \frac{dXt}{Xt} + e^{-t}\sigma^2 X_t^2\, dt$

$= -Y_t dt + 2Y_t\, [0.25\, dt + \sigma\, dW_t] + \sigma^2 Y_t\, dt$

$\frac{dYt}{Yt} = [\, 2 * (0.25) - 1 + \sigma^2\,]dt + 2\sigma\, dWt$

$= [\sigma^2 - 0.5]\, dt + 2\sigma\, dWt$

Therefore

$dYt = [\sigma^2 - 0.5]\, Yt\, dt + 2\sigma\, Yt\, dWt$

ii) The process is martingale if drift is zero. This means $\sigma^2 - 0.5 = 0$

i.e. $\sigma^2 = 0.5$

4.

Let n ex-dividend dates are anticipated for a stock and $t_1<t_2<....<t_n$ are the times before which the stock goes ex-dividend. Dividends are denoted by $d_{1....}d_n$.

If the option is exercised prior to the ex-dividend date then the investor receives $S(t_n) - K$.

If the option is not exercised, the price drops to $S(t_n) - d_n$.

The value of the american option is greater than $S(t_n) - d_n$- Kexp(-r(T-t_n))

It is never optional to exercise the option if $S(t_n) - d_n$- Kexp(-r(T-t_n) >= $S(t_n) - K$ i.e. d_n<=K*(1-exp(-r(T-t_n))

Using this equation: we have K*(1-exp(-r(T-t_n)) = 350*(1-exp(-0.95*(0.8333-0.25))=18.87 and 65*(1-exp(-0.95*(0.8333-0.25)) =10.91.Hence it is never optimal to exercise the american option on the two ex-dividend rates.

5.

The required probability is the probability of the stock price being greater than Rs. 258 in 6 months' time.

The stock price follows Geometric Brownian motion
i.e. St = S0 exp(µ – σˆ2 /2)t + σWt

Therefore Ln (St) follows normal distribution with mean Ln (S0) + (µ – σˆ2 /2)t and variance (σˆ2)t

Implies Ln (St) follows φ(Ln 254 + (0.16-0.35ˆ2/2)*0.5, 0.35*0.5ˆ(1/2))
= φ (5.59, 0.247)

This means [Ln (St) – St)]/ σtˆ(1/2) follows standard normal distribution.

Hence the probability that stock price will be higher than the strike price of Rs. 258 in 6 months time =
1- N(5.55-5.59)/0.247 = 1- N(-0.1364) =0.5542.

The put option is exercised if the stock price is less than Rs. 258 in 6 months time.
The probability of this = 1- 0.5542= 0.4457

6.

i) The given relationship can be written as:

$S_t = S_0 e^{\mu t + \sigma Bt}$

Since St is a function of standard Brownian motion, Bt, applying Ito's Lemma, the SDE for the underlying stochastic process becomes:

dBt = 0 X dt + 1 X dBt

Let $G(t, B_t) = S_t = S_0 e^{\mu t + \sigma Bt}$, then

$dG/dt = \mu S_0 e^{\mu t + \sigma Bt} = \mu S_t$

$dG/dB_t = \sigma S_0 e^{\mu t + \sigma Bt} = \sigma S_t$

$d^2G/dB^2{}_t = \sigma^2 S_0 e^{\mu t + \sigma Bt} = \sigma^2 S_t$

Hence, using Ito's Lemma from Page 46 in the Tables we have:

$dG = [0 \text{ X } \sigma S_t + ½ \text{ X } 1^2 \text{ X } \sigma^2 S_t + \mu S_t] dt + 1 \text{ X } \sigma S_t dB_t$

i.e. $dS_t = (\mu + ½ \sigma^2) S_t dt + \sigma S_t dB_t$

Thus,

$dS_t/S_t = \sigma dB_t + (\mu + ½ \sigma^2) dt$

So, $c_1 = \sigma$ and $c_2 = \mu + ½ \sigma^2$

ii)

The expected value of S_t is:

$E[S_t] = E\,[S_0\, e^{\mu t + \sigma Bt}] = S_0\, e^{\mu t}\, E[e^{\sigma Bt}]$

Since $B_t \sim N\,(0,1)$, its MGF is $E[e^{\theta Bt}] = e^{½\,\theta 2t}$

So, $E[S_t] = S_0\, e^{\mu t}\, X\, e^{½\,\sigma 2t} = S_0\, e^{\mu t + ½\,\sigma 2t}$

The variance of S_t is:

$Var[S_t] = E[S^2{}_t] – (E[S_t])^2$

$= E[S^2{}_0\, e^{2\mu t +}{}_{2\sigma Bt}] – (S_0\, e^{\mu t + ½\,\sigma 2t})^2$

$= S^2{}_0\, e^{2\mu t}\, E[e^{2\sigma Bt}] – S^2{}_0\, e^{2\mu t + \sigma 2t}$

$= S^2{}_0\, e^{2\mu t + 2\sigma 2t} - S^2{}_0\, e^{2\mu t + \sigma 2t}$

$= S^2{}_0\, e^{2\mu t}\, (e^{2\sigma 2t} - e^{\sigma 2t})$

iii)

$Cov[S_{t1}, S_{t2}] = E[S_{t1}, S_{t2}] - E[S_{t1}]\, E[S_{t2}]$

From above,

$E[S_{t1}] = S_0\, e^{\mu t1 + ½ \sigma 2t1}$ and $E[S_{t2}] = S_0\, e^{\mu t2 + ½ \sigma 2t2}$

The expected value of the product is:

$E[S_{t1}, S_{t2}] = E[S_0 \exp(\mu t_1 + \sigma B_{t1})\, S_0 \exp(\mu t_2 + \sigma B_{t2})]$

$$= S^2{}_0\, e^{\mu(t1 + t2)}\, E[\exp(\sigma B_{t1} + \sigma B_{t2})]$$

To evaluate this we need to split B_{t2} into two independent components:

$B_{t2} = B_{t1} + (B_{t2} - B_{t1})$ where $B_{t2} - B_{t1} \sim N(0, t_2 - t_1)$

Hence,

$E[S_{t1}, S_{t2}]$

$= S^2{}_0\, e^{\mu(t1 + t2)}\, E[\exp(\sigma B_{t1} + \sigma \{ B_{t1} + (B_{t2} - B_{t1})\})]$

$= S^2{}_0\, e^{\mu(t1 + t2)}\, E[\exp(2\sigma B_{t1} + \sigma \{ B_{t2} - B_{t1}\})]$

$= S^2{}_0\, e^{\mu(t1 + t2)}\, E[\exp(2\sigma B_{t1})]\, E[\exp \{ B_{t2} - B_{t1}\})]$

$= S^2{}_0\, e^{\mu(t1 + t2)} \exp(2\sigma^2 t_1) \exp [\, ½\, \sigma^2 (t_2 - t_1)]$

$= S^2{}_0\, e^{\mu(t1 + t2)} \exp(\frac{3}{2}\sigma^2 t_1 + \frac{1}{2}\sigma^2 t_2)$

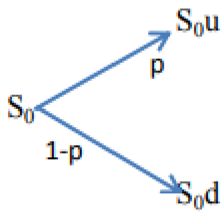
Putting all the equations together:

$$Cov[S_{t1}, S_{t2}] = S^2{}_0\, e^{\mu(t1 + t2)} \exp(\frac{3}{2}\sigma^2 t_1 + \frac{1}{2}\sigma^2 t_2) - S_0\, e^{\mu t1 + ½ \sigma 2t1} .\, S_0\, e^{\mu t2 + ½ \sigma 2t2}$$

$$= S^2{}_0\, e^{\mu(t1 + t2)} (\exp(\frac{3}{2}\sigma^2 t_1) - \exp(\frac{1}{2}\sigma^2 t_1)) \exp(½\, \sigma^2 t_2)$$

7.

i) Setting up the commodity tree using u for up move and d for down move, p is up-step probability:



Where p is the up probability and (1-p) the down probability

Then $E(C_t) = S_0[pu+(1-p)d]$, and

$Var(C_t) = E(C_t^2) - E(C_t)^2$

$= S_0^2\,[pu^2+(1-p)d^2] - S_0^2[pu+(1-p)d]^2$

$= S_0^2\,[pu^2+(1-p)d^2-(pu+(1-p)d)^2]$

$= S_0^2\,[p(1-p)u^2+ p(1-p)d^2-2p(1-p)]$ $\quad (\because d= 1/u)$

$= S_0^2\, p(1-p)(u-d)^2$

Equating moments:

$S_0 e^{rt} = S_0[pu+(1-p)d]$ ______________(A)

And $\sigma^2 S_0^2 t = S_0^2\, p(1-p)(u-d)^2$______________(B)

From (A) we get

$p = \frac{e^{rt}-d}{u-d}$ ________________________(C)

Substituting p into equation (B), we get

$\sigma^2 t = \frac{e^{rt}-d}{u-d}\left(1 - \frac{e^{rt}-d}{u-d}\right)(u-d)^2$

$= -(e^{rt}-d)(e^{rt}-u) = (u+d)\,e^{rt} - (1+e^{2rt})$

Putting d = 1/u, and multiplying through by u we get

$u^2 e^{rt} - u\,(1+e^{2rt}+\sigma^2 t) + e^{rt} = 0$

This is a quadratic in u which can be solved in the usual way.

ii)

a) σ = 0.15, t= 0.25 => u= exp(.15*$\sqrt{.25}$)= exp(.075) = 1.077884, d = 1/u= .92774

The tree is

t=0	t=.25	t=.5	t=.75	
			100.186	Node A
		92.947		
	86.231		86.232	Node B
80		80.001		
	74.22		74.22	Node C
		68.857		
			63.882	Node D

b) r=0, we have p = $\frac{e^{rt}-d}{u-d} = \frac{(1-.927744)}{(1.077884-.927744)}$ = .48126

Discounting back the final payoff at t=.75 to t=0 along the tree using p and (1-p), we get

t=0	t=.25	t=.5	t=.75	
			20.186	Node A
		12.948		
	7.787		6.232	Node B
4.496		2.999		
	1.443		0	Node C
		0		
			0	Node D

Hence value of the call option is 4.496.

c) The lookback call pays the difference between the minimum value and the final value.

Notate paths by U for up and D for down, in order

We get the payoffs

UUU	(100.186 - 80)= 20.186	Node A
UDU	(86.232-80)= 6.232	Node B
UUD	(86.232- 80) =6.232	Node B
UDD	(74.22-74.22) = 0	Node C
DUU	(86.232- 74.22) = 12.012	Node B
DUD	(74.22-74.22) = 0	Node C
DDU	(74.22-68.857) = 5.363	Node C
DDD	(63.882-63.882)= 0	Node D

The lookback payoffs are, for each successful path (i.e. with a non– zero result)

Probabilities of arriving at each node are:

Node A= p^3 = .11147

Node B= $p^2(1-p)$ = .12015

Node C= $p(1-p)^2$ = .12950

Node D= $p(1-p)^3$ = .13959

Hence the tree value of lookback option is:

(.11147*20.186)+ (.12015*[6.232+ 6.232+ 12.012]) + (.12950*5.363)

= 5.8854

8.

i. Consider a stock whose current price is S0 and an option whose current price is f.

We suppose that the option lasts for time T and that during the life of the option the stock price can either move up from S0 to a new level S0u or move down to S0d where u > 1 and d < 1.

Let the payoff be fu if the stock price becomes S0u and fd if stock price becomes S0d Let us construct a portfolio which consists of a short position in the option and a long position in Δ shares.

We calculate the value of Δ that makes the portfolio risk-free. Now if there is an upward movement in the stock, the value of the portfolio becomes ΔS0u – fu and if there is a downward movement of stock, the value of the portfolio becomes ΔS0d – fd

The two portfolios are equal if $\Delta S_0 u - f_u = \Delta S_0 d - f_d$

Or $\Delta = \frac{fu - fd}{S0u - S0d}$ so that the portfolio is risk-free and hence must earn the risk free rate of interest.

This means the present value of such a portfolio is $(\Delta S_0 u - f_u)\exp(-rT)$

Where r is the risk free rate of interest.

The cost of the portfolio is $\Delta S_0 - f$

Since the portfolio grows at a risk free rate, it follows that

$(\Delta S_0 u - f_u)\exp(-rT) = \Delta S_0 - f$

or $f = \Delta S_0 - (\Delta S_0 u - f_u)\exp(-rT)$

Substituting Δ from the earlier equation simplifies to:

$f = e^{-rT}[p\, f_u + (1-p)\, f_d]$ where $p = [e^{rT} - d]/[u-d]$

ii. The option pricing formula does not involve probabilities of stock going up or down although it is natural to assume that the probability of an upward movement in stock increases the value of call option and the value of put option decreases when the probability of stock price goes down.

This is because we are calculating the value of option not in absolute terms but in terms of the value of the underlying stock where the probabilities of future movements (up and down) in the stock already incorporates in the price of the stock. However, it is natural to interpret p as the probability of an up movement in the stock price.

The variable 1-p is then the probability of a down movement such that the above equation can be interpreted as that the value of option today is the expected future value discounted at the risk free rate

iii.

The expected stock price $E(S_T)$ at time T = $pS_0u + (1-p)\, S_0d$ 0.5

or $E(S_T) = p\, S_0(u-d) + S_0d$ ---0.5

Substituting p from above equation in (i) i.e. $p = [e^{rT} - d]/[u-d]$---1

We get $E(S_T) = e^{rT}\, S_0$---0.5 -----1

i.e. stock price grows at a risk free rate or return on a stock is risk free rate

iv. In a risk neutral word individuals do not require compensation for risk or they are indifferent to risk. Hence expected return on all securities and options is the risk free interest rate. Hence value of an option is its expected payoff in a risk neutral discounted at risk free rate.

9.

i) The forward price is given by $F = S \cdot exp(rt)$ where S is the stock price, t is the delivery time and r is the continuously compounded risk-free rate of interest applicable up to time t.

Put-call parity states that: $c + K \cdot exp(-rt) = p + S$ where c and p are the prices of a European call and put option respectively with strike K and time to expiry t and S is the current stock price.

To compute F, we need to find S and r. t is given to be 0.25 years.

Substituting the values from the first two rows of the table in the put-call parity, we get two equations in two unknowns (S and r):

$$13.334 + 70 \cdot exp(-0.25r) = 0.120 + S$$

$$8.869 + 75 \cdot exp(-0.25r) = 0.568 + S$$

Solving the simultaneous equations for S and r, we get:

S = 82 and r = 7%

Therefore, we get the forward price $F = 82 \cdot exp(0.07 * 0.25) = 83.45$

[4]

ii) Let the (continuously compounded, annualized) rate of interest over the next k months be r_k. Then the required forward rate r_F can be found from:

$exp(r_6*0.5) = exp(r_3*0.25)*exp(r_F*0.25)$ or $2*r_6 = r_3 + r_F$

We know that r_3 = 7%.

To find r_6, we substitute values from the last row in the put-call parity relationship and S = 82:

$2.569 + 90*exp(-0.5*r_6) = 7.909 + 82$

Therefore, r_6 = 6% and r_F = 5% [2]

iii) Using the put-call parity for each row in the given table, we get:

6.899 + a*exp(-0.07*0.25) = 1.055 + 82

b + 80*exp(-0.07*0.25) = 1.789 + 82

2.594 + 85*exp(-0.07*0.25) = c + 82

Solving individually, we get:

a = 77.5

b = 5.177

c = 4.119

10.

i) Since interest rates are assumed zero, the risk-neutral up-step probability is given as:

$q = (1 - d) / (u - d)$

where u and d are the sizes of up-step and down-step respectively

For a recombining tree, d = 1/u.

Substituting d = 1/u in the expression for q and simplifying, we get:

q = (1-1/u) / (u – 1/u) = 1 / (u+1)

For no-arbitrage to hold, we must have u > 1 > d.

Then, u > 1 => u + 1 > 2 => q = 1 / (u+1) < ½. Hence proved.

ii)

Since each step is one month and the expiry of the derivative is one year from now. Therefore, a 12-step recombining binomial tree needs to be created, i.e. n = 12.

Further, at time T = 12 months, the stock price will be $S_0 u^k d^{n-k}$ with risk-neutral probability ${}^nC_k\, q^k (1-q)^{n-k}$ where q, the up-step probability is 1/3, u, the up-step size is 2, and d = 1/u = ½.

We know that the derivative has a payoff $\sqrt{\frac{S_T}{S_0}}$ at time T = 12 months.

Thus, the current price of that derivative is: $P = \sum_{k=0}^{n} \sqrt{\frac{S_T}{S_0}} \cdot \frac{n!}{k!\cdot(n-k)!} q^k (1-q)^{n-k}$

Therefore, $P = \sum_{k=0}^{n} \sqrt{\frac{S_0 u^k d^{n-k}}{S_0}} \cdot \frac{n!}{k!\cdot(n-k)!} q^k (1-q)^{n-k} = \sum_{k=0}^{n} \sqrt{u^k d^{n-k}} \cdot \frac{n!}{k!\cdot(n-k)!} q^k (1-q)^{n-k}$

$$P = \sum_{k=0}^{n} u^{\frac{k}{2}} d^{\frac{n-k}{2}} \cdot \frac{n!}{k!\cdot(n-k)!} q^k (1-q)^{n-k} = \sum_{k=0}^{n} 2^{\frac{k}{2}} \left(\frac{1}{2}\right)^{\frac{n-k}{2}} \cdot \frac{n!}{k!\cdot(n-k)!} \left(\frac{1}{3}\right)^k \left(\frac{2}{3}\right)^{n-k}$$

$$P = \sum_{k=0}^{n} 2^{k-\frac{n}{2}} \cdot \frac{n!}{k!\cdot(n-k)!} \frac{2^{n-k}}{3^n} = \sum_{k=0}^{n} 2^{\frac{n}{2}} \cdot \frac{n!}{k!\cdot(n-k)!} \frac{1}{3^n} = 2^{\frac{n}{2}} \frac{1}{3^n} \sum_{k=0}^{n} \cdot \frac{n!}{k!\cdot(n-k)!}$$

$$P = 2^{\frac{n}{2}} \frac{1}{3^n} 2^n = \left(\frac{2\sqrt{2}}{3}\right)^n = \left(\frac{2\sqrt{2}}{3}\right)^{12} = 0.49327$$

11.

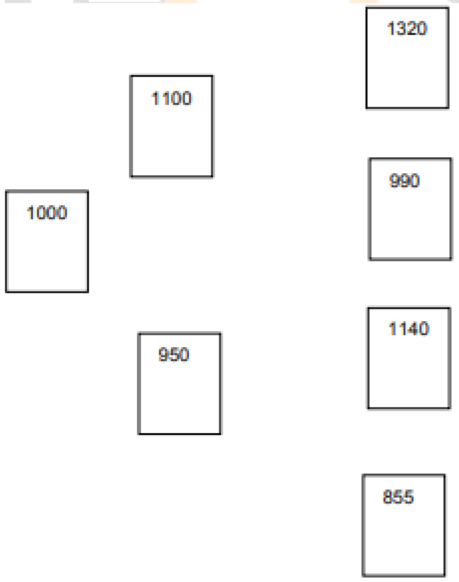
i) A recombining binominal tree or binominal lattice is one in which the sizes of the up-steps and down-steps are assumed to be the same under all states and across all time intervals. i.e., u t (j)=u and d t (j)=d for all times t and states j, with d < exp(r) < u

•.It therefore follows that the risk neutral probability 'q' is also constant at all times and in all states eg. q t (j)=q

•The main advantage of a 'n' period recombining binominal tree is that it has only [n+1] possible states of time as opposed to 2n possible states in a similar non-recombining binominal tree. This greatly reduces the amount of computation time required when using a binominal tree model.

• The main dis-advantage is that the recombining binominal tree implicitly assumes that the volatility and drift parameters of the underlying asset price are constant over time, which assumption is contradicted by empirical evidence

ii) The tree is as:



a) The risk-neutral probabilities at the first and second steps are as follows:

$q_1 = (\exp(0.0175) - 0.95)/(1.10-0.95)$

$= (0.06765)/0.15$

$= 0.4510$

$q_2 = (\exp(0.025) - 0.90)/(1.20-0.90)$

$= 0.41772$

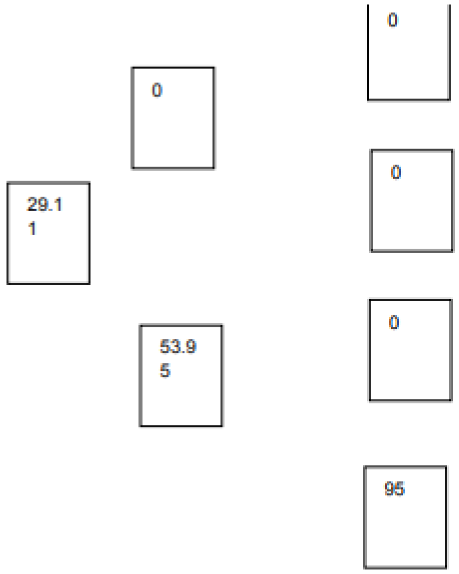
Put payoffs at the expiration date at each of the four possible states of expiry are 0,0,0 and 95.

Working backwards, the value of the option V1 (1) following an up step over the first 3 months is

V1 (1) exp(0.025) = [0.41772* 0] + [0.58228*0]
i.e., V1 (1) = 0

The value of the option V1 (2) following a down step over the first 3 months is:
V1 (2)exp(0.025) = [0.41772* 0] + [0.58228*95]
i.e., V1 (2) = 53.9508

The current value of the put option is:
V0 exp(0.0175) = [0.4510*0] + [0.5490*53.9508]
i.e., V0 =29.105



b) While the proposed modification would produce a more accurate valuation, there would be a lot more parameter values to specify. Appropriate values of u and d would be required for each branch of the tree and values of ‘r’ for each month would be required.

The new tree would have $2^6 = 64$ nodes in the expiry column. This would render the calculations prohibitive to do normally, and would require more programming and calculation time on the computer. • An alternative model that might be more efficient numerically would be a 6-step recombining tree which would have only 7 nodes in the final column.

12.

Given $Z(t)$ is standard Brownian

a. $dU(t) = 2dZ(t) - 0$

$= 0dt + 2dZ(t)$.

Thus, the stochastic process $\{U(t)\}$ has zero drift.

b. $dV(t) = d[Z(t)]^2 - dt$.

$d[Z(t^{)]2} = 2Z(t)dZ(t) + 2/\ 2\ [dZ(t)]^2$

$= 2Z(t)dZ(t) + dt$ by the multiplication rule

Thus, $dV(t) = 2Z(t)dZ(t)$. The stochastic process $\{V(t)\}$ has zero drift.

c. $dW(t) = d[t^2 Z(t)] - 2t\ Z(t)dt$

Because $d[t^2\ Z(t)] = t^2\ dZ(t) + 2tZ(t)dt$, we have

$dW(t) = t^2\ dZ(t)$.

Thus The process $\{W(t)\}$ has zero drift

13.

i. Let S_t/S_0 follows lognormal distribution with parameters $\left(\mu - \frac{1}{2}\sigma^2\right)t$, and $\sigma^2 t$ such the expected return on a stock is μ and volatility is σ

This means Expected value of stock price at the end of first time step $= S_0 e^{\mu\delta t}$. On the tree the expected price at that time $= qS_0u + (1-q)S_0d$

In order to match the expected return on the stock with the tree's parameters we have
$qS_0u + (1-q)S_0d = S_0e^{\mu\delta t}$.
Or $q = (e^{\mu\delta t} - d)/(u-d)$

Volatility σ of a stock price is defined so that $\sigma\sqrt{\delta t}$ is the standard deviation of the return on the stock price in a short period of time δt

The variance of stock price return is
$qu^2 + (1-q)d^2 - [qu + (1-q)d]^2 = \sigma^2\delta t$

Substituting the value of q in the expression above we have
$e^{\mu\delta t}(u+d) - ud - e^{2\mu\delta t} = \sigma^2\delta t$

When higher powers of δt other than δt are ignored.
This implies $u = e^{\sigma\sqrt{\delta t}}$ and $d = 1/u = e^{-\sigma\sqrt{\delta t}}$ **[5]**

ii.

S	200
r	10%
σ	35%
T	2 months
t	1 month = 0.0833
u	1.1063
d	0.9039
q	0.5161
p=(1-q)	0.4839
K	200

t=0	t=1	t=2	Pay off @ K=200
		244.79	0
		221.26	
	221.26		
	210.36		
	0	200.00	0
		206.85	
200.00			
		200.00	6.62
		193.38	
	180.78		
	190.15		
	9.85	163.41	19.22
		180.78	

Note: Bold italics are geometric means of the stock price, reds are the payoffs.

The value of option at time 1 is
$(6.62 \times q + 19.22(1-q))e^{-0.1\times\frac{1}{12}} = 12.61$ this is more than the payoff at time 1. Hence the American option will not be exercised.

The value of put option at time 0 is
$12.61 \times e^{-0.1\times\frac{1}{12}} \times (1-q) = 6.05$ **[4]**

14.

i. Pay off Diagram for European Call:

		51.8
	20.000	561.8
	530.000	
500		503.5
	475	
	0	451.25
		0

u	1.06
d	0.95
r	0.05
t	3 months = 0.25
p	[exp(r*T)-d]/(u-d) = 0.5689
1-p	0.4311

Value of the 6 month European Call = exp(-0.5*0.05)*51.8*0.5689^2 = 16.351

ii. The pay off diagram for European Put

	B	***0***
	0	561.8
	530.000	
500		503.5***6.50***
	475	
	35	451.25
	C	***58.75***

Value of European put option = exp(0.05*.5)*[0.4311^2*58.75+2*0.4311*0.5689*6.50] =13.759

Using Put call parity, Value of put + stock = 500+13.759=513.759

Value of call + discounted price = 16.351+510*exp (-.05*.5) = 513.759

iii. For the American option, the value at final nodes is same for a European Option. At earlier nodes, the value of the option is greater of value calculated at the node and the payoff from early exercise.

We can find that the value at node B= 6.50*exp (-0.25*0.05)*0.4311=2.767.

Value at node C=exp (-0.25*0.05)*(6.50*0.5689 + 58.75*0.4311) = 28.65

However, payoff from immediate exercise is 35 at node C. Hence it is advisable to exercise the put option at this point.

Value of the American put option at time t=0 will be
Exp (-0.05*.25)*(35*0.4311+2.767*0.5689) = 16.46

iv. Expected payoff in 3 months' time is calculated by using real rate of return of 9%. p=0.6614. Hence expected payoff = 20*0.6614= 13.228.Unfortunately, it is not easy to know the correct discount rate to apply to the expected payoff in real world to be able to compute the value of the option. Risk neutral valuation solves this problem as under risk neutral valuation all assets are expected to earn the risk free rate.

15.

i. Let f = $f(S_t,t) = S^k$

$$\frac{\partial f}{\partial S} = kS^{k-1}, \qquad \frac{\partial^2 f}{\partial S^2} = k(k-1)S^{k-2}, \qquad \frac{\partial f}{\partial t} = 0$$

Using Ito calculus, $df = kS^{k-1}dS_t + ½\, k(k-1)S^{k-2}(dS_t)^2$
$= f[k\mu + ½k(k-1)\sigma^2]dt + f[k\sigma]dW_t$

Hence f = S^k follows Geometric Brownian motion, with drift $\mu' = k\mu + ½k\,(k-1)\,\sigma^2$ and volatility $\sigma' = k\sigma$.

Hence $f_t = f_0 e^{(\mu' - ½\sigma'^2)t + \sigma' W_t}$

$This\ means\ \frac{f}{f_0} \sim lognormal\ ((\mu' - ½\sigma'^2)t, \sigma'^2 t)$

The mean and variance of a log normal distribution with parameters μ and σ^2

Is given by $e^{\mu + 1/2\sigma^2}$ and $(e^{\sigma^2} - 1)e^{2\mu + \sigma^2}$

This means E(f) = $f_0 e^{\mu' t}$ and V(f) = $f_0^2 e^{2\mu' t}(\,e^{\sigma'^2 t} - 1)$

ii. Let f = $f(t,S_t) = e^{-rt}S_t$

$$\frac{\partial f}{\partial t} = -re^{-rt}S_t, \qquad \frac{\partial f}{\partial S} = e^{-rt}, \qquad \frac{\partial^2 f}{\partial S^2} = 0$$

By Ito calculus, $df = -re^{-rt}S_t dt + e^{-rt}dS_t$
$= (\mu - r)f dt + \sigma f dW_t$ (on substituting the expression for dS_t)

which is a martingale if and only if $\mu = r$

iii. Combining results from part (a) and (b), we need, for discounted S^k to be a martingale,

$$k\mu + \tfrac{1}{2}k(k-1)\sigma^2 = r$$

For given values of r, σ and k, one can solve for the value of μ for which discounted S^k will be a martingale.