INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Subject: Financial Engineering 1

Chapter: Unit 3 & 4

Category: Assignment Solutions

1.

i) Data: $S_0 = 65, K = 55, \sigma = 25\%\ p.a., T = 0.5\ year, r = 2\%$

Let C_t be the price of the European call.

The Black-Scholes formula returns [½ Mark each]

$d_1 = 1.09$

$d_2 = 0.9132$

$N(d_1) = 0.8621$

$N(d_2) = 0.8194$

Therefore $C_0 = 65 \times 0.8621 - 55e^{-0.02\times0.5} \times 0.8194$ [1]

$= 11.42$ [1]

ii) $delta = \frac{\partial c}{\partial s}$ [1]

[Note to markers: please award ½ mark for stating $N(d_1)$]

iii) In the Black-Scholes model $delta = N(d_1)$ [1]

Using the results from above $delta = 0.8621$ [1]

iv) $delta_{put} = delta_{call} - 1$ [1]

Therefore, $delta_{put} = -0.1379$ [1]

2.

i) For a derivative whose price at time t is $f(t, S_t)$ where S_t is the price of the underlying asset,

- Delta is the rate of change of its price with respect to a change in S_t: $\Delta = \frac{\partial f}{\partial S_t}$
- Vega is the rate of change of its price with respect to a change in the assumed level of volatility of S_t: $\nu = \frac{\partial f}{\partial \sigma}$

[2]

ii) Put-call parity states that: $c + K^*\exp(-r\tau) = p + S$ where c and p are the prices of a European call and put option respectively with strike K and time to expiry τ and S is the current stock price.

Differentiating w.r.t. σ implies $\frac{\partial c}{\partial \sigma} = \frac{\partial p}{\partial \sigma}$, i.e. the vegas are identical.

[1]

iii)

$$d_1 = \frac{\log\frac{S}{K} + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}$$

Therefore, $d_1 = 0.706241$

$$d_2 = d_1 - \sigma\sqrt{\tau}$$

Therefore, $d_2 = 0.456241$

$$c = S\Phi(d_1) - Ke^{-r\tau}\Phi(d_2)$$

Therefore, c = 9.652546

$$p = c + Ke^{-r\tau} - S$$

Therefore, p = 2.214017

[3]

iv)

A portfolio for which the overall delta (i.e. weighted sum of the deltas of the individual assets) is equal to zero is described as delta-hedged or delta-neutral. Such a portfolio is immune to small changes in the price of the underlying asset.

A portfolio for which the overall vega (i.e. weighted sum of the vegas of the individual assets) is equal to zero is described as vega-hedged or vega-neutral. Such a portfolio is immune to small changes in the assumed level of volatility. [2]

v) Let the required portfolio consist of x call options, y put options and z forwards.

The delta and vega for a forward are 1 and 0 respectively and there are no current cashflows.

Thus, for a single unit of each of them, we have:

	Present value / cashflow	Delta	Vega
Call option	c = 9.6525	Δ_c	V_c
Put option	p = 2.2140	Δ_p	V_p
Forward	-	1	-

<u>Vega-neutrality:</u> The vega of a forward is zero. For the portfolio must be vega-neutral, we must have: $x*V_c + y*V_p = 0$.
From part b, we have $V_c = V_p$. Therefore, $(x+y)*V_c = 0$. Therefore, $x+y = 0$. Therefore, $y = -x$.

<u>Delta-neutrality:</u>

We know that Δ of a forward is one. For the portfolio to be delta-neutral, we need: $x*\Delta_c + y*\Delta_p + z = 0$.

Also, $\Delta_p = \Delta_c - 1$ and $y = -x$. Therefore, on simplifying, we get: $x + z = 0$ or $z = -x$.

<u>Overall portfolio:</u>

Thus, we have $x = -y = -z$ and the total portfolio is to be worth \$1000. So we must have:

$x*c + y*p + z*0 = 1000$. Therefore, $x*9.6525 - x*2.2140 = 1000$.

Therefore, $x = 134.4$, $y = z = -134.4$

So our portfolio must consist of:

- Long position of 134 call options
- Short position of 134 put options
- Short position of 134 forwards

[4]

[12 Marks]

3.

(i) $C_t = E(e^{-r(T-t)}C_T | F_t)$ [1]

where F_t denotes the filtration at time $t > 0$, [½]

C_T is the payoff under the derivative [½]

at maturity time T, [½]

C_t is the derivative value at time t, [½]

and the expectation is taken under the risk-neutral martingale measure. [½]
[Max 3]

Data: $S = 50; K = 49; r = 5\%; \sigma = 25\%; T = 0.5$

(ii) The Black-Scholes formula returns:

$d1 = 0.3441$ [½]

$d2 = 0.1673$ [½]

$N(d1) = 0.6346$ [½]

$N(d2) = 0.5664$ [½]

So $\text{Call} = 50 \times 0.6346 - 49e^{-0.05\times 0.50} \times 0.5664 = 4.66$ [2]

(iii) Same as European call (as the stock is non-dividend-paying), i.e. 4.66 [1]

(iv) Using put-call parity (or otherwise):

$p_t = c_t + Ke^{-r(T-t)} - S_t$ [1]

Hence $p_t = 2.45$. [1]

(v) If the stock is dividend-paying, the payment of the dividends would cause the value of the underlying asset to fall – which follows from the no arbitrage principle [1]

Alternatively: in valuing the option we must take account of the fact that dividends are payable on the underlying asset which do not feed through to the holder of the option. [1]

Therefore the price of the European call would decrease… [½]

… since by buying the option instead of the underlying share the investor forgoes the income [½]

Similarly, the price of the European put would increase [½]

The American call would now be more expensive than the European call due to potential early exercise opportunity [1]

[Max 3]

4.

(i) Suppose that Z_t is a standard Brownian motion under P. [1]

Furthermore, suppose that γ_t is a previsible process. [½]

Then there exists a measure Q equivalent to P [½]

and where $\tilde{Z}_t = Z_t + \int_0^t \gamma_s ds$ is a standard Brownian motion under Q. [1]

Conversely, if Z_t is a standard Brownian motion under P and if Q is equivalent to P then there exists a previsible process γ_t such that $\tilde{Z}_t = Z_t + \int_0^t \gamma_s ds$ is a Brownian motion under Q. [1]

(ii) Under the risk-neutral probability measure, the discounted value of asset prices are martingales. [1]

5.

(i) Delta = $\Delta = \Phi(d_1)$

using standard Black-Scholes notation. [1]

(ii) $\Delta = \Phi(d_1) = 0.6179$ means that $d_1 = 0.3$ [1]

So $0.3 = (\log(40/45.91) + (0.02 + 0.5\sigma^2) \times 5) / \sigma\sqrt{5}$ [1]

So $-0.0378 - 0.6708\sigma + 2.5\sigma^2 = 0$ [½]

Solving the quadratic gives $\sigma = 0.3161$ or $\sigma = -0.0478$ [1]

Rejecting the negative root gives $\sigma = 32\%$ *(or may quote variance = 10%)* [½]

(iii) Under the risk-neutral probability measure Q, the fair price of the option is $ce^{-rT} Q(S_1/S_0 < k_S)\, Q(R_1/R_0 < k_R)$ [2]

(iv) Under the Black-Scholes model , if the stocks are perfectly correlated then $S_1/S_0 = R_1/R_0$. [1]

So if $k_S < k_R$ then the option only depends on stock S and has value $ce^{-rT} Q(S_1/S_0 < k_S)$ [1]

Similarly if $k_S > k_R$ then the option only depends on stock R and has value $ce^{-rT} Q(R_1/R_0 < k_R)$ [½]

If $k_S = k_R$ then the option can be defined in terms of the price of either stock as $ce^{-rT} Q(S_1/S_0 < k_S) = ce^{-rT} Q(R_1/R_0 < k_S)$ [½]

So overall the option can be defined in terms of the lower of k_S and k_R, and either of the stock increases, i.e. has value

$ce^{-rT} Q(R_1/R_0 < \min(k_S,k_R)) = ce^{-rT} Q(S_1/S_0 < \min(k_S,k_R))$ [1]

[Max 3]

(v) $ce^{-rT} Q(S_T/S_0 < k_S)\ Q(R_T/R_0 < k_R)$

$= 50e^{-0.02}\ Q(S_T/S_0 < 0.8)\ Q(R_T/R_0 < 0.6)$

$= 50e^{-0.02}\ Q(S_1 < 0.8 \times 40)\ Q(R_1 < 0.6 \times 30)$ [1]

$= 50e^{-0.02}\ (1 - \Phi((\log(S_1/0.8S_1) + (r - 0.5\sigma_S^2))/\sigma_S))\ (1 - \Phi((\log(R_1/0.6R_1) + (r - 0.5\sigma_R^2))/\sigma_R))$ [1]

$= 50e^{-0.02}\ (1 - \Phi((\log(1/0.8) + 0.02 - 0.5 \times 0.32^2)/0.32)\ (1 - \Phi((\log(1/0.6) + 0.02 - 0.5 \times 0.15)/\sqrt{0.15})$ [½]

$= 50e^{-0.02}\ (1 - (0.59982))\ (1 - \Phi(1.1769))$ [½]

$= 50e^{-0.02}\ (1 - 0.7257)\ (1 - 0.88039)$ [1]

$= \$1.61$ (using $\sigma = 0.32$, or \$1.59 using an exact $\sigma = 0.3161$) [1]

[Total 15]

6.

(i) (a) Let f denote the price of a put option,
then $d_1 = (\ln(S_0/K) + (r + ½\sigma^2)T)/ \sigma\sqrt{T}$
and then $\Delta = -\Phi(-d_1) = \Phi(d_1) - 1$.

(b) In this case, we must have $100{,}000\Delta = -24{,}830$ and so $\Delta = -0.25$

(ii) $\Delta = -.2483$ and so $d_1 = 0.68$. It follows (rearranging the expression for d_1) that $(.01575 + .03 + 0.5\sigma^2) = 0.68\sigma$. Solving the quadratic equation we obtain $\sigma = 0.68 \pm \sqrt{0.3709} = 0.07098 = 7.1\%$ (choosing the root less than 1).

(iii) We need to calculate $K e^{-rT}\Phi(-d_2) = e^{-r}\Phi(-d_1 + \sigma\sqrt{T})$
$= 630e^{-0.03}\, \Phi(-0.609)$p $= 630e^{-0.03} * 0.2712 = 165.806$p.

Clearly the option price is 165.806 – 24830 * 640/100,000 = 6.894p.
and the value of the cash holding is 100,000 * 165.806p = £165,806

7.

(i) Denote the individual derivative by f and assume this is written on an underlying security S

Delta = $\partial f/\partial S$
Gamma = $\partial^2 f/\partial S^2$
Vega = $\partial f/\partial \sigma$

(ii) Delta = 0.801

(iii) The hedge is delta = 0.801 shares = and 17.91 – 0.801 * 60 = \$30.15 short in cash.

(iv) Using the approximation $f(S, \sigma + \delta) \approx f(S, \sigma) + \delta df/d\sigma$, we obtain an option price ≈ 17.91 + 29.00 * 0.02 = \$18.49.

8.

(i) Δ is the first partial derivative of the option price with respect to the underlying asset price. [1]

(ii) Using the formula for the Δ, we see that $\Phi(d_1) = 0.42074$ and hence $d_1 = -0.2$.

Thus $-0.2\sigma = -0.0600 + \frac{1}{2}\sigma^2$ or $\frac{1}{2}\sigma^2 + 0.2\sigma - 0.06 = 0$.

Solving the quadratic gives $\sigma = 20\%$ or -60% and rejecting the negative value gives $\sigma = 20\%$. [3]

9.

(i) The PDE is the Black-Scholes PDE:

$$\tfrac{1}{2}\sigma^2 x^2 g_{xx} + (r - q)xg_x - rg + g_t = 0$$

with boundary condition as above: $g(T, x) = f(x)$.

(ii) The proposed solution implies that for this derivative the function g is given by $g(t, x) = (x^n / S_0^{n-1})e^{\mu(T-t)}$, where n is an integer great than 1.

This gives $xg_x = ng$, $x^2 g_{xx} = n(n-1)g$ and $g_t = -\mu g$.

Thus, to solve the PDE we need $\mu = \frac{1}{2}\sigma^2 n(n-1) + (n-1)r - nq$.

A quick check shows that g satisfies the boundary condition: $g(T, x) = x^n/S_0^{n-1}$.

10.

(i) Consider the portfolio which is long one call plus cash of $Ke^{-r(T-t)}$ and short one put.

The portfolio has a payoff at the time of expiry of S_T.

Since this is the value of the stock at time T, the stock price should be the value at any time $t < T$: that is

$$C_t + Ke^{-r(T-t)} - P_t = S_t.$$

(ii) This relationship is known as *put-call parity*.

The Black-Scholes formula gives us that $S_0\,\Phi(d_1)\,Ke^{-rT}\,\Phi(d_2)$, with

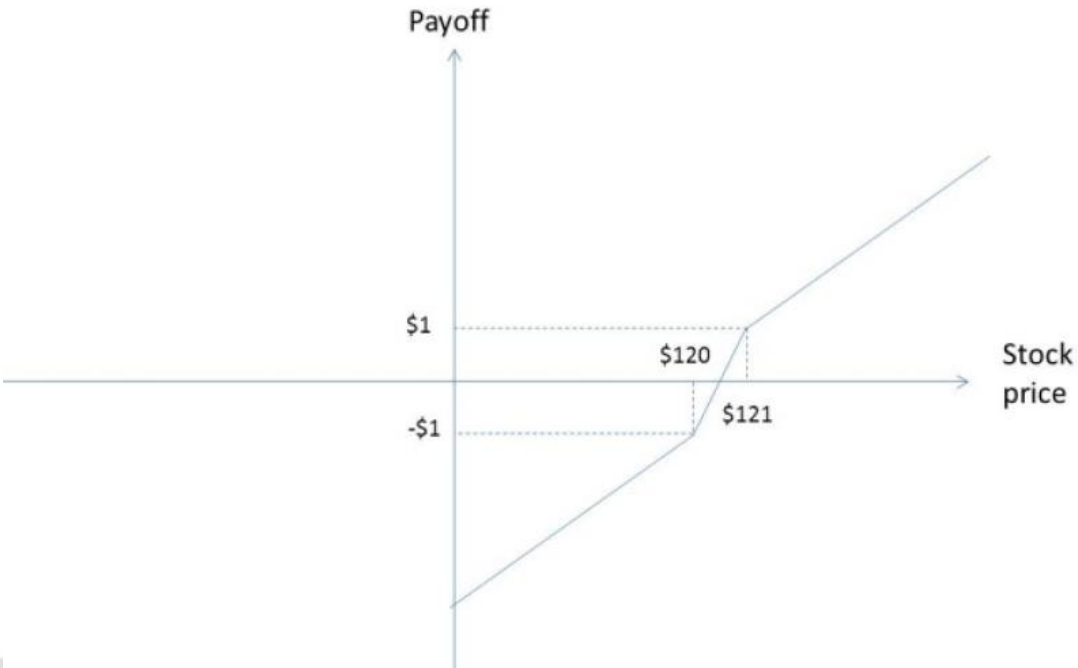
$$S_0 = 110,\ K = 120,\ r = .02,\ T = 1$$

so that

$$d_1 = (\log(S_0 / K) + r + ½\sigma^2 T) / \sigma\,\sqrt{T} = (\log(11/12) + .02 + ½\sigma^2) / \sigma,$$
$$d_2 = d_1 - \sigma.$$

Guessing and repeated interpolation gives $\sigma = 30\%$.

(iii)



(iv) (a) The payoff from the portfolio, D, satisfies

$$S_1 - 121 \le D \le S_1 - 120.$$

It follows that the initial price, V, of the portfolio should satisfy

$$S_0 - 121e^{-r} \le V \le S_0 - 120\, e^{-r},$$

i.e. $-8.604 \le V \le -7.624$.

(b) And this implies that $17.714 \le P_0 \le 18.694$.

(v) The Black-Scholes price (using the formula in the tables) is \$18.35.

11.

(i) The Δ of the call holding must be minus the Δ of the shareholding, which, by definition is – 18673, so the Δ of a call is $\Delta_C = 0.18673$.

(ii) Δ_C for a call is $\Phi(d_1)$, where $d_1 = (\ln(S_0/k) + r + \frac{1}{2}\sigma^2))/\sigma = (\ln(1.1798/1.5) + 0.02 + \frac{1}{2}\sigma^2))/\sigma = -0.22/\sigma + \frac{1}{2}\sigma$.

Now $\Phi(d_1) = 0.18673$ so $d_1 = -0.89$

which implies that

$-0.22 + 0.89\,\sigma + \frac{1}{2}\,\sigma^2 = 0$ so $\sigma = -0.89 \pm (0.89^2 + 0.44)^{\frac{1}{2}}$.
Rejecting the negative root gives a value of $\sigma = 22\%$.

(iii) $d_2 = d_1 - \sigma\sqrt{T} = -1.11$. Thus $P = Ke^{-rT}\,\Phi(-d_2) - S_0\,\Phi(-d_1)$
$= 150e^{-r}\,\Phi(-d_2) - 117.98\Phi(-d_1) = 147.0298\,\Phi(-d_2) - 117.98\Phi(-d_1)$
$= 147.0298 \times 0.8665 - 117.98 \times 0.81327 = \$31.4517$

(iv) Using C to denote the call option, P the put option and S the stock we know that:

$$\Delta_C - \Delta_P = \Delta_S = 1$$
$$\Gamma_C = \Gamma_P \text{ and } \Gamma_S = 0$$

So since we hold 100,000 call options, we must be short 100,000 put options and 100,000 shares to get a gamma and delta neutral portfolio.

Alternative calculation approaches were awarded full marks if candidates reached the right conclusions.

12.

i) The assumptions underlying the Black-Scholes model are as follows:

1. The price of the underlying share follows a geometric Brownian motion. [1/2]

2. There are no risk-free arbitrage opportunities. [1/2]

3. The risk-free rate of interest is constant, the same for all maturities and the same for borrowing or lending. [1/2]

4. Unlimited short selling (that is, negative holdings) is allowed. [1/2]

5. There are no taxes or transaction costs. [1/2]

6. The underlying asset can be traded continuously and in infinitesimally small numbers of units. [1/2]

ii)

Data: $S = 8; K = 9; r = 2\%; \sigma = 20\%;\ T = 0.25$

By the Black-Scholes formula:

$-d_1 = 1.0778$ [0.5]

$-d_2 = 1.1778$ [0.5]

$N(-d_1) = 0.8594$ [0.5]

$N(-d_2) = 0.8806$ [0.5]

Therefore $P_0 = 9e^{-0.02\times0.25} \times 0.8806 - 8 \times 0.8594$ [1]

$= 1.01$ [1]

iii) As interest rates increase in the market, the expected return required by investors in stock tends to increase [0.5]However, the present value of any future cash flow generated by option contracts decreases [0.5] The combined impact of these two effects is to decrease the value of the put option [1] Rho is negative for a put option [0.5] put options become less valuable in times of increasing interest rates because they effectively defer the selling of a share and so delay access to the cash required to obtain the risk-free rate [0.5]

13.

(i) Let j denote the mean yield, then

$1 + j = \exp[\mu + \sigma 2/2] = 1.0757305$

$j = 0.0757305$

We require

$20{,}000\, E(X10) + 150{,}000\, E(S10)$

$= 20{,}000 s_{10} \square\square 150{,}000 (1 \square\square j)^{10}$ at rate j%

$= 20{,}000 \times 14.1961 + 311{,}261.98$

$= 595{,}183.99$

where $X10$ represents the accumulation after 10 years of INR1 p.a. paid in arrears for 10 years and $S10$ represents the accumulation after 10 years of INR1 paid now. **[5]**

(ii) We require $\Pr(Z . S10 \geq \square 600{,}000) = 0.99$

where Z = single amount paid now

$\Pr[S10 \geq \square 600{,}000 / z] = 0.99$

Now $\{\log S10 - 10\mu / \sigma\sqrt{10}\} \sim N(0,1)$

So we want

$\Phi\{\log(600{,}000/z) - 10\mu / \sigma\sqrt{10}\} = 0.01$

So, from tables,

$\log(600{,}000/z) - 10\mu / \sigma\sqrt{10} = -2.326$

So $600{,}000 / Z = \exp\square\square 2.326\, \sigma\sqrt{10} \square\square 10\mu\square\square = 1.139112$

$Z = 526{,}726.25$

[6]

[11 Marks]

14

The expected accumulated value can be calculated as:

$2{,}400\, E(S_5) = 3{,}600$

$2{,}400\, (1 + j)\, 5 = 3{,}600$

⇨ $j = 8.45\%$, where j is the mean rate of interest

The variance of accumulated value can be calculated as:

$\mathrm{var}(2{,}400\, S_5) = 2{,}400^2\, \mathrm{var}(S_5) = 50^2$

$\mathrm{var}(S_5) = 50^2 / 2400^2 = [\, (1 + j)^2 + s^2\,]^5 - (1 + j)^{10}$

$50^2 / 2400^2 = [1.0845^2 + s^2]^5 - 1.0845^{10}$

Solving for s, will give:

$s = 0.006735$, where s is variance of interest rate

As the interest rate follows*Gamma* (α, λ) distribution; we can determine its parameters:

Let i_k be the interest rate in year k.

$j = E(i_k) = \alpha \setminus \lambda = 0.0845$

$s^2 = \mathrm{var}(i_k) = \alpha \setminus \lambda^2 = 0.006735^2$

Solving for parameters:

$\alpha = 157.3$

$\lambda = 1862.23$

15.

(i) $J = 0.05 \text{x}\ 0.4 + 0.07 \text{ x } 0.2 + 0.09 \text{ x } 0.4$

$= 0.07$

Mean accumulation $= 10{,}000 \text{ x } (1 + j)^{10}$

$= 10{,}000 \text{ x } (1.07)^{10}$

$=$ Rs. 19671.5

(ii) $s^2 = 0.05^2 \text{ x } 0.4 + 0.07^2 \text{ x } 0.2 + 0.09^2 \text{ x } 0.4 - 0.07^2$

$= 0.00522 - 0.0049$

$= 0.00032$

Var (accumulation) $= 10{,}000^2 \{(1 + 2j + j2 + s2)^{10} - (1 + j)^{20}\}$

$= 10{,}000^2 \{1 + 2 \text{ x } 0.07 + 0.07^2 + 0.00032)^{10} - (1.07)^{20}\}$

$= 10{,}000^2 \{1.14522)^{10} - (1.07)^{20}\} \qquad = 1{,}082{,}939.70$

$SD(\text{accumulation}) = (1082939.70)^{1/2} =$ Rs.1040.64

(iii) By symmetry $j = 0.07$ (as in (i))

Hence, mean (accumulation) will be same as in (i) (i.e. Rs. 196715.51).

The spread of the yields around the mean is lower than in (i). Hence, the standard deviation of the accumulation will be lower than Rs.1040.64

16. Both models are:

- Continuous-time Markov models
- Ito processes
- One-factor models
- Usually defined in terms of a standard Brownian motion under risk-neutral probability measure

The SDEs defining the two models are similar:

- Vasicek: $dr(t) = \alpha[\mu - r(t)]dt + \sigma dW(t)$
- Hull-White: $dr(t) = \alpha[\mu(t) - r(t)]dt + \sigma dW(t)$

Additionally, both models:

- imply the short-rate is mean-reverting
- imply the future short rate has a normal distribution
- allow negative values for the short rate
- are mathematically tractable, although Hull-White model is algebraically a bit more complicated

Key differences:

Vasicek model is time homogenous (μ constant), but Hull-White model is not (μ time-dependent).

Hull-White model has to be calibrated to match the current pattern of bond prices.

Hull-White model can provide a better fit to historical data.

17.

i. $dr(t) = \alpha(\mu - r(t))dt + \sigma(\sqrt{r(t)}d\tilde{W}(t)$ **or**

$dr(t) = 0.2(0.08 - r(t))dt + 0.1(\sqrt{r(t)}d\tilde{W}(t)$

ii. Revised SDE is

$dr(t) = \alpha(\mu - r(t))dt + \sigma(\sqrt{r(t)}d\tilde{W}(t) + \varphi\, r(t)dt$

$dr(t) = (\alpha - \varphi)(\frac{\alpha\mu}{\alpha-\varphi} - r(t))dt + \sigma(\sqrt{r(t)}d\tilde{W}(t)$

$dr(t) = (0.14)(0.1143 - r(t))dt + 0.1(\sqrt{r(t)}d\tilde{W}(t)$

Hence the revised parameters are as follows:

α' = 0.14, μ' = 0.1143 σ' = 0.1

iii. Bond prices at time 5 and 10:

	Time 5	**Time 10**	**Marks**
$\theta = \sqrt{\alpha'^2 + 2\sigma'^2}$	0.199	0.199	1
$b(\tau) = \frac{2(e^{\theta\tau} - 1)}{(\theta + \alpha')(e^{\theta\tau} - 1) + 2\theta}$	3.494	4.975	2 (1 for each correct b(τ))
$a(\tau) = \frac{2\alpha'\mu'}{\sigma'^2}\ln\left(\frac{2\theta(e^{(\theta+\alpha')\tau/2})}{(\theta + \alpha')(e^{\theta\tau} - 1) + 2\theta}\right)$	-0.1581	-0.5058	2 (1 for each correct a(τ))
$\boldsymbol{B(t,T) = e^{a(\tau)-b(\tau)r(t)}}$	**0.6685**	**0.4257**	2 (1 for each correct B)

18.

1. (Equation of both the models)

2. Both are one factor model

3. BDT is not mean reverting whereas VM is.

4. Volatility parameter is constant for both the models.

5. Interest rates are strictly positive for BDT whereas they can be negative in VM.

6. Interest rates are log normally distributed for BDT whereas they are normally c
for VM.

7. Both the models are simple to calibrate.

8. Both the models cannot be used to price complex derivatives

19.

i) Similarities:

Both are one factor models.

Both have drifts that are deterministic

Both show some degree of local mean reversion

Differences

Black-Karasinski Model	Vasicek Model
Model given by $d(\ln r(t)) = k(t)(\theta(t) - \ln r(t))dt - \sigma(t)dW(t)$	$dr(t) = \alpha(\mu - r(t))dt - \sigma dW(t)$
ln r can take any value but r will always be positive	**The interest rates can go negative as per this model**
Local mean reversion of ln r. Hence r/r_0 shows local mean reversion.	**Local mean reversion of the interest rates**
Time- heterogeneous model	Time-homogenous model
Speed of mean reversion is time dependent	Speed of mean reversion is constant
Complex to model (in comparison to Vasicek Model)	Very easy to model

[5

ii) ln(r) follows a normal distribution hence 'r' follows a log-normal distribution.