



Class: M.Sc. – Sem 2

Subject : Financial Engineering

Chapter: Unit 1 Chapter 2

Chapter Name: Introduction to principles of derivative pricing

Today's Agenda

1. Introduction to Derivatives

1.1 Expectation Based Pricing

1.2 Logical Reasoning Based Pricing

1.3 Arbitrage Pricing

Introduction

- Derivatives are instruments which depend on the other - without the underlying (stock) there could be no future claims. However the connection between the two is complex and uncertain for both to trade efficiently in the same market.
- The apparently random nature of stocks filters through to the claims - they appear random too.
- The study of probability and expectation shows one way of coping with randomness.
- Probabilistic foundations will help to build the strongest possible links between derivative claims and their random underlying stocks.

1.1 Expectation based pricing



Consider playing the following game:

Someone tosses a coin and pays you one dollar for heads and nothing for tails.

What price should you pay for this prize?



1.2 Through Logical Reasoning



- If the coin is fair, then heads and tails are equally likely – about half the time you should win the dollar and the rest of the time you should receive nothing.
- Over enough plays, then, you expect to make about fifty cents a go.
- So paying more than fifty cents seems extravagant and less than fifty cents looks extravagant for the person offering the game.
- Fifty cents, then, seems about right.

1.3 Through Mathematical Expectation



- A probabilistic analysis of the game would observe that although the outcome of each coin toss is essentially random, we could develop a non-random structure to the game.
- We could assume that there was a fixed measure of likelihood attached to the coin tossing, a probability of the coin landing heads or tails of $\frac{1}{2}$.
- Along with the probability there comes the idea of expectation, in this discrete case, the total of each outcome's value weighted by its attached probability. The expected payoff in the game is $\$1 \times \frac{1}{2} + \$0 \times \frac{1}{2} = \$0.50$.
- On this basis, let's look at average profit or loss.

Average Profit or Loss



- If the arithmetical average of outcomes tends towards the mathematical expectation with certainty, then the average profit/loss per game tends towards the mathematical expectation less the price paid to play the game.
- If this difference is positive, then in the long run it is certain that you will end up in profit. And if it is negative, then you land up with overall loss with certainty.
- In the short term of course, nothing can be guaranteed, but over time, expectation wins out.
- Fifty cents is a fair price in this sense.

Is this an enforceable price?



Suppose someone offered you a play of the game for 40 cents in the dollar, but instead of allowing you a number of plays, gave you just one for an arbitrarily large payoff.



Not a rational thought



The strong law of large trails lets you take advantage of them over repeated plays.

40 cents a dollar looks good if we have large number of trails.

But what if there is only one play, it then leaves us to a question mark!!

Thus, this approach fails when it comes to this point. Hence we need to look out for another approach to pricing.

Stocks, Not coins



- What about real stock prices in a real financial market? One widely accepted model holds that stock prices are log-normally distributed.
- Suppose, now, that we have some claim on this stock, some contract that agrees to pay certain amounts of money in certain situations - just as the coin game did. The most natural claim on a stock is the forward: two parties enter into a contract whereby one agrees to give the other the stock at some agreed point in the future in exchange for an amount agreed now.
- The 'pricing question' for the forward stock 'game' is:

What amount should be written into the contract now to pay for the stock one year in the future?

In Formal Notations



- In formal notation - the stock price at time T is given by S_T and the forward payment written into the contract is K , thus the value of the contract at its expiry, that is when the stock transfer actually takes place, is $S_T - K$.
- The time value of money tells us that the value of this claim as of now is $\exp(-rT)(S_T - K)$.
- The strong law (expectation pricing) suggests that the expected value of this random amount, $E(\exp(-rT)(S_T - K))$, should be zero. If it is positive or negative, then long-term use of that pricing should lead to one side's profit.
- Thus one apparently reasonable answer to the pricing question says K should be set so that $E(\exp(-rT)(S_T - K)) = 0$, which happens when $K = E(S_T)$
- Just as with the coin game, this price can only be a suggestion as to the market's trading level and it is not an enforceable price. Also the technique will clearly work for more than just forwards, not when only one forward is considered.

1.3 Arbitrage Pricing



- The price we have just determined for the forward could only be the market price by an unfortunate coincidence. With markets where the stock can be bought and sold freely and arbitrary positive and negative amounts of stock can be maintained without cost, trying to trade forward using the strong law, could lead to disaster - in most cases there would be unlimited interest in selling forward to you at that price.
- Why does the expectation pricing fail so badly with forwards? As mentioned above in the context of the coin game, the strong law cannot enforce a price, it only suggests.
- And in this case, another completely different mechanism does enforce a price.
- The fair price of the contract is $S_0 \exp(rT)$.

Why $S_0 \exp(rT)$?



The Strategy!

- Consider the seller of the contract, obliged to deliver the stock at time T in exchange for some agreed amount. They could borrow S_0 now, buy the stock with it, put the stock in a drawer and just wait. When the contract expires, they have to pay back the loan - which if the continuously compounded rate is r means paying back $S_0 \exp(rT)$, but they have the stock ready to deliver.
- If they wrote less than $S_0 \exp(rT)$ into the contract as the amount for forward payment, then they would lose money *with certainty*.
- But of course, the buyer of the contract can run the scheme in reverse, thus writing *more* than $S_0 \exp(rT)$ into the contract would guarantee them a loss.
- Thus there is an *enforced* price of $S_0 \exp(rT)$.

Arbitrage Price



- Any attempt to strike a different price and offer it into a market would inevitably lead to someone taking advantage of the free money available via the construction procedure. This type of market opportunism is called - arbitrage.
- The price of $S_0 \exp(rT)$ is an arbitrage price - it is justified because any other price could lead to unlimited riskless profits for one party.
- To put it simply, if there is an arbitrage price, any other price is too dangerous to quote. This price can work for one contract and even for n number of contracts.
- All derivatives can be built from the underlying arbitrage pricing theory.

Quick Recap

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