Lecture 3

INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Class: M.Sc. – Sem 2

Subject : Financial Engineering

Chapter: Unit 2 Chapter 1

Chapter Name: Discrete Approach to Pricing

Today's Agenda

1. Binomial Branch Model

1.1 Assumptions of the Binomial Model

1.2 Study the One Period Binomial Model

1.3 Study the two period Binomial Model

1.4 Study the Multiple Period Binomial Model

1.4 Recombining and non recombining tree

2.1 Binomial Representation Theorem

3 Expand the application of the binomial model to the continuous approaches

3.1 The State Price Deflator Approach

Introduction

- We start to develop simple models that can be used to value derivatives. In particular, we use binomial trees or lattices to find the value at time 0 of a derivative contract that provides a payoff at a future date based on the value of a non-dividend-paying share at that future date.
- The analysis throughout this applies the no-arbitrage principle in order to value derivative contracts.

1.1 Assumptions of the Binomial model

In the binomial model it is assumed that:

- there are no trading costs
- there are no taxes
- there are no minimum or maximum units of trading
- stock and bonds can only be bought and sold at discrete times 1, 2, ...
- the principle of no arbitrage applies

A stock and A bond

We at least need something to represent randomness of the stock and something to represent the time-value of money. Any model without them cannot begin to claim any relation to the real financial market.

Consider, then, the simplest possible model with a stock and a bond/cash.

❑ The Stock

- We will use $\boldsymbol{S_t}$ to represent the price of a non-dividend-paying stock at discrete time intervals t (t = 0,1,2,...) . For t > 0, $\boldsymbol{S_t}$ is random.
- Note that in this instance 'stock' specifically means a share or equity as opposed to a bond.
- The stock price $\boldsymbol{S_t}$ is assumed to be a random or stochastic process. So we assume that only two things can happen to the stock in this time: an 'up' move or a 'down' move. Our randomness will have some structure - we will assign probabilities to the up and down move

A stock and A bond

- The Bond

- We also need something to represent the time-value of money - a cash or a bond. The account will value B_t at time *t* per unit invested at time 0. This account is assumed to be risk-free and we will assume that it earns interest at the constant risk-free continuously compounding rate of *r* per annum. Thus, $B_t = e^{rt}$
- B_t increases in an entirely predictable manner as we move through time, *ie* at the continuously compounding rate of *r* per time period.
- At all points in time there are no constraints (positive or negative) on how much we can hold in stock or cash.

1.2 The One-Period Model

The aim in this chapter is to find the value at an arbitrary time 0 – *ie now* – of a derivative that provides a payoff at some future date based on the value of the stock at that future date. As a starting point, we consider a one-period binomial model.

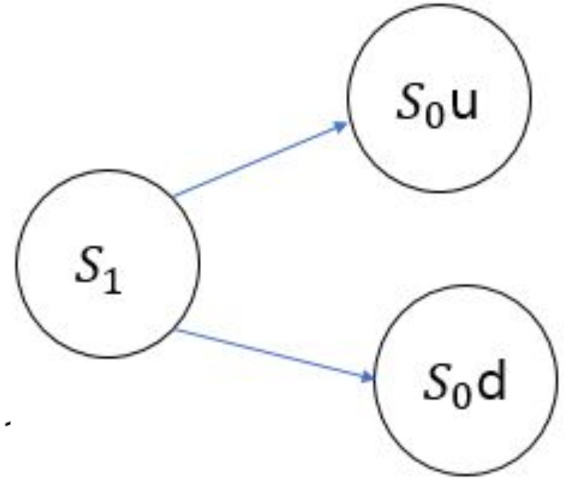
The price of the stock at time o = S_0. As per we assumed, We have two possibilities for the price of the stock at time 1:

S_1 = S_0u - if the price goes up

= S_0d - if the price goes down.

Here *u* is a fixed number bigger than 1 and *d* is a fixed number less than 1

This binomial branch can be represented as:



No Arbitrage

In order to avoid arbitrage we must have $d < e^r < u$.

Task!
If the above condition is not ensured, discuss how this will lead to arbitrage.

Determination of the derivative price

Suppose that we have a derivative which pays *cu* if the price of the underlying stock goes up and *cd* if the price of the underlying stock goes down.

At what price should this derivative trade at time 0?

At time 0, suppose that we hold φ units of stock and ψ units of cash. The value of this portfolio at time 0 is V_0.

- $V_0 = \varphi S_0 + \psi$

One tick later, it would be worth one of two possible values:

- $V_1 = \varphi S_0 u + \psi e^r$ - if stock goes up

  $= \varphi S_0 d + \psi e^r$ - if stock goes down

Determination of the derivative price

Let us choose φ and ψ so that V_1 = *cu* if the stock price goes up and V_1 = *cd* if the stock price goes down. Then:

Cu = $\varphi \boldsymbol{S_0} u + \psi \boldsymbol{e^r}$

Cd = $\varphi \boldsymbol{S_0} d + \psi \boldsymbol{e^r}$

We have two linear equations in two unknowns. We solve this system of equations and find that:

$$\varphi = \frac{Cu - Cd}{S_0(u-d)}$$

$$\psi = e^{-r} \left(\frac{Cd \times u - Cu \times d}{u - d}\right)$$

Determination of the derivative price

Substituting the values for φ and ψ derived above in the equation $V_0 = φS_0 + ψ$ and then simplifying we get,

- $\boldsymbol{V_0} = \boldsymbol{e^{-r}}$ (q x cu + (1 - q) x cd)

where q = $\frac{e^r - d}{u - d}$

Note that the no-arbitrage condition $d < e^r < u$ ensures that $0 < q < 1$.

If we denote the payoff of the derivative at t =1 by the random variable $\boldsymbol{C_1}$, we can write:

- $\boldsymbol{V_0 = e^{-r}\, E_Q[C_1]}$

where *Q* is an artificial probability measure which gives probability *q* to an upward move in prices and 1 - *q* to a downward move. We can see that *q* depends only upon *u*, *d* and *r* and not upon the potential derivative payoffs *cu* and *cd*. Q is not a real-world probability.

Question!

Let's consider a one-period binomial model of a stock whose current price is 40. Suppose that:

- over the single period under consideration, the stock price can either move up to 60 or down to 30
- the actual probability of an up-movement is equal to ½
- the continuously compounded risk-free rate of return is 5% per time period

Find the current value of a one-period European call option, V_0 , that has an exercise price of 45.

Solution

Tree looks like:

$$\begin{tikzcd}[row sep=small] & S_1 = 60 \\ S_0 = 40 \arrow[ur] \arrow[dr] & \\ & S_1 = 30 \end{tikzcd}$$

The risk neutral probabilities and thus option price are:

$$d = \frac{30}{40} = 0.75$$

and: $$u = \frac{60}{40} = 1.5$$

So: $$q = \frac{e^{0.05} - 0.75}{1.5 - 0.75} = 0.40169$$

and: $$1 - q = 0.59831$$

Hence:

$$V_0 = e^{-0.05}\left[0.40169 \times 15 + 0.59831 \times 0\right] = 5.732$$

ie $$C_0 = 5.732$$

Replicating Portfolio

- A replicating portfolio will always precisely reproduce the relevant payoff or cash flow.
- The portfolio (φ ,ψ) is called a replicating portfolio because it replicates, precisely, the payoff at time 1 on the derivative without any risk.
- In other words, $\mathbf{V_1}$ = cu if the stock price goes up and $\mathbf{V_1}$ = cd if the stock price goes down. So, there is never any risk or possibility that $\mathbf{V_1} \neq \mathbf{C_1}$.
- Hence, in a world that is free of arbitrage, the values of the derivative and the replicating portfolio (φ ,ψ) at time 0 must be equal – ie they are both equal to

$$\mathbf{V_0} = \mathbf{e^{-r}} \, \mathbf{E_Q[C_1]}$$

Real-World probability measure P

- Up until now we have not mentioned the real-world probabilities of up and down moves in prices. Let these be p and 1 - p where $0 < p < 1$, defining a probability measure P.
- So P is a set of probabilities that assigns the actual or real-world probability p of an upward jump in the stock price to an up-branch of the binomial tree and 1– p to a down-branch.
- Other than by total coincidence, p will not be equal to q.
- Let us consider the expected stock price at time 1. Under P this is:

$$\mathbf{E_P}\,[\,\mathbf{S_1}] = \mathbf{S_0}\,(pu + (1\text{-}p)d)$$

- This is the expectation of the stock price at time 1 with respect to the real-world probability measure P

Risk-Neutral probability measure Q

Q is the risk-neutral probability measure. Let's see why!

Under Q the expected stock price at time 1 is

$$\mathbf{E_Q}\,[\,\mathbf{S_1}] = \mathbf{S_0}\;(qu + (1\text{-}q)d) = \mathbf{S_0}\;\left(u\,\frac{\mathbf{e^r - d}}{\mathbf{u - d}} + d\,\frac{\mathbf{u - e^r}}{\mathbf{u - d}}\right) \;=\; \mathbf{S_0 e^r}$$

Under Q we see that the expected return on the risky stock is the same as that on a risk-free investment in cash.

In other words, under the probability measure Q, investors are neutral with regard to risk: they require no additional return for taking on more risk.

This is why Q is sometimes referred to as a risk-neutral probability measure.

Task!
What will ideally be the relation between p and q?

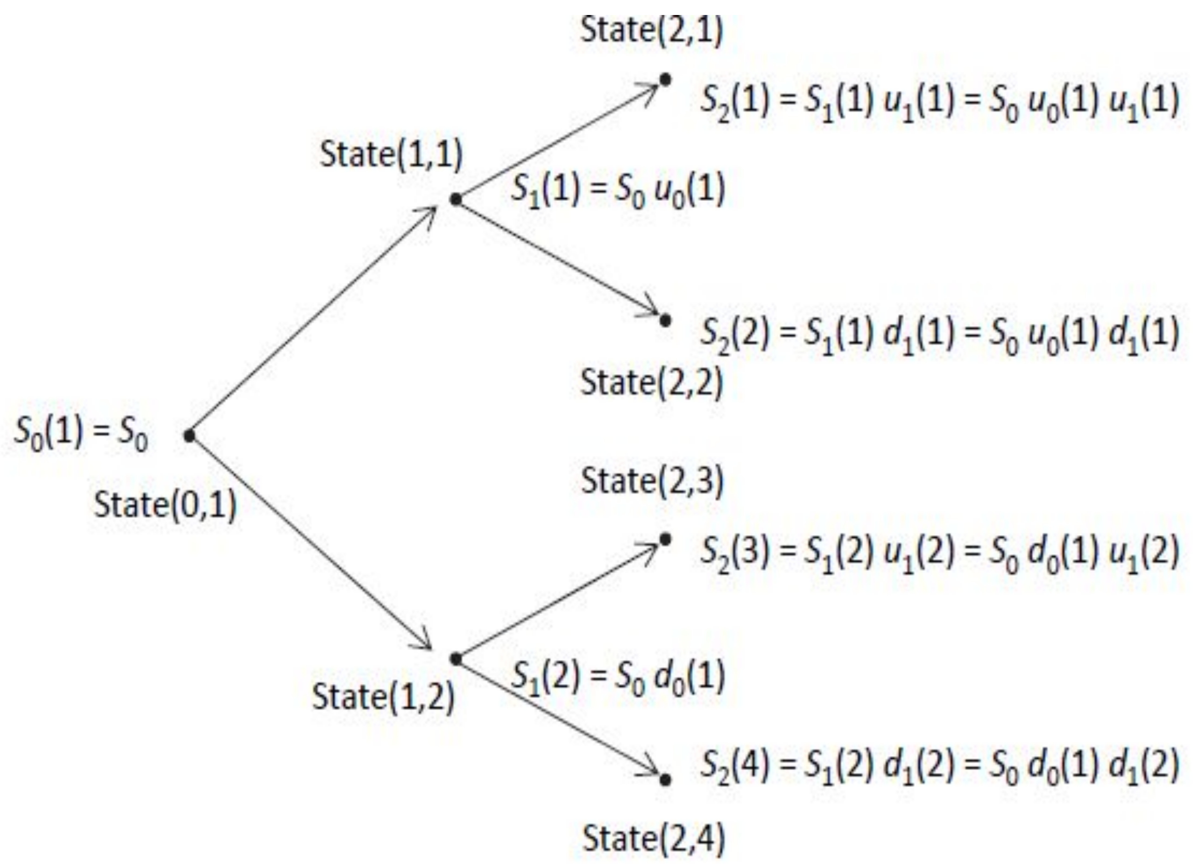
1.3 Two-Period Binomial Model

The next step is to develop a two-period binomial model. It turns out that the ideas discussed above carry over quite naturally to the two-period model.

The most general model is as follows:

Two-period binomial model. Where the price at a particular node is denoted S_t(j) , this means the price in state (t,j) .

The subscripts here denote time and the arguments in brackets denote the vertical position, counting from the top of the tree.



Determination of derivative price

Period 1:

- $S_0 = S_0 u_0(1)$ - if up, to state (1,1)

  $= S_0 d_0(1)$ - if down, to state (1,2)

Period 2: From state (1,1) , *ie* following a price *increase* in the first time interval:

- $S_0 u_0(1) = S_0 u_0(1)\, u_1(1)$ - if up, to state (2,1)

  $= S_0 d_0(1)\, d_1(1)$ - if down, to state (2,2)

Period 2: From state (1,2) , *ie* following a price *decrease* in the first time interval:

- $S_0 d_0(1) = S_0 d_0(1)\, u_1(2)$ - if up, to state (2,3

  $= S_0 d_0(1)\, d_1(2)$ - if down, to state (2,4)

Determination of derivative price

Suppose that the derivative gives a payoff at time 2 of $c_2(j)$ if the price of the stock at time 2 is in state (2,j)

- How do we calculate the price of the derivative at time 0?

We first find the intermediate price at time 1 and then use this price to find the price at time 0. We do this by working backwards from time 2. So we calculate the value of the contract at time 1 for each of the possible states at time 1.

Let V_1(j) be the value of the contract if we are in state *j* at time 1. Then, by analogy with the one-period model:

- $\boldsymbol{V_1(1) = e^{-r}\,(q_1(1)\,c_2(1) + (1 - q_1(1))\,c_2(2))}$
- $\boldsymbol{V_1(2) = e^{-r}\,(q_1(2)\,c_2(3) + (1 - q_1(2))\,c_2(4))}$

where $q_1(1) = \dfrac{e^r - d_1(1)}{u_1(1) - d_1(1)}$ and $q_1(2) = \dfrac{e^r - d_1(2)}{u_1(2) - d_1(2)}$

Determination of derivative price

The price at time 0 is found by treating the values $\boldsymbol{V_1(1)}$ and $\boldsymbol{V_1(2)}$ in the same way as derivative payoffs at time 1.

So:

- $\boldsymbol{V_0}(1) = \boldsymbol{e^{-r}}\,(\boldsymbol{q_0}(1)\,\boldsymbol{V_1(1)} + (1 - \boldsymbol{q_0}(1))\,\boldsymbol{V_1(2)})$

We can generalize these results to n-periods.

Determination of derivative price

- Let C_2 be the random derivative payoff at time 2 (that is, it takes one of the values c_2(j) for *j* = 1,2,3,4 .. Let V_t be the random value of the contract at time *t*. Let F_t be the history of the process up to and including time *t*. Let *Q* be the risk-neutral probability measure.
- So, V_t(j) , the value of the derivative contract if we are in State *j* at time *t*, must depend on *Ft* .
- Then: $\boldsymbol{V_1 = e^{-r}\, E_Q[C_2|F_1]}$
- Likewise:

$$\boldsymbol{V_0 = e^{-r}\, E_Q[V_1|F_0]}$$

$$\boldsymbol{= e^{-r}\, E_Q\,[e^{-r}\, E_Q\{C_2|F_1\}|F_0]}$$

$$\boldsymbol{= e^{-2r}\, E_Q\,[C_2|F_0]}$$

Question

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Consider a two-period Binomial model of a stock whose current price $S0 = 100$. Suppose that:

- over each of the next two periods, the stock price can either move up by 10% or move down by 10%
- the continuously compounded risk-free rate is $r = 8\%$ per period

1) Show that there is no arbitrage in the market.

2) Calculate the price of a one-year European call option with a strike price $K = 100$.

1.4 N-period Binomial tree

All the results derived above can be extended to n-periods in a similar fashion. In this case:

- the risk-neutral up-step probability from State (t, j) is: $q_t(j) = \frac{e^r - d_t(j)}{u_t(j) - d_t(j)}$
- the expectation of the stock price in *n* periods' time, calculated with respect to the risk-neutral measure *Q*, is equal to the current stock price, accumulated at the continuously compounded risk-free rate of return over those *n* periods, ie $E_Q\left[S_{t+n}|F_t\right] = S_t\, e^{rn}$
- the derivative price at time *t* is $V_t = e^{-r(n-t)}\, E_Q[C_n|F_t]$
- the number of units of stock in the replicating portfolio when in State(t, j) at time *t* is: $\varphi_{t+1}(j) = \frac{V_{t+1}(2j-1) - V_{t+1}(2j)}{S_t(j)(u_t(j) - d_t(j))}$ and the amount held in cash in the replicating portfolio when in State (t, j) at time *t* is: $\psi_{t+1}(j) = e^{-r}\left(\frac{V_{t+1}(2j)u_t(j) - V_{t+1}(2j-1)d_t(j)}{(u_t(j) - d_t(j))}\right)$

1.5 Recombining Binomial Tree

Suppose now that we assume the sizes of the up-steps and down-steps are the same in all states. That is:

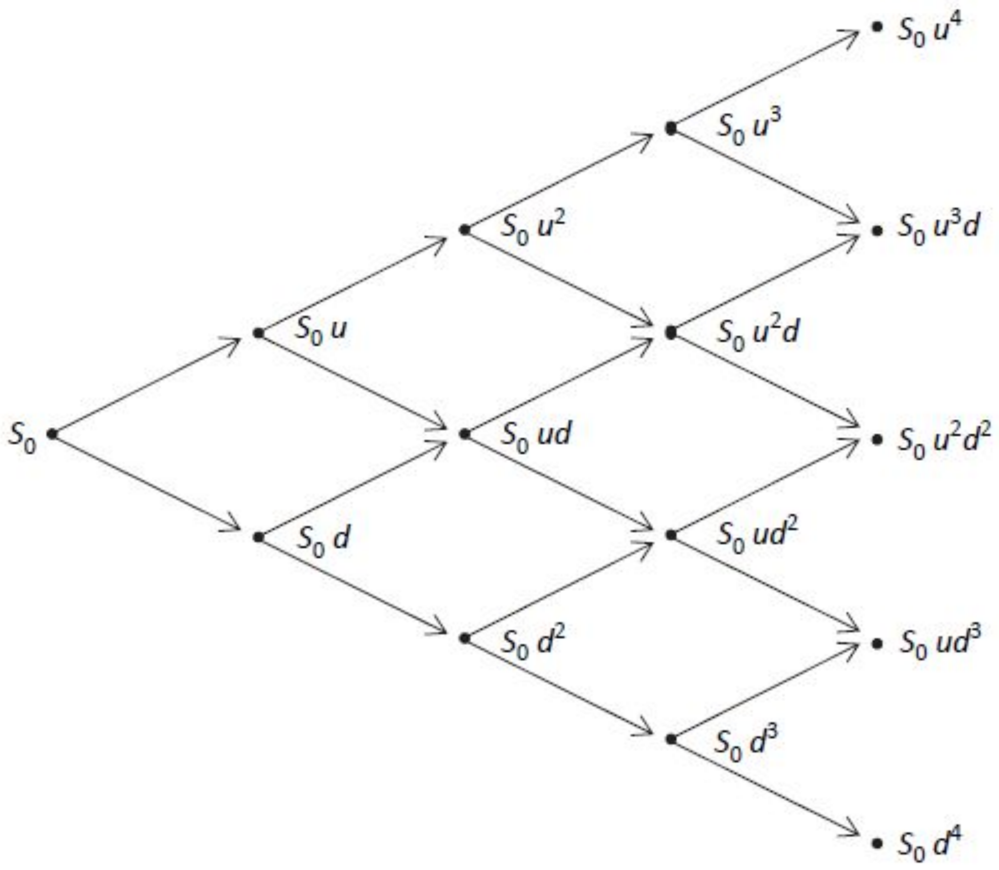
u_t(j) = u and d_t(j) = d . Also q_t(j) = q for all t, j.

Then we have:

- $\boldsymbol{S_t = S_0\, u^{N_t}\, d^{t - N_t}}$

where N_t is the number of up-steps between time 0 and time *t*. This means that we have *n* + 1 possible states at time *n*.

This special form for the *n*-period model allows us to call it a *recombining* binomial tree or a *binomial*



2.1 Binomial Distribution

Under this model the *q*-probabilities are, as stated above, equal, and all steps are made independent of one another. So the number of up-steps up to time *t* , *Nt* , has a binomial distribution with parameters *t* and *q*.

Furthermore, for $0 < t < n$:

- *Nt* is independent of *Nn* – *Nt*
- and *Nn* - *Nt* has a binomial distribution with parameters *n*- *t* and *q*.

The price at time *t* of the derivative is:

- $V_t = e^{-r(n-t)} \sum_{k=0}^{n-t} f(S_t\, u^k\, d^{n-t-k}) \frac{(n-t)!}{k!(n-t-k)!}\; q^k\, (1-q)^{n-t-k}$

Question

A company share price is to be modelled using a **5-step recombining binomial tree**, with each step in the tree representing one day. Each day, it is assumed that the share price: **increases by 2%, or decreases by 1%**. Assume that the **risk-free force of interest is 5.5%** *pa* and that there are **365 days in a year. No dividends are to be paid over the next five days.**

(i) Calculate the risk-neutral probability of an up-step on any given day.

(ii) Calculate the fair price of a 5-day at-the-money call option on £10,000 worth of shares in this company.

A special option is available where the payoff after 5 days is: max { S_5^* -K, 0 } where S_5^* is the arithmetic average share price recorded at the end of each of the 5 days and *K* is the strike price.

(iii) Calculate the fair price of the special option (strike price $K = 1.06S0$) on £10,000 worth of shares in this company.

Solution

(i) ***Risk-neutral probability***

Using the formula for the risk-neutral up-step probability on page 45 of the *Tables*, we have:

$$q = \frac{e^{0.055/365} - 0.99}{1.02 - 0.99} = 0.338357$$

Solution

(ii) *Call option value*

For simplicity, we can assume that we are valuing one call option on a share worth £10,000. We can use the up and down steps to calculate the six possible final share prices in this binomial tree:

$$S_5 = £10{,}000 \times 1.02^5 \times 0.99^0 = £11{,}040.81 \quad \text{for 5 up jumps}$$
$$S_5 = £10{,}000 \times 1.02^4 \times 0.99^1 = £10{,}716.08 \quad \text{for 4 up jumps}$$
$$S_5 = £10{,}000 \times 1.02^3 \times 0.99^2 = £10{,}400.90 \quad \text{for 3 up jumps}$$
$$S_5 = £10{,}000 \times 1.02^2 \times 0.99^3 = £10{,}094.99 \quad \text{for 2 up jumps}$$
$$S_5 = £10{,}000 \times 1.02^1 \times 0.99^4 = £9{,}798.08 \quad \text{for 1 up jump}$$
$$S_5 = £10{,}000 \times 1.02^0 \times 0.99^5 = £9{,}509.90 \quad \text{for 0 up jumps}$$

[2]

Since the option is at-the-money, $K = £10{,}000$, so the payoffs from the call option are:

£1,040.81	for 5 up jumps
£716.08	for 4 up jumps
£400.90	for 3 up jumps
£94.99	for 2 up jumps
£0	for 1 up jump
£0	for 0 up jumps

[1]

So, the fair price of the call option is:

$$V_0 = e^{-5\times 0.055/365} \left\{ \begin{aligned} &1{,}040.81q^5 + 716.08 \times 5q^4(1-q) \\ &+400.90 \times 10q^3(1-q)^2 + 94.99 \times 10q^2(1-q)^3 \end{aligned} \right\}$$

[1]

Substituting in the value of q, this is:

$$V_0 = 0.999247\{4.62 + 31.05 + 67.98 + 31.50\}$$
$$= £135.05$$

[1]

[Total 5]

Solution

(iii) ***Special option value***

Again, for simplicity, we can assume that we are valuing one special option on a share worth £10,000. By investigation, we find that the average share price only exceeds £10,600 if the share price goes up on all 5 days. If we try 4 up jumps and 1 down jump, we get:

$$\frac{1.02+1.02^2+1.02^3+1.02^4+1.02^4\times 0.99}{5}=1.055<1.06 \qquad [1]$$

The average share price over the 5 days, if it goes up every day, is:

$$£10{,}000\times\frac{1.02+1.02^2+1.02^3+1.02^4+1.02^5}{5}=£10{,}616.24 \qquad [1]$$

The payoff of the special option in this case is:

$$\max\{10{,}616.24-10{,}600,0\}=£16.24 \qquad [1]$$

So, the fair price of the special option is:

$$V_0=e^{-5\times 0.055/365}\times 16.24q^5$$

$$=£0.07 \qquad [1]$$

[Total 4]

Pricing of American Options - Question

The market price of a security can be modelled by assuming that it will either increase by 12% or decrease by 15% each month, independently of the price movement in other months. No dividends are payable during the next two months. The continuously compounded monthly risk free rate of interest is 1%. The current market price of the security is 127.

(i) Use the binomial model to calculate the value of a two-month European put option on the security with a strike price of 125.

(ii) Calculate the value of a two-month American put option on the same security with the same strike price.

(iii) Calculate the value of a two-month American call option on the same security with the same strike price.

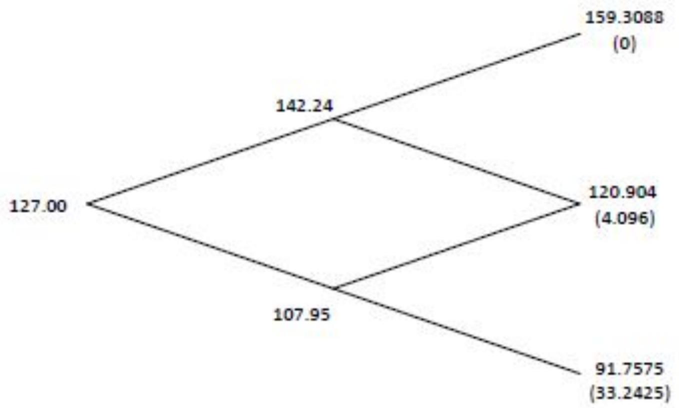
Solution

(i) ***Calculate the value of the European put option***

We are given $u = 1.12$, $d = 0.85$ and $r = 0.01$ in the question. The risk-neutral probability under the binomial model is:

$$q = \frac{e^r - d}{u - d} = \frac{e^{0.01} - 0.85}{1.12 - 0.85} = 0.5928$$

The binomial tree of security prices looks like this, with the final put payoffs



We can discount the expected value of the payoff under the risk-neutral probability to find the initial value of the European put option:

$$e^{-2r}\left(q^2 p_{uu} + 2q(1-q)p_{ud} + (1-q)^2 p_{dd}\right)$$

$$= e^{-0.02}\left(0 + 2 \times 0.5928 \times 0.4072 \times 4.096 + 0.4072^2 \times 33.2425\right)$$

$$= 7.342$$

3.1 The state price deflator approach

Here we will present a different, but equivalent, approach to pricing, this approach is a real-world approach to pricing. It uses the real-world probabilities – "p" while doing the calculations.

So far we have been pricing derivatives on a risk-neutral basis. Consider the period case where we have:

$$\boldsymbol{V_0} = \boldsymbol{e^{-r}}\, \boldsymbol{E_Q[C_1]} = \boldsymbol{e^{-r}}\,(\text{q x cu} + (1 - \text{q}) \text{ x cd})$$

We can re-express this value in terms of the real-world probability p :

$$\boldsymbol{V_0} = \boldsymbol{e^{-r}}\,(\text{p x}\frac{\boldsymbol{q}}{\boldsymbol{p}}\text{ x cu} + (1 - \text{p}) \text{ x} \frac{\boldsymbol{(1-q)}}{\boldsymbol{(1-p)}} \text{ x cd})$$

$$= \boldsymbol{E_P}\,[\,A_1\,V_1\,]$$

where A_1 is a random variable with:

$$A_1 = \boldsymbol{e^{-r}}\,\frac{\boldsymbol{q}}{\boldsymbol{p}} \qquad \text{if } S_1 = S_0\text{u}$$

The state price deflator approach

Thus in this case we have real-world probabilities and a different discount factor. The discount factor $\mathbf{A_1}$ depends on whether the share price goes up or goes down. This means it is random and so we call it a stochastic discount factor.

$\mathbf{A_1}$ is called a state price deflator. It also has a variety of other names:

deflator

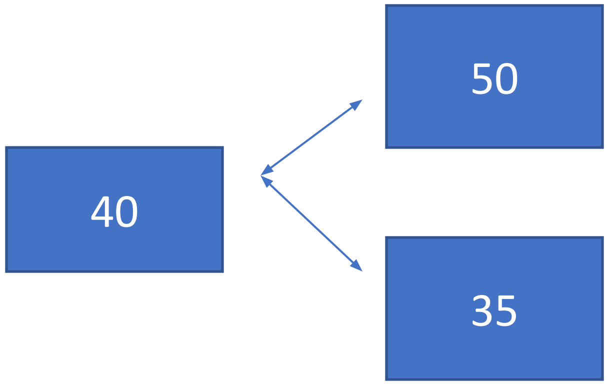
state price density

pricing kernel

stochastic discount factor.

Question

Find the no of shares and cash required to replicate the derivative payoff for a put option with strike price of 50 for the share price process. Risk free rate is 5%



Solution

Payoff	Share Price	
0	50 (So*u)	Upmove
15	35 (So*d)	Downmove

Let ψ *be the amount of cash and* ϕ *be the number of shares*

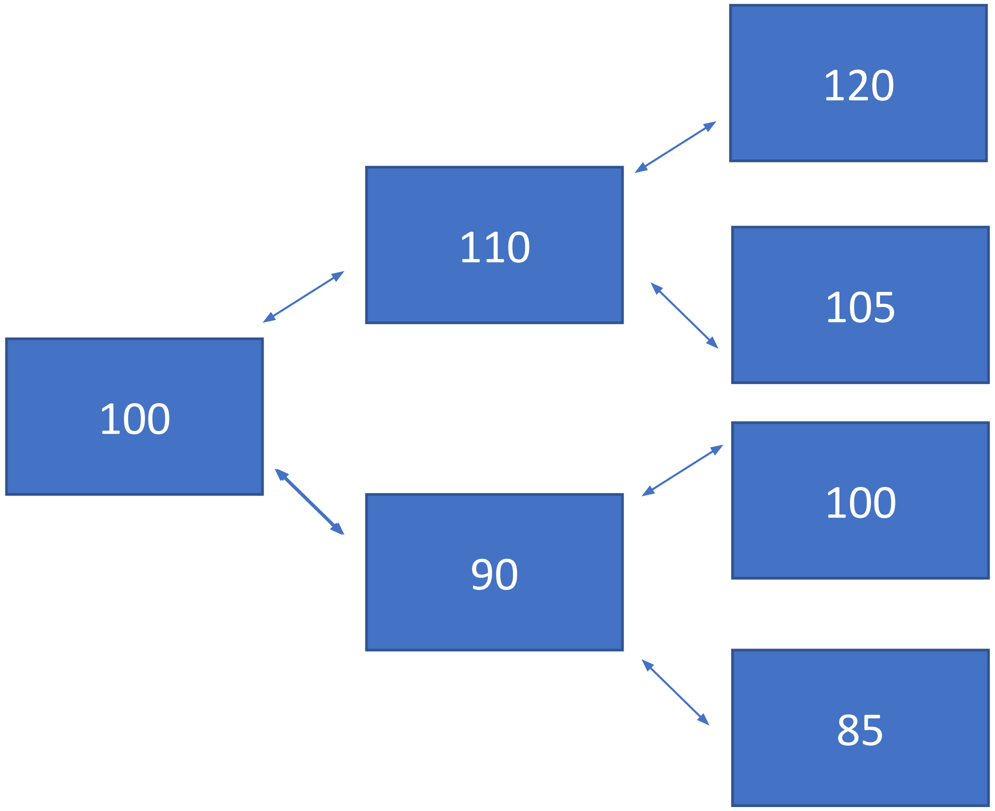
0 = $\phi * So * u + \psi * e^r$

15 = $\phi * So * d + \psi * e^r$

Solving the above equation we get $\phi =$ -1 , $\psi = 47.56$

Example 2 (Homework)

Find the no of shares and cash required to replicate the derivative payoff for a call option with strike price of 100 for the share price process. Risk free rate is 5%



Question

The movement of a share price over the next two months is to be modelled using a two-period recombining binomial model. Over each month, it is assumed that the share price will either increase or decrease by 10%.

(i) Over each month, the risk-neutral probability of an up-step is $q = 0.55$. Calculate the monthly risk-free force of interest r that has been used to arrive at this figure.

(ii) The current share price is 1. The annualised expected force of return on the share is $\mu = 30\%$. Calculate the state-price deflators in each of the three possible final states of the share price.

Question contd

(iii) Calculate the value of each of the following two-month derivatives:

(a) a derivative with payoff profile (1,0,0)

(b) a derivative with payoff profile (0,1,0)

(c) a derivative with payoff profile (0,0,1(

(d) a European call option with a strike price of K = 0.95

(e) a European put option with a strike price of K = 1.05

(f) a derivative whose payoff is 2x|S-0.98| , where S is the share price at the end of the two months.

Note - A payoff profile of (x,y,z) means that the derivative returns x if the share price goes up twice, y if the share price goes up once and down once, and z if the share price goes down twice

Solution

(i) *The risk-free force of interest r*

Under the risk-neutral probability measure we have:

$$E_Q[S_1] = S_0 e^r$$

$$\Leftrightarrow \quad q \times 1.1 S_0 + (1-q) \times 0.9 S_0 = S_0 e^r$$

Dividing through by S_0, substituting in the given value of q and solving for r gives:

$$r = \log(0.55 \times 1.1 + 0.45 \times 0.9)$$

$$= \log(1.01) = 0.995\%$$

(ii) *State price deflators*

The state price deflators for each of the three final possible states can be derived from the formulae:

$$A_2(1) = e^{-2r}\frac{q^2}{p^2} \qquad A_2(2) = e^{-2r}\frac{2q(1-q)}{2p(1-p)} \qquad A_2(3) = e^{-2r}\frac{(1-q)^2}{(1-p)^2}$$ [:

We can use the annualised expected force of return on the share, μ, to calculate the real-world probability of an up-step p:

$$E_P[S_1] = S_0 e^{\mu/12}$$

$$\Leftrightarrow \quad p \times 1.1 + (1-p) \times 0.9 = e^{\mu/12} = e^{0.3/12}$$

$$\Leftrightarrow \quad p = \frac{e^{0.3/12} - 0.9}{1.1 - 0.9} = 0.62658$$ [:

Using this value of p, and noting that, since $r = \log(1.01)$, the monthly effective risk-free interest rate is 1%, the state price deflators are:

$$A_2(1) = \frac{1}{1.01^2}\frac{0.55^2}{0.62658^2} = 0.75533$$

$$A_2(2) = \frac{1}{1.01^2}\frac{2 \times 0.55 \times 0.45}{2 \times 0.62658 \times 0.37342} = 1.03695$$

$$A_2(3) = \frac{1}{1.01^2}\frac{0.45^2}{0.37342^2} = 1.42356$$ [:

Solution

(iii)(a) *Payoff* $(1,0,0)$

$$e^{-2r}q^2 = \frac{0.55^2}{1.01^2} = 0.2965$$

(iii)(b) *Payoff* $(0,1,0)$

$$e^{-2r}2q(1-q) = \frac{2\times0.55\times0.45}{1.01^2} = 0.4852$$

(iii)(c) *Payoff* $(0,0,1)$

$$e^{-2r}(1-q)^2 = \frac{0.45^2}{1.01^2} = 0.1985$$

We could alternatively have calculated the values in (a), (b) and (c) using

$$A_2(1)\times p^2 \ , \ A_2(2)\times 2p(1-p) \ , \ A_2(3)\times(1-p)^2$$

Solution

(iii)(d) *Call option, strike price* $K = 0.95$

Depending on the final price of the share, the payoff for this option will be:

$$\max(1\times1.1^2 - 0.95, 0) = 0.26,$$

$$\max(1\times1.1\times0.9 - 0.95, 0) = 0.04, \text{ or}$$

$$\max(1\times0.9^2 - 0.95, 0) = 0.$$

So, the value is:

$$0.26\times0.2965 + 0.04\times0.4852 = 0.0965$$

(iii)(e) *Put option, strike price* $K = 1.05$

Depending on the final price of the share, the payoff for this option will be:

$$\max(1.05 - 1\times1.1^2, 0) = 0,$$

$$\max(1.05 - 1\times1.1\times0.9, 0) = 0.06, \text{ or}$$

$$\max(1.05 - 1\times0.9^2, 0) = 0.24.$$

So, the value is:

$$0.06\times0.4852 + 0.24\times0.1985 = 0.0768$$

(iii)(f) *Payoff* $2\times|S - 0.98|$

Depending on the final price of the share, the payoff for this option will be:

$$2\times\left|1\times1.1^2 - 0.98\right| = 0.46,$$

$$2\times\left|1\times1.1\times0.9 - 0.98\right| = 0.02, \text{ or}$$

$$2\times\left|1\times0.9^2 - 0.98\right| = 0.34.$$

So, the value is:

$$0.46\times0.2965 + 0.02\times0.4852 + 0.34\times0.1985 = 0.2136$$