Lecture 1

INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Class: M.Sc. – Sem 2

Subject : Financial Engineering

Chapter: Unit 2 Chapter 1

Chapter Name : Wiener Processes and ITO's Lemma

Stochastic Process

What is a continuous process? Three small-scale principles guide us.

- Firstly, the value can change at any time and from moment to moment.
- Secondly, the actual values taken can be expressed in arbitrarily fine fractions - any real number can be taken as a value.
- And lastly the process changes continuously - the value cannot make instantaneous jumps
- **A stochastic process** is a sequence of values of some quantity where the future values cannot be predicted with certainty.
- Ahead of here we are concerned with continuous-time stochastic processes that have applications in financial economics.

Introduction to Brownian motion

Near 1900, Bachelier adapted an approach that was fairly conventional at the time; he would model an asset price as a random walk. At the start of his thesis he argues that:

- **At a given instant the market believes neither in a rise nor in a fall of the true price.**

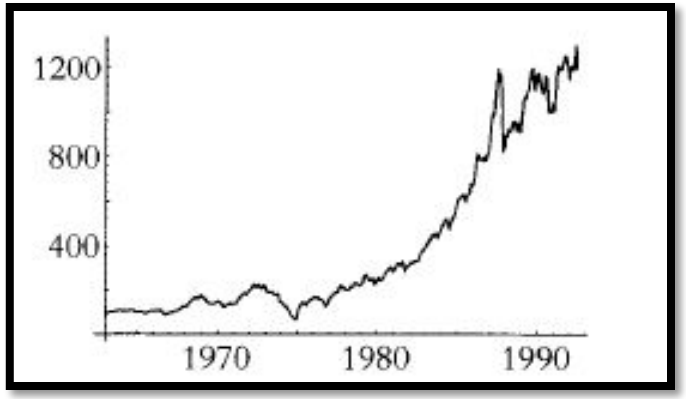
Which means that:

- **The mathematical expectation of the speculator is zero.**

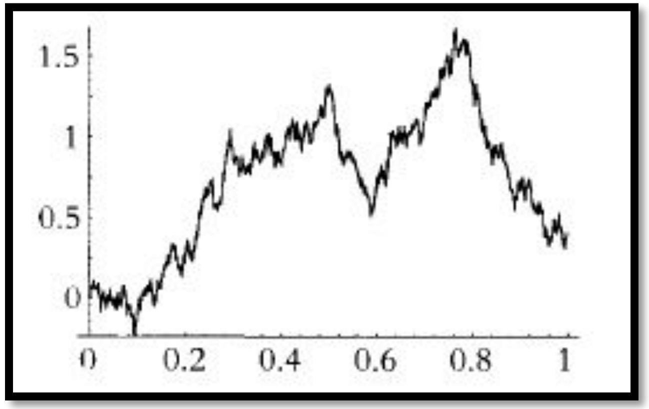
His innovation was to consider the **random walk to be continuous**, **rather than a discrete-time random walk**. This is analogous to **moving from the binomial to the normal distribution.**

Let's Look In

Bachelier who analysed the motion of the Paris stock exchange, people have gone further and compared the prices to one particular continuous process - the process followed by a randomly moving gas particle, or Brownian motion.



UK FTA index, 1963–92



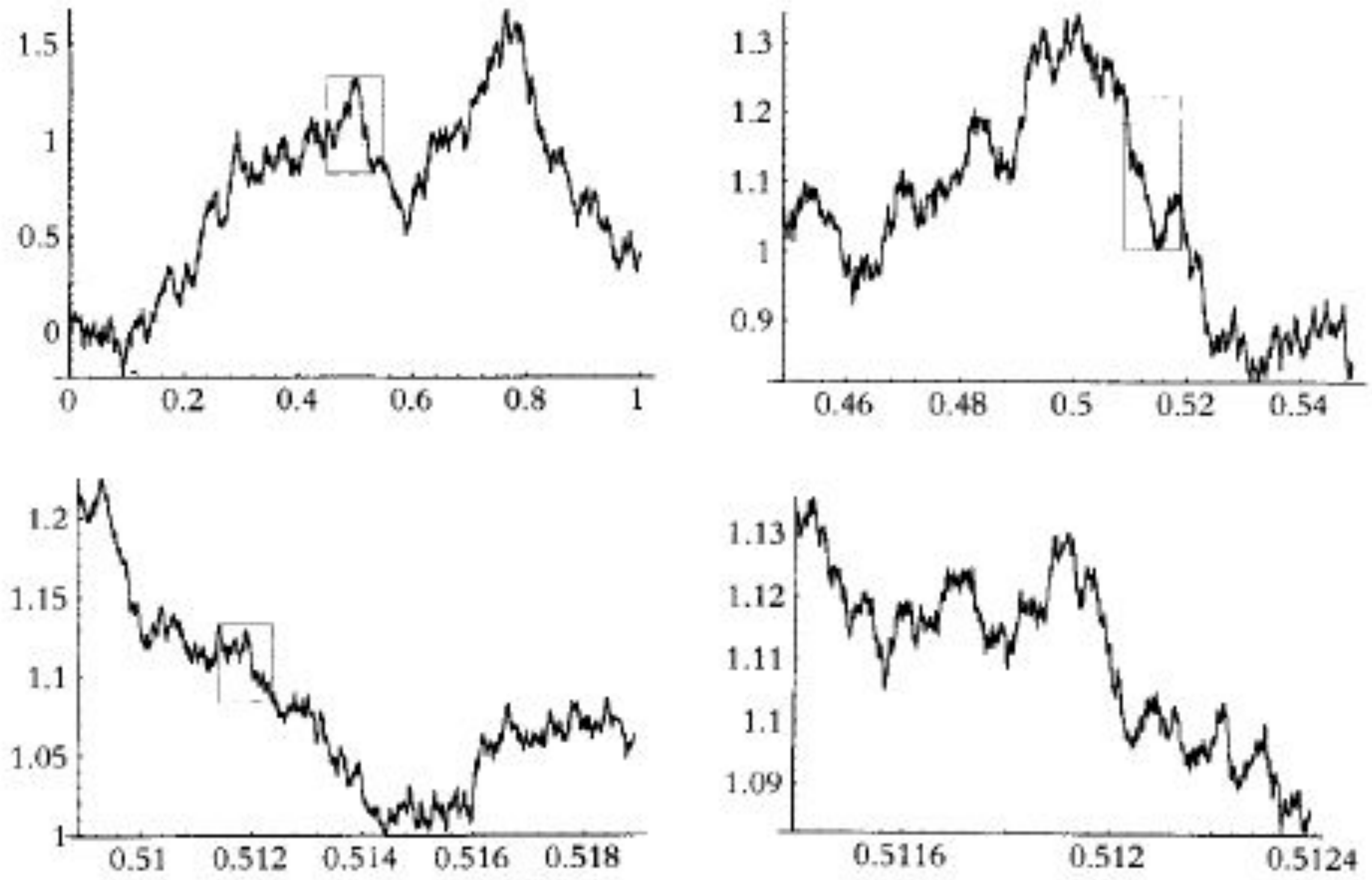
Brownian motion

Locally **the likeness can be striking - both display the same jaggedness, and the same similarity** under scale changes - **the jaggedness never smooths out as the magnification increases.**

Thus, Brownian Motion is an effective component to build the model ahead.

Zooming in on Brownian Motion

We see that the graphs don't become smooth irrespective of how much ever we zoom in.



Wiener process (standard Brownian motion)

A stochastic process W_t t ≥ 0 is a Wiener process if:

(i) $\boldsymbol{W_0}$ = 0

(ii) W_t has **continuous sample paths**.

(iii) The process follows a **normal distribution**, $\boldsymbol{W_t}$ ~ N(0, t).

(iv) For any 0 ≤ s < *t* the **increment** $\boldsymbol{W_t}$ - $\boldsymbol{W_s}$ **is normally distributed**, W_t - W_s ~ *N*(0,*t* - *s*). i.e Mean = 0 and Variance = Time lag (t-s in this case)

v) W_t has **independent increments**, that is for any sequence of times $0 \le t_1 < t_2 < \dots < t_n$ we have that the increments $W_{t_n} - W_{t_{n-1}}, \dots, W_{t_2} - W_{t_1}$ are independent random variables.

vi) The **increments are stationary** – stable/not changing. Mean remains zero and variance depends on the time lag in consideration.

Certain Dissimilarities

- However successful the Brownian motion model may be for describing the movement of market indices in the short run, it is useless in the long run, if only for the reason that a **standard Brownian motion is certain to become negative eventually.** The **stock prices cannot be negative**.
- Brownian motion wanders. **It has mean zero**, whereas **the stock of a company normally grows at some rate.** Thus BM tends to 0 whereas the asset prices represent an increasing graph.
- The **volatility of stock prices is quite high** (in absolute terms), where as **Brownian motion represents low volatility** (in absolute terms).

Thus we parameterize the Brownian motion, to insert drift and high volatility into the picture.

Parameterization Of the Brownian Motion

- We **insert a drift/ trend - μ** – on the **basis on expectation** of the asset prices.
- We also **insert a high volatility by σ** . The volatility can be controlled, since it is introduced as variance, which is **dependent on the time lag**.
- We have both the drift and volatility dependent on the time lag.
- Thus we get **the General Brownian Motion (represented by Z)** as:

$$Z_t - Z_s \sim N(\mu.(t-s), \sigma^2 x(t-s))$$

Stock Price Modelling

In order to model asset prices we need to define some starting point, such as Z_0 = ____.
(since the Brownian motion has starting point as zero)

Thus we finally get the distribution as:

Z_t - Z_s ~ Z_0+ N($\mu.(t-s), \sigma^2$x(t-s))

SBM to GBM and GBM to SBM

- Moving from the Standard BM to the General BM.

N (0, t) x σ + μt + $\boldsymbol{Z_0}$ → $\boldsymbol{Z_0}$ + N (μt, σ^2t)

- Moving from the General BM to the Standard BM.

We move from GBM to SBM in the same way as we covert normal variable to a standard normal variable.

We have $\boldsymbol{Z_t} = \boldsymbol{Z_0} + \mu t + \sigma \boldsymbol{W_t}$

Therefore, $\boldsymbol{W_t} = \dfrac{\boldsymbol{Z_t} - \boldsymbol{Z_0} - \mu t}{\sigma}$

Geometric Brownian motion

We also need to adjust the Brownian motion for the negativity factor, as stock prices don't go negative!

We consider,

$\boldsymbol{S_t = e^{Z_t}}$

Where Z_t is the Brownian motion process $Z_t = Z_0 + \sigma W_t + \mu t$.

Thus S_t which is called Geometric Brownian motion, is lognormally distributed with parameters $\boldsymbol{Z_0 + \mu t}$ and $\boldsymbol{\sigma^2 t}$.

So the values of $\log S_t$ are normally distributed with mean $Z_0 + \mu t$ and variance $\sigma^2 t$.

Properties of Geometric Brownian

- The most important property of S_t is $S_t \geq 0$, for all t.
- From the properties of the lognormal distribution we also have:

E[S_t] = exp((Z_0 + μt) +1/2 $\sigma^2 t$) and V[S_t] = E^2[S_t] (exp($\sigma^2 t$)-1)

- The increments of S_t are of the form S_t - S_s = $\boldsymbol{e^{Z_t}}$ - $\boldsymbol{e^{Z_s}}$
- The log-return log$\frac{S_t}{S_s}$ from time *s* to time *t* is given by log$\frac{S_t}{S_s}$ = log$\frac{\boldsymbol{e^{Z_t}}}{\boldsymbol{e^{Z_s}}}$ = Z_t - Z_s.
- It follows by the independent increments property of Brownian motion that the log-returns, and hence the returns themselves, are independent over disjoint time periods.

Additional properties of Brownian motion

- Standard Brownian motion has a number of other properties inherited from the simple symmetric random walk. A simple symmetric random walk is a discrete-time stochastic process.

$X_n = \sum_{i=1}^{n} Z_i$ where Z_i = +1 or -1 with equal probability.

- Many of the properties of standard Brownian motion can be demonstrated using the following decomposition. For $s < t$:

$$W_t = W_s + (W_t - W_s)$$

a decomposition in which the first term is known at time *s* and the second is independent of everything up to and including time *s*

Covariance of a Wiener Process

An important property is that of the covariance between its values s >0 and t > s

Cov (W_s, W_t) = E[$(W_s$ - E[W_s]) $(W_t$ - E[W_t])]

= E[$W_s(W_s + (W_t - W_s)$)]

This follows from the fact that $E[Wt\,] = E[Ws] = 0$, and then by applying the decomposition.

Cov (W_s, W_t) = E[W_s^2] + E[W_s]E[$(W_t - W_s$)]

= Var(W_s) + 00

= s

- In general, Cov $(\boldsymbol{W_s}, \boldsymbol{W_t})$ = min{s, t}
- The importance of this result is that, in fact, if a stochastic process has the property that:

Cov (W_s, W_t) = min{s, t}, then the process X_t is a Wiener process

Scaled Wiener process

Given a positive constant c and a Wiener process W_t define the stochastic process X_t by:

$$X_t = \sqrt{c}W_{t/c}$$

The 'clock' of the process X_t has been scaled by a factor c. For example, the process has been slowed down and magnified if $c > 1$ (and speeded up and shrunk if $c < 1$).

- Prove that X_t is Weiner process

Scaled Wiener process

$$
\begin{aligned}
Cov(X_{t+u}, X_t) &= Cov\left(\sqrt{c}W_{\frac{t+u}{c}}, \sqrt{c}W_{\frac{t}{c}}\right) \\
&= c \times Cov\left(W_{\frac{t+u}{c}}, W_{\frac{t}{c}}\right) \\
&= c \times \min\left\{\frac{t+u}{c}, \frac{t}{c}\right\} \\
&= c\frac{t}{c} \\
&= t
\end{aligned}
$$

assuming $u > 0$.

Since $Cov(X_{t+u}, X_t) = \min\{t+u, t\}$, X_t is a Wiener process.

Time-inverted Wiener process

Given a Wiener process W_t define the stochastic process X_t by:

$X_t = t.W_{1/t}$

The time-inverted Wiener process is itself a Wiener process.

❖ Prove the above statement.

Time-inverted Wiener process

Then we have:

$$Cov(X_{t+u}, X_t) = Cov\left((t+u)W_{\frac{1}{t+u}}, tW_{\frac{1}{t}}\right)$$

$$= (t+u)t \times Cov\left(W_{\frac{1}{t+u}}, W_{\frac{1}{t}}\right)$$

Since $1/(t+u) < 1/t$ and by the covariance of Wiener processes:

$$Cov(X_{t+u}, X_t) = (t+u)t \times \min\left\{\frac{1}{t+u}, \frac{1}{t}\right\}$$

$$= (t+u)t\frac{1}{t+u}$$

$$= t$$

Therefore X_t is also a Wiener process.

Martingales

We have the following definitions of continuous-time martingales.

Given a filtered probability space (Ω, F, F_t, P) a stochastic process X_t is called a martingale with respect to the filtration, F_t, if:

- X_t is adapted to F_t
- E[|X_t|] < ∞ for all t
- E [X_t | F_s] = X_s for all s ≤ t.

The first condition is just a technicality to ensure that the process value can be known with certainty at time *t*, and the second is to guarantee that *Xt* is integrable.

In most questions we are only concerned with the last condition and we'll assume the first two hold

Supermartingale & Submartingale

- Given $s \leq t$, a *supermartingale* is such that: $\mathrm{E}\,[\boldsymbol{X_t} \mid \boldsymbol{F_S}] \leq \boldsymbol{X_S}$
- Given $s \leq t$, *a submartingale is such that:* $\mathrm{E}\,[\boldsymbol{X_t} \mid \boldsymbol{F_S}] \geq \boldsymbol{X_S}$

A process which is both a supermartingale and a submartingale must therefore be a martingale.

Wiener processes are martingales

We try to prove the third condition required for martingales.

Consider,

$$E\,[\,W_t \mid F_s^W\,] = E\,[W_s + (W_t - W_s) \mid F_s^W]$$

$$= E\,[\,W_s \mid F_s^W\,] + E\,[\,(W_t - W_s) \mid F_s^W\,]$$

Since increments are independent and $W_t - W_s \sim N(0, t\text{-}s)$:

$$\mathbf{E\,[\,W_t \mid F_s^W\,] = W_s}$$

The Wiener process is a martingale with respect to its natural filtration, noting that $E[|W_t|] < \infty$ since $W_t < \infty$ almost surely.

Try it yourself!

Consider the stochastic process defined by W_t^2 - t.

Prove whether it is a martingale or not with respect to its natural filtration.

Solution

$$E\left[W_t^2 - t \mid F_s^W\right] = E\left[(W_s + (W_t - W_s))^2 \mid F_s^W\right] - t$$

$$= E\left[W_s^2 \mid F_s^W\right] + 2E\left[W_s(W_t - W_s) \mid F_s^W\right] + E\left[(W_t - W_s)^2 \mid F_s^W\right] - t$$

$$= W_s^2 - s$$

so it is a martingale with respect to its natural filtration.

Homework!!

Consider the stochastic process defined by $\exp(\lambda W_t - \frac{1}{2}\lambda^2 t)$.

Prove that this process is a martingale.

HINT: [Consider Z as a normally distributed variable and make use of the MGF of a normal]

Stochastic Interest Rates

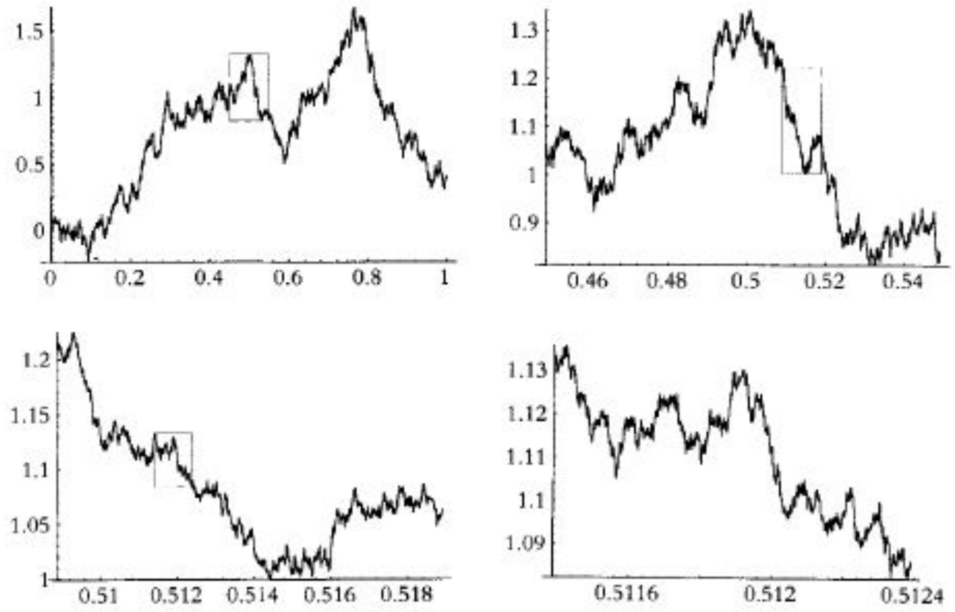
Initially the calculation has assumed that interest rates are constant throughout the term. However in practice, Interest rates are never constant.

Thus, we define $\boldsymbol{i_t}$ = interest applicable for the period from time t-1 to time t.

- i_t will be a stochastic random variable (discrete or continuous) and different for each year.
- Since i_t is a stochastic R.V. it will have some expectation and variance.
- The AV($\boldsymbol{i_t}$) and PV($\boldsymbol{i_t}$) are also random variables.
- We will assume i_t follows a normal distribution with some parameters

Weiner Process is not differentiable

Earlier we saw that how much ever you zoom in to the BM process it does not become a well-behaved function. The jaggedness remains intact.



As a result the **Brownian motion is differentiable no where.**

How do we solve this problem then?

Solving the Problem

To solve the problem of differentiability we consider the increments of the BM.

We know that increments of a BM are stationary, independent and normally distributed.

We have:

X_t - X_s ~ $N(0,t - s)$

If we consider an infinitely small increment, even then we have

X_{t+dt} - X_t ~ $N(0, dt)$ – the distribution of the increments remain normal irrespective of how much ever you zoom in.

Thus we get the solution for differentiation as X_{t+dt} - X_t gives dX_t.

For the General BM

Consider a small increment B_{t+dt} - B_t.

The distribution is as follows: B_{t+dt} - B_t ~ N(μdt , σ^2dt)

Therefore, dB_t ~ N(μdt + σ^2dt)

dB_t = μdt + N(0, σ^2dt)

dB_t = μdt + σ N(0, dt)

d$\boldsymbol{B_t}$ = $\boldsymbol{\mu dt}$ + $\boldsymbol{\sigma}$ d$\boldsymbol{X_t}$ (we just saw that dX_t~ *N*(0, *dt*) *)*

This is the Stochastic Differential Equation. (SDE)

Integration

Under integration we add up the (infinitesimal) increments (independent).

An integral in which we are **integrating with respect to Brownian motion** is called an **Ito integral**.

We have: $\int_0^t dW_t = W_t - W_0 = W_t$ (thus W_t is the process)

It is worth noting that the increments of Brownian motion are just normal random variables, and furthermore, the increments over disjoint time periods are independent.

In 'adding' up the increments dW_t we are effectively **summing independent normal random variables**. Moreover, the increment dW_t should have a $N(0,dt)$ distribution. Again this is consistent with the value of the integral (W_t), which has a $N(0,t)$ distribution.

SDE to the Process

1) Consider where μ and σ are both constants, meaning that B has constant volatility and drift, the SDE for B is $\mathrm{d}\boldsymbol{B_t} = \boldsymbol{\mu dt} + \boldsymbol{\sigma}\,\mathrm{d}\boldsymbol{X_t}$

Integrating on both sides we have:

$$\int_0^T dB_t = \int_0^T \mu dt + \int_0^T \sigma\, dX_t$$

$$B_T - B_0 = \mu T + \sigma \int_0^T dX_t$$

$$B_T - B_0 = \mu T + \sigma\,[X_T - X_0]$$

We have $X_0 = 0$.

$\mathbf{B_T = B_0 + \mu T + \sigma X_T}$ → Integral form

$\mathbf{B_T} \sim N\,(\mathbf{B_0 + \mu T, \sigma^2 T})$ → Distributional form (since $\mathbf{X_T} \sim N(0, T)$)

SDE to the Process

2) Consider now a stochastic process where the drift and volatility are both dependent on time. The SDE is as:

$\mathrm{d}\boldsymbol{Y_t} = \boldsymbol{\mu(t)dt + \sigma(t)dX_t}$

Suppose that $\mu(t) = t^2$ (a deterministic fn)

Integrating both sides we get;

$\int_0^T \mathbf{dY_t} = \int_0^T \boldsymbol{\mu}\,(\boldsymbol{t})\ \mathrm{dt} + \int_0^T \boldsymbol{\sigma(t)}\ \mathrm{d}\boldsymbol{X_t}$

$Y_T - Y_0 = \frac{t^3}{3} + \int_0^T \sigma(t)\ \mathrm{d}X_t$

{First term - Taking the example of $\mu(t) = t^2$ and integrating it. Second term is an Ito integral}.

Solving $\int_0^T \sigma(t)\ \mathrm{d}X_t$:

We have $\sigma(t)\mathrm{d}X_t \sim \mathrm{N}(0, \sigma^2 dt)$

Thus under integration we are adding the normal variables.

$\int_0^T \boldsymbol{\sigma(t)}\ \mathrm{d}\boldsymbol{X_t} \sim \mathrm{N}\,(0, \int_0^T \boldsymbol{\sigma^2}\ \mathrm{dt})$

- $\mathrm{E}[\int_0^T \sigma(t)\ \mathrm{d}X_t] = \int_0^T \sigma(t)\ \mathrm{E}[\mathrm{d}X_t] = 0$
- $\mathrm{V}[\int_0^T \sigma(t)\ \mathrm{d}X_t] = \int_0^T V\,[\sigma(t)\ \mathrm{d}X_t] = \int_0^T \sigma^2(t)\ \mathrm{V}\,[\mathrm{d}X_t] = \int_0^T \sigma^2(t)\ dt$

  (since independent)

SDE to the Process

3) Consider now that the drift and volatility both depend on time and also the current value of the process. Therefore now both are random stochastic variables. The SDE is given as:

$$\mathrm{d}\boldsymbol{Z_t} = \boldsymbol{\mu}\,(\boldsymbol{t},\boldsymbol{Z_t})\,\boldsymbol{dt} + \boldsymbol{\sigma}(\boldsymbol{t},\boldsymbol{Z_t})\,\mathrm{d}\boldsymbol{X_t}$$

Integrating on both sides

$$\int_0^T \mathrm{d}Z_t = \int_0^T \mu\,(t,Z_t)\,dt + \int_0^T \sigma(t,Z_t)\,\mathrm{d}X_t$$

Z_T - Z_0 = Stochastic term   Ito integral of a stochastic function

(time integral)   (outside the reach of our syllabus)

Process to SDE

Going from the process to the Stochastic differential equation we make use of Ito's Lemma.

First Consider the Taylor's theorem

We first write down Taylor's theorem to for a small change h, where h → 0.

$$f(x+h) = f(x) + h.f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots\ldots$$

$$f(x+h) - f(x) = h.f'(x) + \frac{h^2}{2!} f''(x) + \ldots\ldots.$$

Now second term onwards on the right side turn to be 0 as h → 0. Thus we have,

$$f(x+h) - f(x) = h.f'(x)$$

What does this become if we replace the function x by the non-differentiable Wiener process X_t?
Lets see!

Process to SDE

The analysis starts in much the same way. We can write Taylor's theorem to second-order as:

$$f(X_t + dX_t) = f(X_t) + dX_t \,.\, f'(X_t) + \frac{(dx_t)^2}{2!} \,.\, f''(X_t) + \ldots..$$

$$f(X_t + dX_t) - f(X_t) = dX_t \,.\, f'(X_t) + \frac{(dx_t)^2}{2!} \,.\, f''(X_t) + \ldots..$$

With Wiener processes, it turns out that the second-order term $(dX_t)^2$ cannot be ignored. In fact, it must be changed to *dt* ,

i.e. $(dX_t)^2$ = dt. This is not rigorous, but is a useful rule of thumb.

What we end up with is therefore:

$$df(\boldsymbol{X_t}) = d\boldsymbol{X_t} \,.\, f'(\boldsymbol{X_t}) + \tfrac{1}{2} f''(\boldsymbol{X_t})\, dt$$

This is Ito's Lemma for functions of Wiener processes, *ie* it tells us how to differentiate functions of standard Brownian motion. Note, however, that this statement must be interpreted in terms of integrals, since standard Brownian motion is not differentiable

Question

Find the stochastic differential equation for $W_t{}^2$.

Solution:

Applying the above formula we have:

$$d\left(W_t^2\right) = 2W_t dW_t + ½ 2dt = 2W_t dW_t + dt$$

Process to SDE – Type 1

First we consider that there is only a single factor involved which is the Weiner process X_t.

A process B_t satisfies the stochastic differential equation

$dB_t = \mu dt + \sigma\, dX_t$ where X_t is a standard Brownian motion.

Deduce the stochastic differential equation for the process $B_t{}^3$.

By Ito's Lemma:

$$df(B_t{}^3.) = dB_t \,.\, f'(B_t) + \frac{(dB_t)^2}{2!} f''(B_t)$$

Now $f'(B_t) = 3B_t{}^2$ and $f''(B_t) = 6B_t$. Therefore we have

$$df(B_t{}^3.) = dB_t \,.\, 3B_t{}^2 + 3\,(dB_t)^2\, B_t$$

Now substituting $dB_t = \mu dt + \sigma\, dX_t$, we get

$$df(B_t{}^3.) = (\mu dt + \sigma\, dX_t)\, 3B_t{}^2 + 3\,(\mu dt + \sigma\, dX_t)^2\, B_t$$

$$df(B_t{}^3.) = 3B_t{}^2\, \mu dt + 3B_t{}^2\, \sigma\, dX_t + \sigma^2 dt\, 3B_t$$

$$df(B_t{}^3.) = (3B_t{}^2\, \mu + \sigma^2\, 3B_t)\, dt + 3B_t{}^2\, \sigma\, dX_t$$

Question

A stochastic process X_t satisfies the stochastic differential equation $dX_t = \mu_t dt + \sigma_t dB_t$, where B_t is a standard Brownian motion.

Find the stochastic differential equations for each of the following processes:

i) $G_t = e^{X_t}$

ii) V_t = (1+X_t)^-1

Solution

i) Here the function we are applying Ito's Lemma to is $f(x)=e^x$, with $f'(x)=e^x$ and $f''(x)=e^x$.

So we get:

$$dG_t=\sigma_t e^{X_t}dB_t+[\mu_t e^{X_t}+\tfrac{1}{2}\sigma_t^2 e^{X_t}]dt$$

$$=\sigma_t G_t dB_t+[\mu_t+\tfrac{1}{2}\sigma_t^2]G_t dt$$

ii) Here the function we are applying Ito's Lemma to is $f(x)=(1+x)^{-1}$, with $f'(x)=-(1+x)^{-2}$ and $f''(x)=2(1+x)^{-3}$. So we get:

$$dV_t=-\sigma_t(1+X_t)^{-2}dB_t+[-\mu_t(1+X_t)^{-2}+\sigma_t^2(1+X_t)^{-3}]dt$$

$$=-\sigma_t V_t^2 dB_t+[-\mu_t V_t^2+\sigma_t^2 V_t^3]dt$$

Process to SDE – Type 2

Now we consider a multifactor function. Suppose we have Weiner process X_t and time t (i.e. f(X_t,t)). Here we write the Ito's Lemma in two variables following the same logic as earlier:

$$f(X_t + dX_t\ ,\ t + dt) = f(X_t,t) + dX_t \cdot \frac{d\, f(x_t,t)}{dx_t} + dt \cdot \frac{d\, f(x_t,t)}{dt} + \frac{(dx_t)^2}{2!} \cdot \frac{d^2 f(x_t,t)}{dx_t 2} + \frac{(dt)^2}{2!} \cdot \frac{d^2 f(x_t,t)}{dt^2} + dt.\ dX_t \cdot \frac{d^2 f(x_t,t)}{dt.\ dx_t}$$

Combining Ito's formula with the Taylor expansion of *f* (*x*,*t*) we can deduce the following 'rules': $(dX_t)^2$ = dt and dX_tdt = $(dt)^2$ = 0

$$d\, f(\boldsymbol{X_t},t) = dt \cdot \frac{d\, f(x_t,t)}{dt} + d\boldsymbol{X_t} \cdot \frac{d\, f(x_t,t)}{dx_t} + \frac{(dx_t)^2}{2!} \cdot \frac{d^2 f(x_t,t)}{dx_t 2}$$

This is actually just an application of Taylor's theorem in two variables.

Martingale

We know that for the process to be a martingale it should fulfill the condition that $E[X_t | F_S] = X_s$ Thus we should have that the drift in the process = 0.

Therefore the driftless Ito process is a martingale.

To check for martingale, find the Ito's Lemma and check for dt = 0

The mean-reverting process

The mean-reverting process, defined by the SDE

$$\mathrm{d}Y_t = \gamma(\mu - Y_t)\mathrm{d}t + \sigma \mathrm{d}W_t$$

In the mean-reverting process, the process is pulled back to some equilibrium level μ, at a rate determined by $\gamma > 0$. Note that this process can go negative.

We can investigate:

$$\mathrm{d}Y_t + \gamma Y_t \mathrm{d}t = \gamma\mu \mathrm{d}t + \sigma \mathrm{d}W_t$$

Multiply by $e^{\gamma t}$ throughout

$$\mathrm{d}Y_t e^{\gamma t} + e^{\gamma t}\gamma Y_t \mathrm{d}t = e^{\gamma t}\,\gamma\mu \mathrm{d}t + e^{\gamma t}\,\sigma \mathrm{d}W_t$$

$$\mathrm{d}(e^{\gamma t} Y_t) = e^{\gamma t}\,\gamma\mu \mathrm{d}t + e^{\gamma t}\,\sigma \mathrm{d}W_t$$

The mean-reverting process

Integrating both sides between 0 to T

$\int_0^T d(e^{\gamma t} Y_t) = \int_0^T e^{\gamma t}\, \gamma\mu dt + \int_0^T e^{\gamma t}\, \sigma dW_t$

$e^{\gamma T} Y_T - Y_0 = \mu[e^{\gamma T} - 1] + \int_0^T e^{\gamma t}\, \sigma dW_t$

$Y_T = Y_0 e^{-\gamma T} + \mu[1 - e^{-\gamma T}] + e^{-\gamma T}\sigma \int_0^T e^{\gamma t}\, dW_t$

Now consider $e^{-\gamma T}\sigma \int_0^T e^{\gamma t}\, dW_t$, here $\int_0^T e^{\gamma t}\, dW_t$ is an Ito integral.

$dW_t \sim N(0, dt)$ -> $e^{\gamma t}\, dW_t \sim N(0, e^{2\gamma t} dt)$ -> $\int_0^T e^{\gamma t}\, dW_t \sim N(0, \int_0^T e^{2\gamma t}\, dt)$

$e^{-\gamma T}\sigma \int_0^T e^{\gamma t}\, dW_t \sim N(0, \sigma^2[\frac{1-e^{-2\gamma T}}{2\gamma}])$

Therefore finally, $\boldsymbol{Y_T \sim N(Y_0 e^{-\gamma T} + \mu[1 - e^{-\gamma T}], \sigma^2[\frac{1-e^{-2\gamma T}}{2\gamma}])}$

Square root mean-reverting process

The process defined by the SDE:

$$dY_t = \gamma(\mu - Y_t)dt + \sigma\sqrt{Y_t}\,dW_t$$

with $\mu\,,\sigma > 0$ is known as the CIR, Feller or 'square root mean-reverting' process.

If parameters satisfy $\sigma^2 \leq 2\,\gamma\mu$ the process is positive. If the process hits zero, its volatility disappears and its drift is positive, the process deterministically moves away from zero and spends 'no time' at zero (*ie* the time spent at zero has measure zero). This is a very useful property in modelling asset prices.

There is no closed form solution for Y_t.

Share Price Process

Consider General BM with drift η_t and volatility $\boldsymbol{\sigma}$. It is given by the process $B_t = B_0 + \eta t + \sigma \,\mathrm{d}X_t$

Geometric BM is given as $S_t = e^{B_t} = e^{B_0} + \eta t + \sigma \,\mathrm{d}x_t$

What is SDE for this process? We use Ito's Lemma:

$$\mathrm{d}\, f(\boldsymbol{X_t},t) = \mathrm{d}t \,.\, \frac{d\, f(x_t,t)}{\mathrm{d}t} + \mathrm{d}\boldsymbol{X_t} \,.\, \frac{d\, f(x_t,t)}{\mathrm{d}x_t} + \frac{(\mathrm{d}x_t)^2}{2!} \,.\, \frac{d^2 f(x_t,t)}{\mathrm{d}x_{t}2}$$

Here:

$$\frac{d\, f(x_t,t)}{\mathrm{d}t} = (e^{B_0} + \eta t + \sigma \,\mathrm{d}x_t).\eta \;=\; S_t.\,\eta$$

$$\frac{d\, f(x_t,t)}{\mathrm{d}x_t} = (e^{B_0} + \eta t + \sigma \,\mathrm{d}x_t).\,\sigma = \;S_t.\,\sigma$$

$$\frac{d^2 f(x_t,t)}{\mathrm{d}x_{t}2} = (e^{B_0} + \eta t + \sigma \,\mathrm{d}x_t).\,\sigma^2 = \;S_t.\,\sigma^2$$

Share Price Process

d $f(\boldsymbol{X_t}, t) = d\boldsymbol{S_t} = (\boldsymbol{\eta} + \frac{\sigma^2}{2})\, \boldsymbol{S_t}\, dt + \boldsymbol{\sigma}\, \boldsymbol{S_t}\, d\boldsymbol{X_t}$

Now we let $\mu = (\eta + \frac{\sigma^2}{2})$, we have

$d\boldsymbol{S_t} = \mu\, \boldsymbol{S_t}\, dt + \boldsymbol{\sigma}\, \boldsymbol{S_t}\, d\boldsymbol{X_t}$

This is the SDE for the share price process.

Share Price Process

Now consider the function $\ln S_t$.

What is the SDE for this process if the process S_t evolves as $dS_t = \mu S_t \, dt + \sigma S_t \, dX_t$.

Solution: Let $f(S_t) = \ln(S_t)$ and apply Taylor's theorem to give:

$$
\begin{aligned}
df(S_t) &= \frac{\partial f}{\partial S_t} dS_t + \frac{1}{2}\frac{\partial^2 f}{\partial S_t^2}(dS_t)^2 + \ldots \\
&= \frac{1}{S_t} dS_t - \frac{1}{2S_t^2}(dS_t)^2 + \ldots \\
&= \frac{S_t}{S_t}(\mu dt + \sigma dW_t) - \frac{S_t^2 \sigma^2}{2S_t^2} dt \\
&= \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW_t
\end{aligned}
$$

Share Price Process

Now we move from the SDE to the process for the share price.

Integrating the SDE on both sides between time 0 and T gives:

$$\int_0^T d\, lnS_t = (\mu - ½\, \sigma^2) \int_0^T dt + \sigma \int_0^T \mathrm{d}X_t$$

$$\ln S_T - \ln S_0 = (\mu - ½\, \sigma^2)\, \mathrm{T} + \sigma\, X_T$$

$$\ln S_T = \ln S_0 + (\mu - ½\, \sigma^2)\, \mathrm{T} + \sigma\, X_T$$

$$\boldsymbol{S_T = S_0\, e^{(\mu - ½\, \sigma^2)\, \mathrm{T} + \sigma\, X_T}}$$

This is the solution to the geometric Brownian motion SDE $\mathrm{d}S_t = \mu\, S_t\, \mathrm{dt} + \sigma\, S_t\, \mathrm{d}X_t$., and is a standard share price model.

Now we look at the distribution

$$\frac{\boldsymbol{S_T}}{\boldsymbol{S_0}} \sim \mathrm{logN}\,((\mu - ½\, \boldsymbol{\sigma^2})\, \mathrm{T}, \boldsymbol{\sigma^2}\mathrm{T})$$

$$\ln \boldsymbol{S_T} \sim \mathrm{N}\,(\ln \boldsymbol{S_0} + (\mu - ½\, \boldsymbol{\sigma^2})\, \mathrm{T}, \boldsymbol{\sigma^2}\mathrm{T})$$

Quick Recap

- A stochastic process W_t t ≥ 0 is a Wiener process if:

(i) $\boldsymbol{W_0}$ = 0

(ii) W_t has **continuous sample paths**.

(iii) The process follows a **normal distribution, $\boldsymbol{W_t}$ ~ N(0, t).**

(iv) For any 0 ≤ s < t the **increment $\boldsymbol{W_t}$ - $\boldsymbol{W_s}$ is normally distributed,** W_t - W_s ~ N(0,t - s). i.e Mean = 0 and Variance = Time lag (t-s in this case)

v) W_t has **independent increments,** that is for any sequence of times 0 ≤ t_1 < t_2 < .... < t_n we have that the increments W_{t_n} - $W_{t_{n-1}}$, ..., W_{t_2} - W_{t_1} are independent random variables.

vi) The **increments are stationar**y – stable/not changing. Mean remains zero and variance depends on the time lag in consideration.

- $\boldsymbol{Z_t}$ - $\boldsymbol{Z_s}$ ~ N($\boldsymbol{\mu.(t-s)}$, $\boldsymbol{\sigma^2}$x(t-s))

- N (0, t) x σ + μt + $\boldsymbol{Z_0}$ → $\boldsymbol{Z_0}$ + N (μt, σ^2t)

  $\boldsymbol{Z_t}$ = $\boldsymbol{Z_0}$ + μt + σ$\boldsymbol{W_t}$

  Therefore, $\boldsymbol{W_t} = \dfrac{\boldsymbol{Z_t} - \boldsymbol{Z_0} - \mu t}{\sigma}$

- Given a filtered probability space (Ω, F, F_t, P) a stochastic process X_t is called a martingale with respect to the filtration, F_t, if:

• $\boldsymbol{X_t}$ is adapted to $\boldsymbol{F_t}$

• E[|$\boldsymbol{X_t}$|] < ∞ for all t

• E [$\boldsymbol{X_t}$ | $\boldsymbol{F_S}$] = $\boldsymbol{X_S}$ for all s ≤ t.

Quick Recap

- Given $s \leq t$, a *supermartingale* is such that: E $[\boldsymbol{X_t} \mid \boldsymbol{F_S}] \leq \boldsymbol{X_S}$
- Given $s \leq t$, *a submartingale is such that:* E $[\boldsymbol{X_t} \mid \boldsymbol{F_S}] \geq \boldsymbol{X_S}$
- The General Brownian Motion

  $$d\boldsymbol{B_t} = \boldsymbol{\mu dt} + \boldsymbol{\sigma}\, d\boldsymbol{X_t}$$

- We know that for the process to be a martingale it should fulfill the condition that $E[X_t \mid F_S] = X_s$ Thus we should have that the drift in the process = 0.

Therefore the driftless Ito process is a martingale.

To check for martingale, find the Ito's Lemma and check for dt = 0

- Mean Reverting Process

$$\boldsymbol{Y_T} \sim N\left(\boldsymbol{Y_0 e^{-\gamma T}} + \boldsymbol{\mu[1 - e^{-\gamma T}]}, \boldsymbol{\sigma^2}\left[\frac{1-e^{-2\gamma T}}{2\gamma}\right]\right)$$

- Share Price Process

$$\ln \boldsymbol{S_T} \sim N\left(\ln \boldsymbol{S_0} + (\mu - ½\, \boldsymbol{\sigma^2})\, T, \boldsymbol{\sigma^2} T\right)$$