Lecture 6

INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Class: M.Sc. – Sem 2

Subject : Financial Engineering

Chapter: Unit 3 Chapter 1

Chapter Name : The Black Scholes Model

Today’s Agenda

1. Black Scholes Model
   1.1 Assumptions under the Black Scholes model
   1.2 Black Scholes Formula
   1.3 The PDE approach
2. The Risk Free Portfolio
3. Black Scholes PDE
4. Greeks
   4.1 Delta

1.1 The assumptions underlying the Black-Scholes model

- The price of the underlying share follows a **geometric Brownian motion,** i.e. the share price **changes continuously** through time according to the stochastic differential equation $dS_t = S_t(\mu dt + \sigma dZ_t)$
- There are **no risk-free arbitrage opportunities**.
- The **risk-free rate of interest is constant**, the same for all maturities and the same for borrowing or lending.
- **Unlimited short selling** (that is, negative holdings) is allowed.
- There are **no taxes or transaction costs**.
- The underlying asset can be **traded continuously and in infinitesimally small numbers of units.**

1.2 The Black-Scholes formula

Let $f(t, S_t)$ be the price at time t of a call option given that the current share price is $S_t = s$, the time of maturity is $T > t$, the exercise price is K.

For such a call option: [The Black Scholes formula]

$$f(t, S_t) = S_t\Phi(d_1) - Ke^{-r(T-t)}\,\Phi(d_2)$$

where:

$$d_1 = \frac{\log(\frac{S_t}{K})+(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma(T-t)}$$

$$d_2 = d_1 - \sigma(T-t)$$

and $\Phi(z)$ is the cumulative distribution function of the standard normal distribution.

This is given on page 47 of the Tables

1.3 The PDE approach

We will give proof of this result using the partial differential equation (PDE) approach.

An expression for df $(t, \boldsymbol{S_t})$

We first use Ito's Lemma to write a stochastic differential equation (SDE) for the change in the derivative price as a function of the change in the share price. Here df (t, S_t) means the change in the value of the derivative over a very small time period.

Given the Ito process:

$dS_t = S_t(\mu dt + \sigma dZ_t)$

then applying Ito's Lemma to the function f $(t, \boldsymbol{S_t})$, we have:

$$df\,(t, S_t) = \frac{df}{ds}\sigma S_t dZ_t + [\frac{df}{dt} + \frac{df}{ds}\mu S_t + ½\,\frac{d^2 f}{ds^2}\sigma^2\, S_t^2]\,dt$$

2. The risk-free portfolio

Suppose that at any time t , $0 \leq t \leq T$, we hold the following portfolio:

- minus one derivative
- plus $\frac{df}{ds}(t, S_t)$ shares

Let $V(t, S_t)$ be the value of this portfolio. That is:

$$V(t, S_t) = - f(t, S_t) + \frac{df}{ds} S_t$$

The pure investment gain over the period $[t, t + dt]$ is the change in the value of the minus one derivative plus the change in the value of the holding of $\frac{df}{ds}$ units of the share. That is:

$$- df(t, S_t) + \frac{df}{ds} dS_t$$

Note, $\frac{df}{ds}$ is constant over the interval $[t, t + dt]$.

2. The risk-free portfolio

Therefore we have:

$$- \mathrm{d}f(t, S_t) + \frac{df}{ds}\,\mathrm{d}S_t = - \{\frac{df}{ds}\sigma S_t \mathrm{d}Z_t + [\frac{df}{dt} + \frac{df}{ds}\mu S_t + ½\,\frac{d^2 f}{ds^2}\sigma^2\, S_t^2]\,\mathrm{dt}\,\} + \frac{df}{ds}\,[S_t \mu \mathrm{dt} + \sigma S_t \mathrm{d}Z_t]$$

After cancelling some terms on the right-hand side of the equation we are left with:

$$-\mathrm{d}f(t, S_t) + \frac{df}{ds}\,\mathrm{d}S_t = (\,- \frac{df}{dt} - ½\,\frac{d^2 f}{ds^2}\sigma^2\, S_t^2\,)\,\mathrm{dt}$$

Now note that the expression for $-\mathrm{d}f(t, S_t) + \frac{df}{ds}\,\mathrm{d}S_t$ involves dt but not $\mathrm{d}Z_t$ so that the instantaneous investment gain over the short interval t to $t + dt$ is risk-free.

Given that the market is assumed to be arbitrage-free, this rate of interest must be the same as the risk-free rate of interest on the cash bond.

(If this was not true then arbitrage opportunities would arise by going long in cash and short in the portfolio (or vice versa) with zero cost initially and a certain, risk-free profit an instant later.)

2. The risk-free portfolio

Therefore we must have, for all *t* and S_t> 0 , the alternative expression:

$$-df\,(t, S_t) + \frac{df}{ds}\, dS_t = r\, V\,(t, S_t)\, dt$$

How do we get the right-hand side of this equation???

The value of the portfolio must accumulate to $e^{rdt}V_t$ during the short time interval *dt*. If we expand the exponential as a series, and ignore second-order and higher-order terms (because *dt* is infinitesimal), then we see that the accumulated value is $V_t(1 + rdt) = V_t + rV_t dt$. The change in the value must therefore be $rV_t dt$.

3. The Black-Scholes PDE

Recall that $V(t, S_t) = -f(t, S_t) + \frac{df}{ds} S_t$ is the value of the portfolio. We now have two different expressions for $-df(t, S_t) + \frac{df}{ds} dS_t$. If we equate these, we get:

$$\left(-\frac{df}{dt} - \tfrac{1}{2}\frac{d^2 f}{ds^2}\sigma^2 S_t^2\right) dt = r\left(-f(t, S_t) + \frac{df}{ds} S_t\right) dt$$

Therefore we have:

$$\boldsymbol{\frac{df}{dt} + rS_t\frac{df}{ds} + \tfrac{1}{2}\frac{d^2 f}{ds^2}\sigma^2 S_t^2} = rf$$

This is known as the Black-Scholes PDE.

So, we have a non-stochastic partial differential equation (PDE) that can be solved to determine the value of the derivative.

3. Boundary Conditions

The value of the derivative is found by specifying appropriate boundary conditions and solving the PDE.

The boundary conditions are

$f(T, S_T) = \max\{S_T - K, 0\}$ for a call option

$f(T, S_T) = \max\{K - S_T, 0\}$ for a put option

4. In terms of Greeks

We now return to the intuitive interpretation of the Greeks, this time within the context of the Black-Scholes PDE.

From the Black-Scholes PDE, we have:

$$\frac{df}{dt} + rS_t\frac{df}{ds} + ½\frac{d^2f}{ds^2}\sigma^2 S_t^2 = rf$$

i.e.

$$\Theta + rS_t\Delta + ½\,\sigma^2 S_t^2\,\gamma = rf$$

If the delta and gamma of a portfolio are both zero then Θ is the risk-free rate of growth of the portfolio.

4. The Delta

Delta

It is worth noting the following results relating to delta, Δ. The following results are derived by differentiating the Black-Scholes and Garman-Kohlhagen (for dividend paying stock – given on page 47 of tables) formulae with respect to S_t:

- For a European call option on a non-dividend-paying share, Δ = $\Phi(d_1)$
- For a European put option on a non-dividend-paying share, Δ = -$\Phi(-d_1)$
- For a European call option on a dividend-paying share, with a continuously-compounded dividend yield, q, Δ = $e^{-q(T-t)}\ \Phi(d_1)$
- For a European put option on a dividend-paying share, with a continuously-compounded dividend yield, q, Δ = -$e^{-q(T-t)}\ \Phi(-d_1)$

Question – Finding one unknown

An investor buys, for a premium of 187.06, a call option on a non-dividend-paying stock whose current price is 5,000. The strike price of the call is 5,250 and the time to expiry is 6 months. The risk-free rate of return is 5% *pa* continuously compounded.

The Black-Scholes formula for the price of a call option on a non-dividend-paying share is assumed to hold.

(i) Calculate the price of a put option with the same time to maturity and strike price as the call.

(ii) The investor buys a put option with strike price 4,750 with the same time to maturity. Calculate the price of the put option if the implied volatility were the same as that in (i).

[You need to estimate the implied volatility to within 1% *pa* of the correct value.]

Solution

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(i) ***Calculate the price of a put option with strike price 5,250***

The put-call parity relationship states that:

$$c_t + Ke^{-r(T-t)} = p_t + S_t$$

Substituting in the values given:

$$187.06 + 5{,}250e^{-0.05\times ½} = p_t + 5{,}000$$

$$\Rightarrow \quad p_t = 307.44$$

Solution

(ii) ***Calculate the price of a put option with strike price 4,750***

We first need to estimate the implied volatility of the stock using the information given in the question. This can be done by trial and improvement. We start by substituting the parameter values into the Black-Scholes formula:

$$c_t = 5{,}000\Phi(d_1) - 5{,}250e^{-0.05\times½}\Phi(d_2)$$

where $d_1 = \dfrac{\log\left(\dfrac{5{,}000}{5{,}250}\right) + \left(0.05 + ½\sigma^2\right)\times ½}{\sigma\sqrt{½}}$ and $d_2 = d_1 - \sigma\sqrt{½}$ [1]

In most instances, $\sigma = 0.2$ is a reasonable starting point for the interpolation. If $\sigma = 0.2$, then substituting both this value and the other parameter values into the Black-Scholes formula gives:

$$d_1 = -0.0975 \text{ and } d_2 = -0.2389$$ [½]

$$\Rightarrow \quad c_t = 5{,}000\underbrace{\Phi(-0.0975)}_{0.4612} - 5{,}250e^{-0.05\times½}\underbrace{\Phi(-0.2389)}_{0.4056} = 229.18$$ [1]

This is above the actual price of 187.06, so we need to try a lower value of σ. If we try $\sigma = 0.1$, then we obtain:

$$d_1 = -0.3011 \text{ and } d_2 = -0.3718$$ [½]

$$\Rightarrow \quad c_t = 5{,}000\underbrace{\Phi(-0.3011)}_{0.3817} - 5{,}250e^{-0.05\times½}\underbrace{\Phi(-0.3718)}_{0.3550} = 90.77$$ [1]

As the two call option prices straddle the actual price of 187.06, we can interpolate between the two values of σ to obtain an estimate for the implied volatility:

$$\frac{187.06 - 90.77}{229.18 - 90.77} \approx \frac{\sigma - 0.1}{0.2 - 0.1} \Rightarrow \sigma \simeq 17\%$$ [1]

We can now use this estimate of σ to determine the price of a put option with a strike price of 4,750:

$$d_1 = \frac{\log\left(\dfrac{5{,}000}{4{,}750}\right) + \left(0.05 + ½\times 0.17^2\right)\times ½}{0.17\times\sqrt{½}} = 0.6948$$ [½]

and:

$$d_2 = d_1 - 0.17\times\sqrt{½} = 0.5746$$ [½]

So the price of the 4,750 put option is:

$$p_t = 4{,}750e^{-0.05\times½}\Phi(-0.5746) - 5{,}000\Phi(-0.6948) = 92.11$$ [1]

[Total 7]

Question – Pricing exotic options

A building society issues a one-year bond that entitles the holder to the return on a weighted-average share index (ABC500) up to a maximum level of 30% growth over the year. The bond has a guaranteed minimum level of return so that investors will receive at least x% of their initial investment back. Investors cannot redeem their bonds prior to the end of the year.

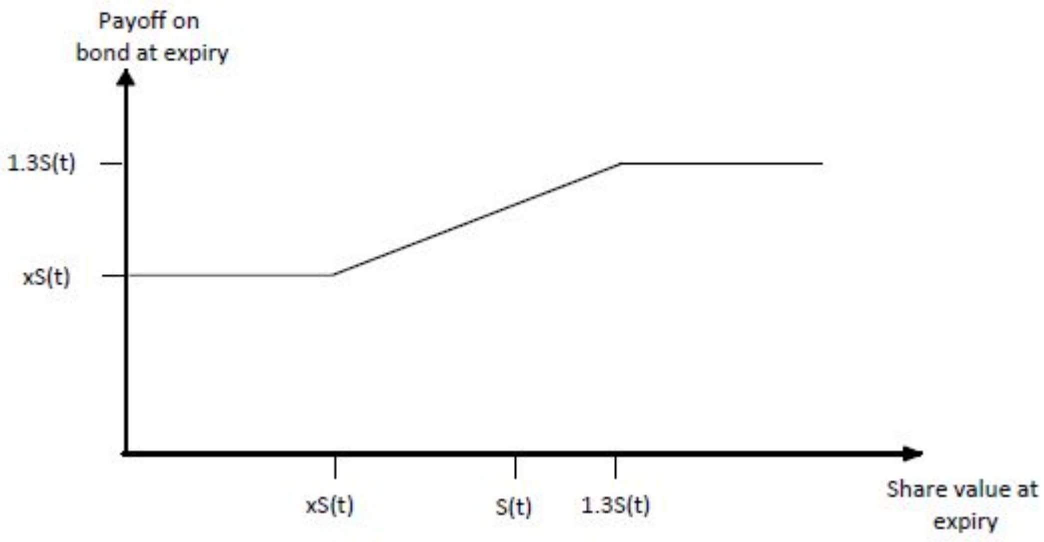
(i) Explain how the building society can use a combination of call and put options to prevent making a loss on these bonds.

(ii) The volatility of the ABC500 index is 30% *pa* and the continuously compounded risk-free rate of return is 4% *pa*. Assuming no dividends, use the Black-Scholes pricing formulae to determine the value of x (to the nearest 1%) that the building society should choose to make neither a profit nor a loss.

Solution

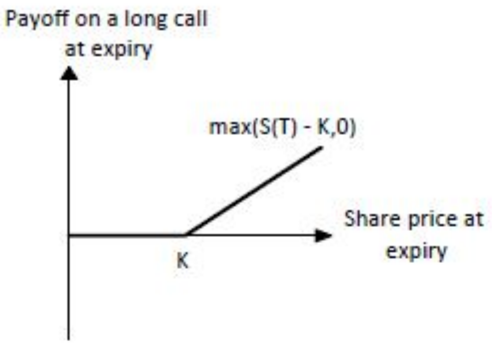
(i) Preventing a loss

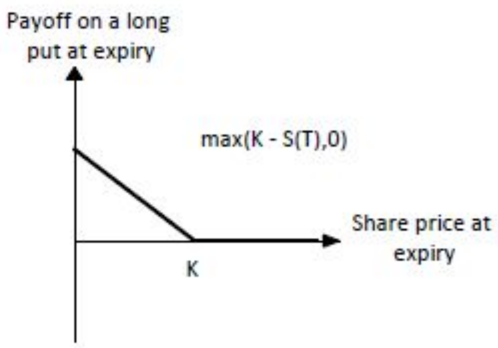
A helpful way of tackling tricky Black-Scholes questions such as this is to draw a graph comparing the payoff on the bond at expiry with the value of the underlying asset at expiry. In the diagram below, S(t) is the value of the initial investment.

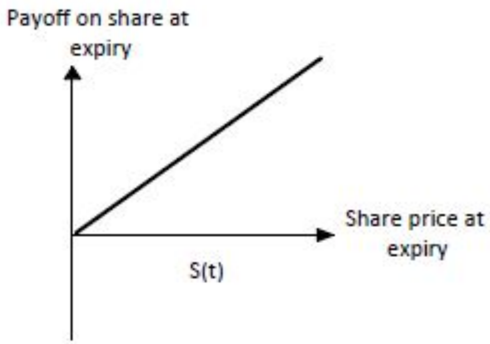


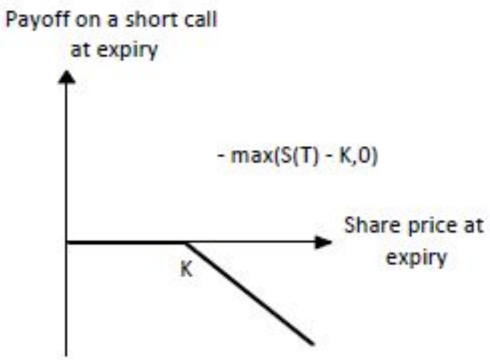
Solution

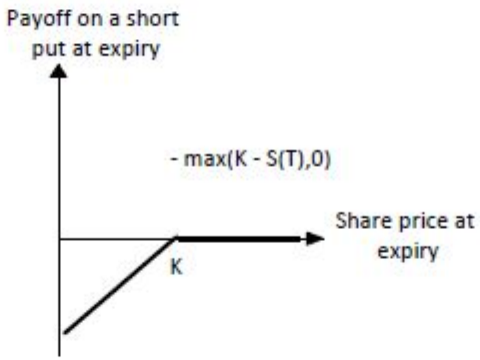
We can compare the shape of this graph against graphs for the payoffs on call and put options, and on the underlying shares.











Solution

The graphs demonstrate that a combination of the underlying shares, a long put with exercise price xSt and a short call with exercise price 1.3St will replicate the graph and hence the payoff on the bond.

If an investor buys a bond the building society can invest the money in the ABC500 so that it is not exposed to movements in the ABC500 index. However, the building society is guaranteeing that investors will receive at least x% of their initial investment back. The building society can hedge this loss by buying a put option on the index with a strike price of x% of the current share price. This put option will cost money – let's say p .

The building society is also limiting the investors' return to 130% of their initial investment. They can do this by selling call options with a strike price of 130% of the current share price. This call option will be priced at c , say. If $c \geq p$ then the building society will not make a loss.

Solution

(ii) *No profit or loss*

If $c = p$ then the building society will not make a profit or a loss. So the problem requires us to work out the price of the call option c and then work out the value of x such that $c = p$.

We will be using the Black-Scholes formula to price the options and, because the numbers are all relative, we can assume that the initial index price is 100, say.

Using the formulae on page 47 of the *Tables*, we can calculate the price of a call option:

$$d_1 = \frac{\ln(S_t/K) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$= \frac{\ln\left(\frac{100}{130}\right) + \left(0.04 + \frac{0.3^2}{2}\right)\times 1}{0.3\sqrt{1}} = -0.5912$$

$$d_2 = d_1 - \sigma\sqrt{T-t} = -0.5912 - 0.3\sqrt{1} = -0.8912$$

$$c = S_t\Phi(d_1) - K\Phi(d_2)e^{-r(T-t)}$$

$$= 100\times\Phi(-0.5912) - 130\times\Phi(-0.8912)e^{-0.04}$$

$$= 100\times 0.277 - 130e^{-0.04}\times 0.186$$

$$= 4.44$$

Solution

We now need to work out the value of x so that p = 4.44 . We will try K= 90 to begin with:

$$d_1 = \frac{\ln(S_t/K) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$= \frac{\ln\left(\frac{100}{90}\right) + \left(0.04 + \frac{0.3^2}{2}\right)\times 1}{0.3\sqrt{1}} = 0.6345$$

$$d_2 = d_1 - \sigma\sqrt{T-t} = 0.6345 - 0.3\sqrt{1} = 0.3345$$

$$p = K\Phi(-d_2)e^{-r(T-t)} - S_t\Phi(-d_1)$$

$$= 90\times\Phi(-0.3345)e^{-0.04} - 100\times\Phi(-0.6345)$$

$$= 90e^{-0.04}\times 0.369 - 100\times 0.263$$

$$= 5.62$$

This is higher than the required value, and so we try a lower value for the strike price, say $K = 80$:

$$d_1 = \frac{\ln(S_t/K) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$= \frac{\ln\left(\frac{100}{80}\right) + \left(0.04 + \frac{0.3^2}{2}\right)\times 1}{0.3\sqrt{1}} = 1.0271$$

$$d_2 = d_1 - \sigma\sqrt{T-t} = 1.0271 - 0.3\sqrt{1} = 0.7271$$

$$p = K\Phi(-d_2)e^{-r(T-t)} - S_t\Phi(-d_1)$$

$$= 80\times\Phi(-0.7271)e^{-0.04} - 100\times\Phi(-1.0271)$$

$$= 80e^{-0.04}\times 0.234 - 100\times 0.152$$

$$= 2.74$$

Solution

We require a put option with a premium of $p = 4.44$. So we linearly interpolate to find the value of K that will give us this:

$$K = 80 + (90 - 80) \times \frac{4.44 - 2.74}{5.62 - 2.74} = 85.90 \qquad [1]$$

So $x = 86\%$ approximately. So, the building society can use $x = 86\%$ and the discussed hedging portfolio to avoid making a loss.

In fact, the exact figure is 86.43%, which is still 86% to the nearest 1%.

Quick Recap

- Under the Black Scholes model the price of the underlying share follows a **geometric Brownian motion**, i.e. the share price **changes continuously** through time according to the stochastic differential equation $dS_t = S_t(\mu dt + \sigma dZ_t)$

- The **risk-free rate of interest is constant**, the same for all maturities and the same for borrowing or lending.

- Let $f(t, S_t)$ be the price at time t of a call option given that the current share price is $S_t = s$, the time of maturity is $T > t$, the exercise price is K.

For such a call option: [The Black Scholes formula]

$$f(t, S_t) = S_t\Phi(d_1) - Ke^{-r(T-t)}\,\Phi(d_2)$$

- The Black Scholes PDE $\frac{df}{dt} + rS_t\frac{df}{ds} + ½\frac{d^2f}{ds^2}\sigma^2 S_t^2 = rf$

Quick Recap

- The boundary conditions are

  $f(T, S_T)$ = max{S_T - K,0} for a call option

  $f(T, S_T)$ = max{K - S_T, 0} for a put option

- For a European call option on a non-dividend-paying share, $\Delta = \Phi(d_1)$
- For a European put option on a non-dividend-paying share, $\Delta = -\Phi(-d_1)$
- For a European call option on a dividend-paying share, with a continuously-compounded dividend yield, q, $\Delta = e^{-q(T-t)}\,\Phi(d_1)$
- For a European put option on a dividend-paying share, with a continuously-compounded dividend yield, q, $\Delta = -e^{-q(T-t)}\,\Phi(-d_1)$