Lecture 1

INSTITUTE OF ACTUARIAL
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Chapter Name: Stochastic Interest Rate Models

Today's Agenda

1. Introduction
   1. Fixed Interest Rate Model
   2. Varying Interest Rate Model
   3. Accumulated Value
   4. Moments of Sn
   5. Moments of An

2. The Log Normal Distribution

1.1 Introduction

The calculations we have done so far have been based on the assumption that future interest rates will take definite values that are known in advance. This is the deterministic approach. In this chapter we will study stochastic interest rate models where future interest rates are assumed to be random. We cannot specify in advance precisely what interest rates will apply. Instead, we can make an assumption about the statistical distribution of future interest rates considered as a random variable.

Preliminary Remarks

Financial contracts are often of a long-term nature. Accordingly, at the outset of many contracts there may be considerable uncertainty about the economic and investment conditions which will prevail over the duration of the contract. Thus, for example, if it is desired to determine premium rates on the basis of one fixed rate of interest, it is nearly always necessary to adopt a conservative basis for the rate to be used in any calculations.

1.1 Introduction

An alternative approach to recognizing the uncertainty that in reality exists is provided by the use of stochastic interest rate models. In such models no single interest rate is used. Variations in the rate of interest are allowed for by the application of probability theory. Possibly one of the simplest models is that in which each year the rate of interest obtained is independent of the rates of interest in all previous years and takes one of a finite set of values, each value having a constant probability of being the actual rate for the year.

1.1 Fixed Interest Rate Model

At this stage we consider briefly an elementary example, which – although necessarily artificial – provides a simple introduction to the probabilistic ideas implicit in the use of stochastic interest rate models.
Suppose that an investor wishes to invest a lump sum of P into a fund which grows under the action of compound interest at a constant rate for n years. This constant rate of interest is not known now, but will be determined immediately after the investment has been made.
The accumulated value of the sum will, of course, be dependent on the rate of interest. In assessing this value before the interest rate is known, it could be assumed that the mean interest rate will apply. However, the accumulated value using the mean rate of interest will not equal the mean accumulated value. In algebraic terms:

$$P(1+\sum_{j=1}^{k}(i_j p_j) = !P(\sum_{j=1}^{k} p_j(1+i_j)^n)$$

Where

i_j is the jth of k possible rates of interest

p_j is the probability of the rate of interest ij

1.2 Varying Interest Rate Model

In our previous example the effective annual rate of interest was fixed throughout the duration of the investment. A more flexible model is provided by assuming that over each single year the annual yield on invested funds will be one of a specified set of values or lie within some specified range of values, the yield in any particular year being independent of the yields in all previous years and being determined by a given probability distribution.

This model is often called the varying interest rate model. The main difference between this and the fixed interest rate model is that, in the varying rate model, the interest rates can be different in each future year, whereas, in the fixed rate model, the same (unknown) interest rate will apply in each future year.

1.3 Accumulated Value

Measure time in years. Consider the time interval [0, n] subdivided into successive periods [0,1],[1,2],....[n-1,n]. For t = 1,2,..., n let i_t be the yield obtainable over the t th year, ie the period [t-1,t].Assume that money is invested only at the beginning of each year. Let Ft denote the accumulated amount at time t of all money invested before time t and let Pt be the amount of money invested at time t. Then, for t = 1,2,3,...:

$$F_t = (1 + i_t)(F_{t-1} + P_{t-1})$$

It follows from this equation that a single investment of 1 at time 0 will accumulate at time n to:

$$S_n = (1 + i_1)(1 + i_2) \ldots (i + i_n)$$

Similarly a series of annual investments, each of amount 1, at times 0,1,2, n -1,n will accumulate at time n to:

$$\begin{aligned} A_n = {} & (1 + i_1)(1 + i_2)(1 + i_3) \ldots (1 + i_n) \\ & + (1 + i_2)(1 + i_3) \ldots (1 + i_n) \\ & + \ldots \\ & + (1 + i_n) \end{aligned}$$

Note that An and Sn are random variables, each with its own probability distribution function.

1.4 Moments of S_n

Let's consider the kth moment of Sn

$$(S_n)^k = \prod_{t=1}(1+i_t)^k$$

$$E[S_n^k] = E[\prod_{t=1}^{n}(1+i_t)^k$$
$$= \prod_{t=1} E[(1+i_t)^k]$$

Suppose that the yield each year has mean j and variance s^2 . Then, letting k=1 in Equation we have $\mathrm{E}[S_n] = \prod_{t=1} E[(1+i_t)]$

$$= \prod_{t=1}(1+E[i_t])$$
$$= (1+j)^n$$

since, for each value of t $E[i_t] = j$

With k = 2

$$E(S_n^2] = \prod_{t=1} E[(1+2i_t+i_t^2)]$$
$$= \prod_{t=1}(1+2E[i_t]+E[i_t^2])$$
$$= (1+2j+j^2+s^2)^n$$

1.4 Moments of S_n contd...

Since, for each value of t:
$E[i_t^2] = (E[i_t])^2 + var[i_t] = j^2 + s^2$

The variance of S_n is :
$var[S_n] = E[S_n^2] - (E[S_n])^2$

This means that
$var(S_n) = [(1+j)^2 + s^2]^n - (1+j)^{2n}$

1.4 Moments of A_n

Remember that An is a random variable that represents the accumulated value at time n of a series of annual investments, each of amount 1, at times 0,1,2,.... n-1.
$i_1, i_2, \dots, i_n$ are independent random variables, each with a mean j and a variance s^2.

$$
\begin{aligned}
A_{n-1} = {} & (1+i_1)(1+i_2)(1+i_3)\dots(1+i_{n-1}) \\
& +(1+i_2)(1+i_3)\dots(1+i_{n-1}) \\
& +\dots \\
& +(1+i_{n-1})
\end{aligned}
$$

and :

$$
\begin{aligned}
A_n = {} & (1+i_1)(1+i_2)(1+i_3)\dots(1+i_n) \\
& +(1+i_2)(1+i_3)\dots(1+i_n) \\
& +\dots \\
& +(1+i_n)
\end{aligned}
$$

1.4 Moments of A_n contd...

$$A_n = (1 + i_n)(1 + A_{n-1})$$

Equation (1.8) can also be deduced easily by general reasoning. A_{n-1}, is the accumulated value at time n-1 of a series of annual payments, each of amount 1, at times 0,1,2,... n-2. The value, at time n-1, of the same series of payments together with an extra payment at time n-1 is $1+A_{n-1}$. Accumulating this value forward to time n gives $(1 + i_n)(1 + A_{n-1})$ and this is equivalent to A_n.

1.4 Moments of A_n contd...

The usefulness of Equation (1.8) lies in the fact that, since A_{n-1} depends only on the values $i_1, i_2, \ldots, i_{n-1}$, the random variables i_n and A_{n-1} are independent. (By assumption the yields each year are independent of one another.) Accordingly, Equation (1.7) permits the development of a recurrence relation from which may be found the moments of A_n . We illustrate this approach by obtaining the mean and variance of A_n.

Let

$\mu_n = E[A_n]$

And let

$m_n = E[A_n^2]$

Since : $A_1 = 1 + i_1$

It follows that

$E[A_1] = E[1 + i_1] = 1 + E[i_1] = 1 + j$

$\mu_1 = 1 + j$

$m_1 = E[A_1^2] = E[(1 + i_1)^2] = 1 + 2E[i_1] + E[i_1^2]$

$m_1 = 1 + 2j + j^2 + s^2$

1.4 Moments of A_n contd...

$\mu_n = (1+j)(1+\mu_{n-1}) \qquad n \geq 2$

Applying the recursive formula repeatedly for each year, gives:

$\mu_n = (1+j)[1+\mu_{n-1}] = (1+j) + (1+j)\mu_{n-1}$

$\mu_n = (1+j) + (1+j)^2 + (1+j)^3 + \cdots + (1+j)^n$

which is the formula for $s(due)_{\bar{n}|}$, calculated at the expected interest rate j

Thus the expected value of An is simply $s(due)_{\bar{n}|}$, calculated at the mean rate of interest.

Since:

$A_n^2 = (1 + 2i_n + i_n^2)(1 + 2A_{n-1} + A_{n-1}^2)$

By taking expectations we obtain, for $n \geq 2$

$m_n = (1 + 2j + j^2 + s^2)(1 + 2\mu_{n-1} + m_{n-1})$

As the value of μ_{n-1} is known provided a recurrence relation for the calculation successively of $m_2, m_3, m_4, \ldots$. The variance of An may be obtained as:

$var[A_n] = m_n - \mu_n^2$

2. The Log-Normal Distribution

In general a theoretical analysis of the distribution functions for An and Sn is somewhat difficult, even in the relatively simple situation when the yields each year are independent and identically distributed. There is, however, one special case for which an exact analysis of the distribution function for Sn is particularly simple.

Because of the compounding effect of interest, the accumulated value of an investment bond grows multiplicatively. This makes the log-normal distribution a natural choice for modelling the annual growth factors 1+i , since a log-normal random variable can take any positive value and has the following multiplicative property:

If $X_1 \sim \log N\,(\mu_1, \sigma_1^2)$ $X_2 \sim \log N(\mu_2, \sigma_2^2)$ are independent random variables then;

$$X_1 X_2 \sim \log N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$$

2. The Log-Normal Distribution contd..

Suppose that the random variable $\log(1+i_t)$ is normally distributed with mean μ and variance σ^2. In this case, the variable $(1+i_t)$ is said to have a log-normal distribution with parameters μ and σ^2

$$\log S_n = \sum_{t=1} \log(1+i_t)$$

The sum of a set of independent normal random variables is itself a normal random variable. Hence, when the random variables $(1+i_t)$ $(t \geq 1)$ are independent and each has a log-normal distribution with parameters and μ and σ^2, the random variable Sn has a log-normal distribution with parameters $n\mu$ and nσ^2.

i.e $\log S_n \sim N(n\mu, n\sigma^2)$

Since the distribution function of a log-normal variable is readily written down in terms of its two parameters, in the particular case when the distribution function for the yield each year is log-normal we have a simple expression for the distribution function of Sn

2. The Log-Normal Distribution contd..

Similarly for the present value of a sum of 1 due at the end of n years:

$$V_n = (1 + i_1)^{-1} \dots (1 + i_n)^{-1}$$
$$\log V_n = -\log(1 + i_1) - \cdots - \log(1 + i_n)$$

Since, for each value of t, log($1 + i_t$) it is normally distributed with mean μ and variance σ2 , each term on the right hand side of the above equation is normally distributed with mean -μ and variance σ^2 . Also the terms are independently distributed. So, logVn is normally distributed with mean -nμ and variance nσ^2 . That is, Vn has log-normal distribution with parameters $-n\mu \; and \; n\sigma^2$

By statistically modelling Vn , it is possible to answer questions such as:

- to a given point in time, for a specified confidence interval, what is the range of values for an accumulated investment
- what is the maximum loss which will be incurred with a given level of probability

Question

Calculate the mean and variance of the accumulated value of an initial investment of £40,000 at the end of 25 years if the annual rates of return are assumed to conform to the varying interest rate model and follow a Gamma (16,200) distribution. (You can find formulae for the mean and variance of a Gamma (α, λ) distribution on page 12 of the Tables.)

Solution

Let i_k be the return in year k. Therefore:

$$i_k \sim Gamma(16, 200)$$

Using the formulae for the mean and variance of the gamma distribution from page 12 of the *Tables*:

$$j = E(i_k) = \frac{\alpha}{\lambda} = \frac{16}{200} = 0.08$$

$$s^2 = \text{var}(i_k) = \frac{\alpha}{\lambda^2} = \frac{16}{200^2} = (0.02)^2$$

So, the mean of the accumulated amount is:

$$40,000E(S_{25}) = 40,000(1+j)^{25} = 40,000 \times 1.08^{25} = £273,900$$

and the variance is:

$$\begin{aligned} \text{var}(40,000S_{25}) &= 40,000^2 \ \text{var}(S_{25}) \\ &= 40,000^2 \left([(1+j)^2 + s^2]^{25} - (1+j)^{50}\right) \\ &= 40,000^2[(1.08^2 + 0.02^2)^{25} - (1.08)^{50}] \\ &= (£25,400)^2 \end{aligned}$$

(If you keep exact values during the calculation, you should get £273,939 and $(£25,417)^2$.)

Question

Derive expressions for the mean and variance of the accumulated value of 1 unit after n years for the fixed interest rate model, assuming that the annual growth rate has a lognormal distribution with parameters $\mu \ and \ \sigma^2$

Solution

For the fixed interest rate model:

$$S_n = (1+i)^n$$

If $1+i \sim \log N(\mu, \sigma^2)$, then:

$$\log(1+i) \sim N(\mu, \sigma^2)$$

So:

$$\log(1+i)^n = n\log(1+i) \sim N(n\mu, n^2\sigma^2)$$

So:

$$S_n = (1+i)^n \sim \log N(n\mu, n^2\sigma^2)$$

Using the formulae for the mean and variance of the log-normal distribution:

$$E(S_n) = e^{n\mu + \frac{1}{2}n^2\sigma^2}$$

$$\text{var}(S_n) = e^{2n\mu + n^2\sigma^2}(e^{n^2\sigma^2} - 1)$$

Quick Recap

- $P(1 + \sum_{j=1}^{k}(i_j p_j) = !\, P(\sum_{j=1}^{k} p_j (1 + i_j)^n)$

Where

i_j is the jth of k possible rates of interest

p_j is the probability of the rate of interest ij

- $F_t = (1 + i_t)(F_{t-1} + P_{t-1})$
- $S_n = (1 + i_1)(1 + i_2) \ldots (i + i_n)$
- $A_n = (1 + i_1)(1 + i_2)(1 + i_3) \ldots (1 + i_n)$
  $+(1 + i_2)(1 + i_3) \ldots (1 + i_n)$
  $+\ldots$
  $+ (1 + i_n)$
- $\prod_{t=1} E[(1 + i_t)]$
  $= \prod_{t=1}(1 + E[i_t])$
  $= (1 + j)^n$
- $var(S_n) = [(1 + j)^2 + s^2]^n - (1 + j)^{2n}$

Quick Recap

- The Log Normal Distribution
- $\log S_n = \sum_{t=1} \log(1 + i_t)$
- $\log S_n \sim N(n\mu\,, n\sigma^2)$