Lecture 1

INSTITUTE OF ACTUARIAL
& QUANTITATIVE STUDIES

**Class:** M.Sc. – Sem 2

**Subject :** Financial Engineering

**Chapter:** Unit 4 Chapter 2

**Chapter Name:** Models for term structure of interest rates

# Today's Agenda

1. Zero Coupon Bonds

2. Yield Curves

3. Short Rates and Forward Rates

4. Equilibrium Models
   - 4.1 The Vasicek Model
   - 4.2 The Cox-Ingersoll-Ross Model
   - 4.3 The Hull White Model

# 1 Zero-coupon bonds

The multitude of traded instruments leads to the first challenge in interest rate modelling: the multitude of definitions of interest rates.

- The basic debt instrument is the discount bond (or, equivalently, the zero-coupon bond). This is an asset that will pay one unit of currency at time T and is traded at time t < T. If the interest rate, R, is constant between t and T then we can say that the price of the discount bond purchased at t and maturing at T is given by P(t,T) where:

$$P(t,T) = \frac{1}{\left(1+R(t,T)\right)^{T-t}}$$

- The spot rate R(t ,T) is the effective rate of interest applicable over the period from time t to time T that is implied by the market prices at time t.

# 1 Zero-coupon bonds

Observe that P(T,T) = 1 and for all t <T , P(t , T) < P(T,T) = 1. We define $\tau = T - t$ in what follows.

- The discrete bond yield calculated from discount bond prices is:

$$R(t, t+\tau) = \frac{1}{P(t,t+\tau)^{1/\tau}} - 1$$

- If a 'spot' rate is paid m times a year, then:

$$\frac{1}{P(0,n)} = \left(1 + \frac{R}{m}\right)^{nm}$$

The limit as m → ∞ is a continuously compounded rate, r(t,T) ('force of interest'), such that:

$$e^{-r(t,T)\tau} = P(t,T) = \frac{1}{\left(1+R(t,T)\right)^{\tau}}$$

The continuously compounded bond yields is calculated as

$$r(t,T) = -\frac{\ln P(t,T)}{\tau}$$

## 2 Yield Curves

- Fixing t=0 and plotting yield, $R(0,t)$ or $r(0,t)$ , against maturity', T, gives the yield curve which gives information on the term structure, how interest rates for different maturities are related. Typically, the yield curve increases with maturity, reflecting uncertainty about far-future rates. However, if current rates are unusually high, the yield curve can be downward sloping, and is inverted.

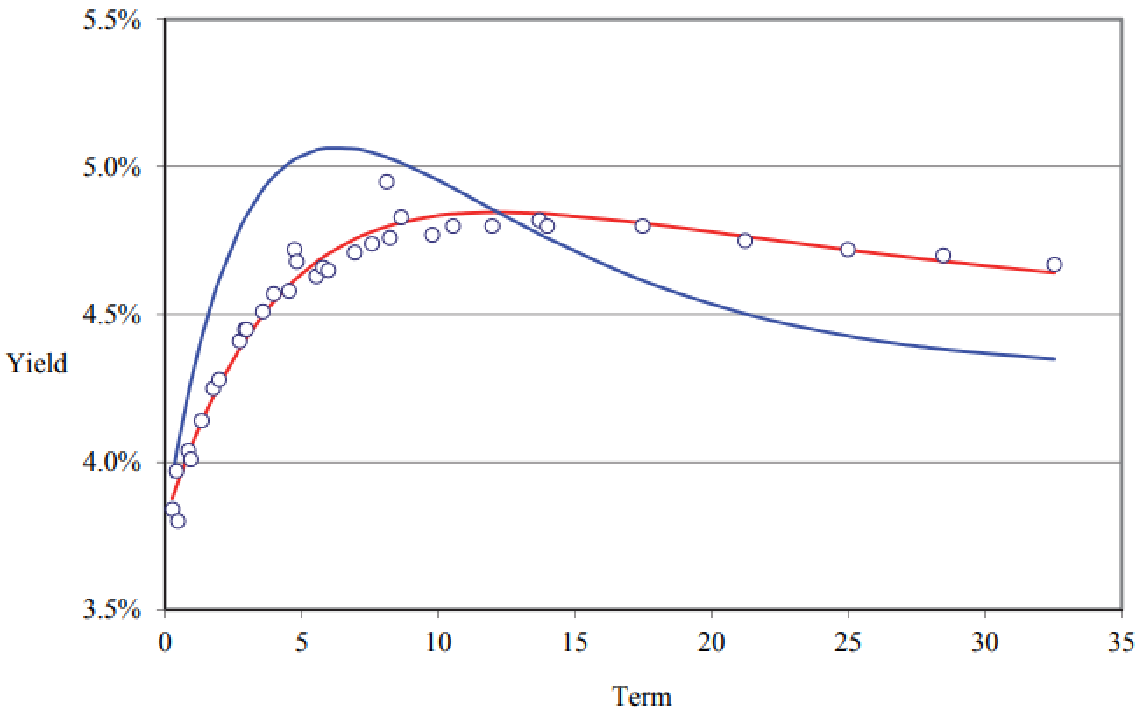
- There are various theories explaining the shape of the yield curve. The expectations theory argues that the long-term rate is determined purely by current and future expected short-term rates, so that the expected final value of investing in a sequence of short-term bonds equals the final value of wealth from investing in long-term bonds.

# 2 Yield Curves

- There are various theories explaining the shape of the yield curve. The expectations theory argues that the long-term rate is determined purely by current and future expected short-term rates, so that the expected final value of investing in a sequence of short-term bonds equals the final value of wealth from investing in long-term bonds.

- The liquidity preference theory argues that lenders want to lend short term while borrowers wish to borrow long term, and so forward rates are higher than expected future zero rates (and yield curves are upward sloping).

# 2 Yield Curves




UK yield curves 31 December 2003

# 3 Short Rates and Forward Rates

➢ Short Rates

>The short or instantaneous rate, r(t), is the interest rate charged today for a very short period (i.e. overnight). This is defined (equivalently) as:

$$r(t) = r(t, t+\delta) \sim R(t, t+\delta)$$

where $\delta$ is a small positive quantity. So the short rate r (t) is the force of interest that applies in the market at time t for an infinitesimally small period of time $\delta$. Using the relationship developed in the opening section we have

$$r(t) = \frac{\partial}{\partial \delta} \ln P(t, t+\delta)$$

The short rate is often the basis of some interest rate models; however, it will not generate, on its own, discount bond prices.

# 3 Short Rates and Forward Rates

- Forward Rates

The forward rate, F (0,t,T) if discretely compounded and f (0,t,T) if continuously compounded, relates to a loan starting at time t, for the fixed forward rate, the forward rate, repaid at maturity, T. It involves three times, the time at which the forward rate agreement is entered into (typically 0), the start time of the forward rate, t and the maturity of the forward rate agreement, T.

The law of one-price/the no-arbitrage principle, implies:

$$F(0,t,T) = \left(\frac{P(0,t)}{P(0,T)}\right)^{\frac{1}{T-t}} - 1$$

# 3 Short Rates and Forward Rates

➢ Forward Rates

The forward rate, F (0,t,T) if discretely compounded and f (0,t,T) if continuously compounded, relates to a loan starting at time t, for the fixed forward rate, the forward rate, repaid at maturity, T. It involves three times, the time at which the forward rate agreement is entered into (typically 0), the start time of the forward rate, t and the maturity of the forward rate agreement, T.

The law of one-price/the no-arbitrage principle, implies:

$$F(0,t,T) = \left(\frac{P(0,t)}{P(0,T)}\right)^{\frac{1}{T-t}} - 1$$

# Question

What do you think term structure models are used for?

Answers –

The main uses of term structure (interest rate) models are:

- by bond traders looking to identify and exploit price inconsistencies
- for calculating the price of interest rate derivatives
- by investors with a portfolio involving bonds or loans who want to set up a hedged position
- for asset-liability modelling.

# 4 Equlibrium Models

- Equilibrium models start with a theory about the economy, such that interest rates revert to some long-run average, are positive or their volatility is constant or geometric. Based on the model for (typically) the short rate, the implications for the pricing of assets is worked out. Examples of equilibrium models are Rendleman and Bartter, Vasicek and Cox Ingersoll-Ross.
- Being based on ‘economic fundamentals’, equilibrium models rarely reproduce observed term structures. This is unsatisfactory.
- No-arbitrage models use the term structure as an input and are formulated to adhere to the no-arbitrage principle. An example of a no-arbitrage model is the Hull-White (one- and two factor).

## 1.1 Risk Neutral Approach to Pricing

- The risk- neutral approach to pricing

  We will assume that the short rate is driven by an Ito diffusion:

  $$dr_t = \mu(t, r_t)dt + \sigma(t, r_t)d\widetilde{W}_t$$

  where $\mu(t, r_t)$ is the drift parameter

  $\sigma(t, r_t)$ is the voltality parameter

  Wt is a Wiener process under the martingale measure.

# 4.1 The Vasicek Model

- Vasicek assumes that :
  $dr_t = \alpha(\mu - r_t)dt + \sigma dW_t^{\sim}$
  for constants $\alpha > 0$ , $\mu$ and, $\alpha$.

Here $\mu$ represents the 'mean' level of the short rate. If the short rate grows (driven by the stochastic term) the drift becomes negative, pulling the rate back to $\mu$ . The speed of the 'reversion' is determined by $\alpha$. If $\alpha$ is high, the reversion will be very quick.

Instantaneous Forward Rates

$$f(t,T) = r(t)e^{-\alpha\tau} + \left(\mu - \frac{\sigma^2}{2\alpha^2}\right)(1 - e^{-\alpha\tau}) + \frac{\sigma^2}{2\alpha^2}(1 - e^{-\alpha\tau})e^{-\alpha\tau}$$

# 4.1 The Vasicek Model

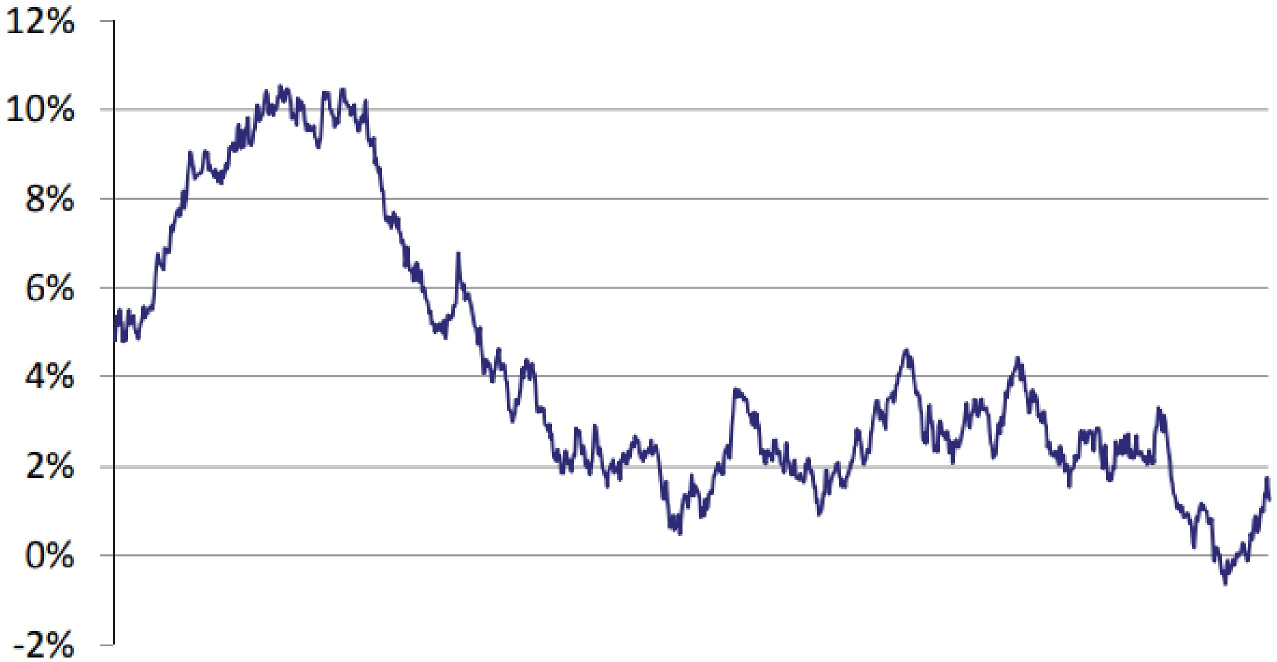
- Vasicek assumes that :
  $dr_t = \alpha(\mu - r_t)dt + \sigma dW_t^{\sim}$
  for constants $\alpha > 0$ , $\mu$ and, $\alpha$.

Here $\mu$ represents the 'mean' level of the short rate. If the short rate grows (driven by the stochastic term) the drift becomes negative, pulling the rate back to $\mu$ . The speed of the 'reversion' is determined by $\alpha$. If $\alpha$ is high, the reversion will be very quick.

Instantaneous Forward Rates

$$f(t,T) = r(t)e^{-\alpha\tau} + \left(\mu - \frac{\sigma^2}{2\alpha^2}\right)(1 - e^{-\alpha\tau}) + \frac{\sigma^2}{2\alpha^2}(1 - e^{-\alpha\tau})e^{-\alpha\tau}$$

# 4.1 The Vasicek Model




Example simulation of short rate from the Vasicek model

# 4.2 The Cox Ingersoll Ross Model

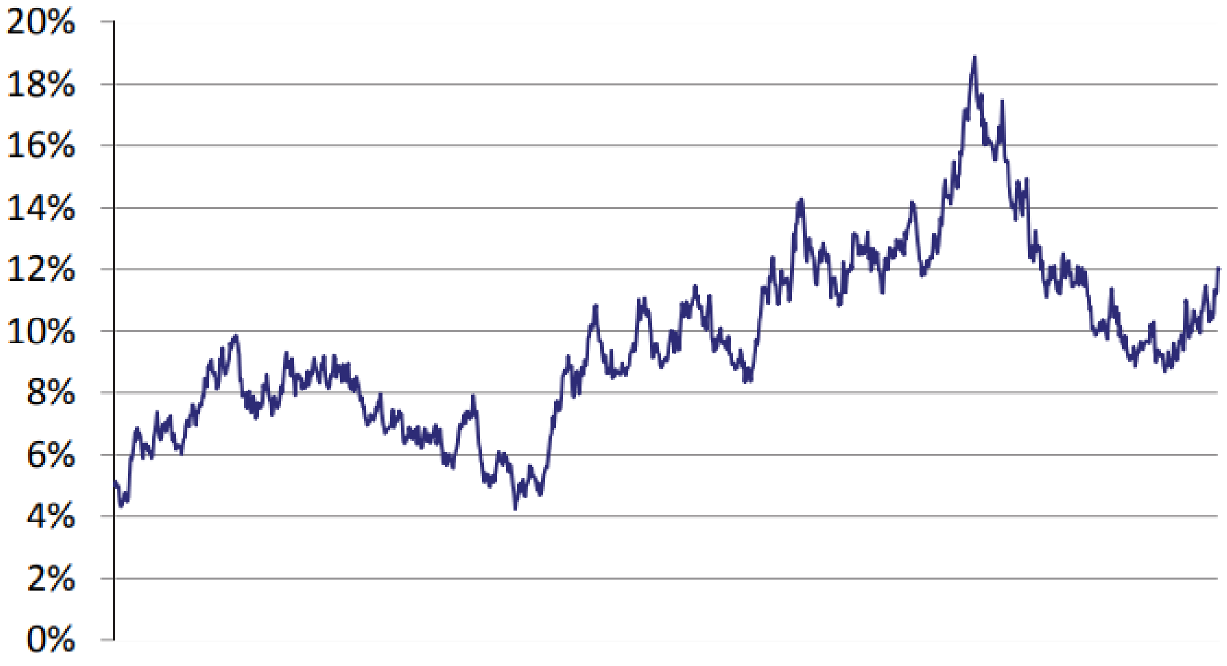
- In Vasicek's model (and Hull-White, below) interest rates are not strictly positive. This assumption is not ideal for a short-rate model. CIR use the Feller, or square root mean reverting process which is positive (it can instantaneously touch 0 but immediately rebounds):

  $$dr_t = \alpha(\mu - r_t)dt + \sigma\sqrt{r_t}dW_t$$

  The associated PDE is

  $$\frac{\partial g(t,r_t)}{\partial t} + \frac{\partial g(t,r_t)}{\partial r_t} * \alpha(\mu - r_t) + \frac{1}{2} * \frac{\partial^2 g(t,r_t)}{\partial r_t^2} * \sigma^2 r_t - r_t g(t,r_t) = 0$$

# 4.2 The Cox Ingersoll Ross Model




Simulation from Cox-Ingersoll-Ross model

# 4.3 The Hull White model

- The Hull-White model is an extension of Vasicek where the mean-reversion level, $\mu$, is a deterministic function of time:

$$dr_t = \alpha(\mu(t) - r_t)dt + \sigma dW_t$$

for $\alpha > 0$ and $\sigma$

This yield a PDE similar to vasicek,

$$\frac{\partial g(t,r_t)}{\partial t} + \frac{\partial g(t,r_t)}{\partial r_t}\alpha(\mu(t) - r_t) + \frac{1}{2} * \frac{\partial^2 g(t,r_t)}{\partial r_t^2}\sigma^2 - r_t g(t,r_t) = 0$$

# Question

If $A(t)$ is a strictly positive supermartingale, then zero-coupon bond prices can be modelled using the formula $B(t,T) = \dfrac{E_P[A(T)\,|\,F_t]}{A(t)}$, where $P$ is a suitably-chosen probability measure.

(i) (a) Express mathematically the fact that $A(t)$ is a strictly positive supermartingale.

(b) Verify that the function $A(t) = e^{-0.05t+0.02W(t)}$, where $W(t)$ denotes standard Brownian motion, satisfies the properties in (i)(a).

(c) State why the supermartingale property is required.

(d) Write down the name given to this type of process. [7]

(ii) By writing $A(t)$ in the form $A(t) = e^{X(t)}$, or otherwise, show that $A(t)$ satisfies a stochastic differential equation of the form:

$$dA(t) = A(t)[\mu_A(t)dt + \sigma_A(t)dW(t)]$$

State the forms of the functions $\mu_A(t)$ and $\sigma_A(t)$. [4]

# Question

(iii) (a) Write down or derive a formula for $P(t,T)$ based on the process $A(t)$ specified in (i)(b).

(b) Write down expressions for the instantaneous forward rate $f(t,T)$ and the spot rate $r(t,T)$ based on this model.

(c) State one problem that this model of interest rates has. [4]

(iv) Calculate the prices at time 5 according to the model in (ii) of the following risk-free bonds:

(a) a 10-year zero-coupon bond

(b) a 10-year bond that pays a coupon of 5% at the end of each year. [4]

[Total 19]

# Solution

(i)(a) ***Express these properties mathematically***

'Strictly positive' simply means that:

$A(t) > 0$ for all times $t$ [1]

The 'supermartingale' property means that, whenever $t < T$ :

$E_P[A(T)\,|\,F_t] \leq A(t)$ [1]

(i)(b) ***Verify that this function has these properties***

The presence of the exponential function ensures that this function is strictly positive. [½]

Since $A(t)$ is strictly positive, the supermartingale property is equivalent to:

$\dfrac{E_P[A(T)\,|\,F_t]}{A(t)} < 1$ [½]

# Solution

With the definition given for $A(t)$, the left-hand side is:

$$LHS = \frac{E_P[A(T) \mid F_t]}{A(t)}$$

$$= e^{0.05t - 0.02W(t)} E_P\left[ e^{-0.05T + 0.02W(T)} \middle| F_t \right]$$

$$= e^{-0.05(T-t)} E_P\left[ e^{0.02[W(T)-W(t)]} \middle| F_t \right] \qquad [1]$$

Because of the independent increments property of Brownian motion, we can drop the $F_t$. We can then use the fact that $W(T) - W(t) \sim N(0, T-t)$ under $P$ to evaluate the expectation on the right-hand side, which corresponds to an MGF based on a normal distribution. Using the formula given on page 11 of the *Tables*, we get:

$$LHS = e^{-0.05(T-t)} e^{½(0.02)^2(T-t)} = e^{-0.0498(T-t)} \qquad [½]$$

When $t < T$ (which we have assumed throughout), this is indeed less than 1. So the supermartingale property is satisfied. [½]

# Solution

(i)(c) ***State why the supermartingale property is required***

The supermartingale property is equivalent to:

$$\frac{E_P[A(T)\,|\,F_t]}{A(t)} < 1$$

The left-hand side matches the formula for the bond price $B(t,T)$. So this property ensures that the price of a zero-coupon bond is always less than 1. This is equivalent to prohibiting negative interest rates. [1]

(i)(d) ***Name of the process***

The process $A(t)$ is a state price deflator. [1]

[Total 7]

(ii) ***Stochastic differential equation for A(t)***

We can write:

$$A(t) = e^{X(t)}$$

where $X(t) = -0.05t + 0.02W(t)$, so that $dX(t) = -0.05dt + 0.02dW(t)$. [1]

So, using a Taylor Series expansion, we can write:

$$dA(t) = d\left[e^{X(t)}\right] = e^{X(t)}dX(t) + \tfrac{1}{2}e^{X(t)}[dX(t)]^2$$

$$= A(t)\left\{dX(t) + \tfrac{1}{2}[dX(t)]^2\right\}$$ [1]

# Solution

Substituting the SDE for $X(t)$ gives:

$$dA(t) = A(t)\left\{-0.05dt + 0.02dW(t) + \tfrac{1}{2}[-0.05dt + 0.02dW(t)]^2\right\}$$

Simplifying using the $2 \times 2$ multiplication grid, we get:

$$dA(t) = A(t)\left\{-0.05dt + 0.02dW(t) + \tfrac{1}{2}(0.02)^2 dt\right\}$$

$$= A(t)\left\{-0.0498dt + 0.02dW(t)\right\}$$ [1]

So, in this case, the drift and volatility coefficients are:

$\mu_A(t) = -0.0498$ and $\sigma_A(t) = 0.02$ [1]

[Total 4]

(iii)(a) ***Formula for*** $\boldsymbol{P(t,T)}$

We have already evaluated the formula for $P(t,T)$ in part (i)(b), which gave:

$$P(t,T) = \frac{E_P[A(T)|F_t]}{A(t)} = e^{-0.0498(T-t)}$$ [1]

# Solution

(iii)(b) ***Expressions for $f(t,T)$ and $r(t,T)$***

The instantaneous forward rate is:

$$f(t,T) = -\frac{\partial}{\partial T}\log P(t,T) = -\frac{\partial}{\partial T}[-0.0498(T-t)] = 0.0498$$

*ie* a constant rate of 4.98%. [1]

The spot rate therefore also takes a constant value of 4.98%. [1]

(iii)(c) ***One problem with this model***

A model with constant interest rates over all terms is not arbitrage-free. This would be a serious problem if the model was used in practical applications. [1]

[Total 4]

(iv)(a) ***Price of a zero-coupon bond***

According to this model, the price at time 5 (or indeed, at *any* time) of a 10-year zero-coupon bond is:

$$P(5,15) = e^{-0.0498(15-5)} = e^{-0.498} = 0.6077$$

*ie* 60.77 per 100 nominal. [2]

# Solution

(iv)(b) ***Price of a 5% annual coupon bond***

The price of a 5% annual coupon bond per 100 nominal is:

$$P = 5[v + v^2 + \cdots + v^{10}] + 100v^{10}$$

where $v = e^{-0.0498} = 0.95142$.

Evaluating the sum as a geometric progression, we get:

$$P = 5v\left(\frac{1 - v^{10}}{1 - v}\right) + 60.77 = 5(7.682) + 60.77 = 99.18$$ [2]

[Total 4]

# Quick Recap

- The multitude of traded instruments leads to the first challenge in interest rate modelling: the multitude of definitions of interest rates.
- Fixing t=0 and plotting yield, R(0,t) or r(0,t) , against maturity', T, gives the yield curve which gives information on the term structure, how interest rates for different maturities are related. Typically, the yield curve increases with maturity, reflecting uncertainty about far-future rates. However, if current rates are unusually high, the yield curve can be downward sloping, and is inverted.
- The short or instantaneous rate, r(t), is the interest rate charged today for a very short period (i.e. overnight). This is defined (equivalently) as:
  $r(t) = r(t, t+\delta) \sim R(t, t+\delta)$
- The forward rate, F (0,t,T) if discretely compounded and f (0,t,T) if continuously compounded, relates to a loan starting at time t, for the fixed forward rate, the forward rate, repaid at maturity, T. It involves three times, the time at which the forward rate agreement is entered into (typically 0), the start time of the forward rate, t and the maturity of the forward rate agreement, T.
- Equilibrium models start with a theory about the economy, such that interest rates revert to some long-run average, are positive or their volatility is constant or geometric. Based on the model for (typically) the short rate, the implications for the pricing of assets is worked out. Examples of equilibrium models are Rendleman and Bartter, Vasicek and Cox Ingersoll-Ross.

# Quick Recap

| Model | Dynamics | $r_t > 0$ for all $t$ | Distribution of $r_t$ |
|---|---|---|---|
| Vasicek | $dr_t = \alpha(\mu - r_t)dt + \sigma d\tilde{W}_t$ | No | Normal |
| CIR | $dr_t = \alpha(\mu - r_t)dt + \sigma\sqrt{r_t}\,d\tilde{W}_t$ | Yes | Non-central chi-squared |
| Hull-White – Vasicek | $dr_t = \alpha(\mu(t) - r_t)dt + \sigma d\tilde{W}_t$ | No | Normal |
| Hull-White – CIR | $dr_t = \alpha(\mu(t) - r_t)dt + \sigma\sqrt{r_t}\,d\tilde{W}_t$ | Yes | Non-central chi-squared |