INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Subject: Financial Engineering 1

Chapter:

Category: Assignment 1 Questions

1. For a 2-period recombining binomial model for S_t, the price of a non-dividend paying stock at times t = 0,1, and 2 is

$$S_t = \begin{cases} S_{t-1}.u, with\ probability\ p \\ S_{t-1}.d, with\ probability\ 1-p \end{cases}$$

The state price deflator is :

$$A_t = \begin{cases} 0.761, when\ S_t = S_{t-1}.u \\ 1.522, when\ S_t = S_{t-1}.d \end{cases}$$

Risk free rate is 5% p.a. The price of a derivative that pays 1 at time 2, if $S_2 < S_0$.

i) Calculate the value of p.
ii) Calculate the value of q.
iii) Calculate the price at time 0 of a derivative that pays 1 at time 2 if $S_2 > S_0$ using risk neutral probability measure.

2. i) Find E [|W (t)|] using first principles if W (t) is a standard Brownian motion.

ii) Share price of ABC Ltd and XYZ Ltd are at the same price at the beginning of a trading day. Let X(t) denote the rupee amount by which the Share price of ABC Ltd. exceeds Share price of XYZ Ltd when t percent of the trading day has elapsed.

{X(t) $0 \le t \le 1$} is modelled as an Arithmetic Brownian motion process with drift 0 and variance parameter $\sigma^2 = 0.3695$.

After 75% of the trading has elapsed, ABC Ltd's share price is Rs. 40.25 and XYZ Ltd 's share price is Rs. 39.75.

Find the probability that ABC Ltd's share price is higher than XYZ Ltd 's share price at the end of the day.

3. i) Explain the following terms in the context of the binomial model:
  a) Risk neutral probability measure
  b) Recombining binary tree

ii) Consider a 3-period binomial model with following parameters u = 1.2, d = 0.9, $S_0 = 60$, r = 11% per period, K = 60.
Calculate the price of a standard European Call with maturity date at the end of the three periods.

4. Consider a two-period recombining binomial model for S_t the price of a non-dividend paying security at times t = 0, 1 and 2, with real world dynamics:

$S_{t+1} = S_t$ u with probability p

$= S_t$ d with probability 1–p

u >d>0

There also exists a risk-free instrument that offers a continuously compounded rate of return of 7% per period. The state price deflator in this model after one period is:

$A_1 = 0.7610$ when $S_1 = S_0$ u

$= 1.5220$ when $S_1 = S_0$ d

i) Calculate the value of 'p' and the risk-neutral probability measure 'q'.

ii) Calculate 'u' and 'd'.

iii) Calculate the price at time 0 of a derivative that pays 1 at time 2 if S_2 greater than or equal to S_0 using the risk-neutral probability measure derived in part (i).

5. i) Explain what is meant by recombining binomial tree. State the main advantage of using it in modelling the share price movements?

ii) An investor wants to determine the value of 6-months European put option on a non-dividend paying share using a two-step binomial tree.

The current price of the share is ₹550, and the payoff is Max(500-S_T,0) where S_T is the price of the share in 6 months. The investor assumes that for the first period (first 3 months) the share price will increase by 10% or decrease by 5% and the continuously compounding risk-free rate of interest is 2% (per 3 months).

For the next 3 months the expected risk-free rate is assumed to be 3% (per 3 months) and the share price to increase by 20% or decrease by 10%.

a) Calculate the value of the European put option using two-step binomial tree.

b) Calculate the value of exotic American put option if the payoff is Max(500-t/6S_t ,0) where S_t is the share price at time t (eg. t is 3 at first-step).

6. The process X has the stochastic differential equation

$$dX_t = \alpha\, \mu\, (T\text{-}t)\, dt + \sigma \sqrt{(T - t)}\, dZ_t,$$

where $\alpha > 0$ & μ are fixed parameters and Z is a standard Brownian motion under Q.

The function f is given by

$$f(x, t) = e^{(m(T-t)-x)}$$

where m is a differentiable function

Find $\frac{\partial m}{\partial t}$ if f(x,t) is a martingale

7. The stochastic process X follows the SDE given by

$$\frac{dX_t}{X_t} = 0.25\, dt + \sigma\, dW_t$$

where W is a standard Brownian motion.

Consider the new process Y defined by $Y_t = f(t, X_t)$ where

$$f(t, x) = e^{-t} x^2$$

i) Write an expression for dY_t

ii) Under what condition will the process be a martingale?

8. A stock price follows Geometric Brownian Motion with an expected return of 16%and a volatility of 35%. The current price is Rs. 254.

What is the probability that a European call option on the stock with an exercise price of Rs. 258 and a maturity date in six months will be exercised?

What is the probability that a European put option on the stock with the same exercise price and maturity will be exercised?

9. The market price of a certain share is modelled as Geometric Brownian motion.

The price S_t at time $t \geq 0$ satisfies:

$$\log(S_t/S_0) = \mu\, t + \sigma\, B_t$$

where B_t is a standard Brownian motion and μ and σ are constants

i) Show that dS_t can be written as: $dS_t/S_t = x\, dB_t + y\, dt$ where x and y are constants to be specified.

ii) Derive $E[S_t]$ and $Var[S_t]$

iii) Derive $cov[S_{t1}, S_{t2}]$ where $0 < t_1 < t_2$

10. A commodity of price C is assumed to follow the process:

$$dC = \mu C dt + \sigma C dWt$$

where μ and σ are positive constants and Wt is a standard Brownian motion. The continuously compounded risk-free interest rate r is a constant.

You wish to value a special type of option.
You construct a recombining binomial tree algorithm using a proportionate "up step" u and "down step" d for each small-time interval Δt, and the stock price at time 0 is S_0.

i) Specify fully the first step of the binomial process, giving formulae for the up anddown probabilities and step sizes u and d.

The initial commodity price is 80, σ is 15% per annum and r=0. You may assume thatu can be approximated by $\exp(\sigma\sqrt{\Delta t})$ for small Δt

ii) For the tree specified in (i)

a) Draw three steps of the tree with quarter- year time steps and calculate the commodity price at each node.

b) Using this tree, calculate the price of a 9- month European call option with anat-the-money strike.

c) By considering each possible path in the tree, evaluate the price of a 9- month European lookback call option, where the lookback period includes time 0.
Note that lookback call pays the difference between the minimum value and the final value of any asset price.

11. Let {Z(t)} be a standard Brownian motion. You are given

a. $U(t) = 2Z(t) - 2$

b. $V(t) = [Z(t)]^2 - t$

c. $W(t) = t^2 Z(t) - 2\int_0^t sZ(s)ds$

Derive the SDEs and explain which of the processes defined above has / have zerodrift?

12. Suppose the stock price S follows geometric Brownian motion with expected returnμ and volatility σ:

$$dS = \mu S dt + \sigma S dz$$

Where dz is a wiener process

The risk-free rate of interest (with continuous compounding) is r.

i) Determine the process followed by the variable S^K(where k is a positive integer).Find the expected return and variance of S^K.

For the same measure corresponding to dz, show that the stock price discountedat risk-free rate is a martingale only for a specific value of μ.

ii) Can you make any similar claim about the process followed by S^K in part (i)?
What conditions are needed?