INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Subject: Financial Engineering 1

Chapter:

Category: Assignment 1 Solutions

Answer 1 –

i) Given: Derivative price = 0.1448,

payoff = 1 at time 2 if S2 < So

$$A = \begin{cases} 0.761, S_t = uS_{t-1} \\ 1.522, S_t = dS_{t-1} \end{cases}$$

Calculating value of p:

derivative pays after 2 downward moves

0.1448 = A1^2 *(1-p)^2

0.1448/ (1.522^2) = (1-p)^2

P = 0.749983

ii) **Calculating value of q:**

A1 = 0.716 for an upward move

A1 = exp(-r) * (q/p)

0.716 = exp (-0.05) * q/ 0.749983

q = 0.56452

iii) **Derivative price at time 0, if payoff = 1 for S2>So, using q**

If candidates are assuming ud =1, which is generally accepted assumption,

price = o.6^2 * e^(-0.05x2) *1

price = 0.3257 [3]

Or

If the candidates assume specifically for recombining binomial tree and use the condition ud = du. This gives payoffs for upper and middle nodes.

price = 0.6^2 * exp(-0.05x2) + 2 x 0.6 x 0.4exp(-0.05x2)

price = 0.7601 [3]

Answer 2 –

i) Let Z ~ N(0, 1). Then, since W(t) ~ N(0,t) we get [0.5]

$E[|W(t)|] = E[\,|\sqrt{t}Z(t)|]$ [1]

$= \sqrt{t}E[|Z(t)|]$ [1]

$= \sqrt{t}\cdot 2\int_0^{\infty}\frac{e^{-\frac{z2}{2}}}{\sqrt{2\pi}}z\,dz$ (from standard normal integration, since the absolute is used, integration is from 0 to infinity multiplied by 2 times) [2]

$= \sqrt{t}\cdot 2\int_0^{\infty}\frac{e^{-w}}{\sqrt{2\pi}}dw$ (by w = z2/2, dw = z dz)

[2]

$= \frac{\sqrt{t}\cdot 2}{\sqrt{2\pi}} * [e^{-w}]$ ∞ & 0 [1]

$= \frac{\sqrt{2t}}{\sqrt{\pi}}$ [0.5]

ii) ABC Ltd's price is higher than XYZ Ltd 's price at the end of the day would be equivalent to:

X(t) >0

If X(t) is an Arithmetic Brownian motion process

Then

$E[X_T] = x + \mu T$ |

Where,

x = X(3/4) = 40.25 – 39.75 = 0.50

μ =0 |

T = (1- ¾) |

Var $[X_T]$ = $\sigma^2 T$ = 0.3695/4 = 0.092375 [

P (X(1) ≥ 0) = P {(X(1) – E[X_T]) / Var [X_T] ≥ (0 – 0.5) / $\sqrt{0.092375}$ }

= 1 – P(Z ≤ –1.6451) = 0.95

Answer 3-

i) a) Risk neutral probability measure –

Q is the risk-neutral probability measure, which gives the risk-neutral probability q to an upward move in prices and 1-q to a downward move.

The risk-neutral probabilities ensure that the underlying security yields an expected return equal to the risk-free rate

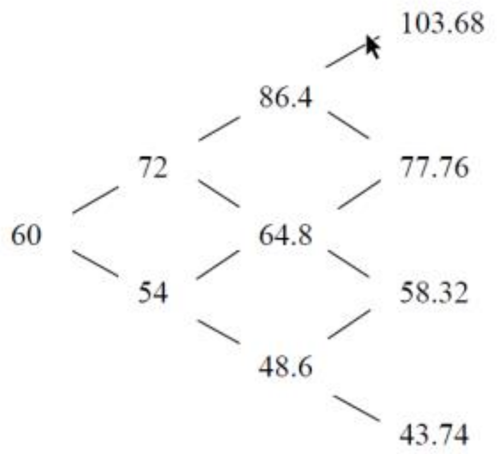
Under the probability measure Q, investors are neutral with regard to risk, they require no additional return for taking on more risk.

b) Recombining binary tree A recombining binomial tree (or binomial lattice) is one in which values of u and d, and consequently the risk-neutral probabilities, are the same in all states.

With such models:

- the volume of computation required is greatly reduced
- Nt , the number of up-steps up to time t, has a binomial distribution with parameters t and q

ii)



$$C_{uuu} = \left(u^3 S_0 - K\right)^+;$$
$$C_{uud} = \left(u^2 d S_0 - K\right)^+;$$
$$C_{udd} = \left(u d^2 S_0 - K\right)^+;$$
$$C_{ddd} = \left(d^3 S_0 - K\right)^+.$$

and in the lowest state,

$$C_2(dd) = \frac{1}{1+r}\left[qC_{udd} + (1-q)C_{ddd}\right],$$

where the risk-neutral probability of an upward move i

$$q = \frac{(1+r)-d}{u-d}.$$

At time 1, we get in the upper state,

$$C_1(u) = \frac{1}{1+r}\left[qC_2(uu) + (1-q)C_2(ud)\right],$$

and in the lower state,

$$C_1(d) = \frac{1}{1+r}\left[qC_2(ud) + (1-q)C_2(dd)\right].$$

At time 0,

$$C_0 = \frac{1}{1+r}\left[qC_1(u) + (1-q)C_1(d)\right].$$

Answer 4

i)

$0.7610 = e^{-r}q/p$ if $S1 = S0u$

$1.5220 = e^{-r}(1 - q)/(1 - p)$ if $S1 = S0d$

Solving the 2 equations with 2 unknown gives

$e^{r} * .7610 * p = (1\text{-}1.5220\,(1 - p) * e^{r})$

$= 1 - 1.5220 * e^{r} + 1.5220 * p * e^{r}$

$1.5220 * e^{r} - 1 = p * e^{r} * (1.5220 - .7610)$

This gives p = 0.7748 and q = 0.63236.

ii)

$(e^{r} - d)/(u - d) = q$

Using the definition for a recombining model ; u x d = 1

Rearranging the terms gives;

$(e^{r} - d)/(1/d - d) = q$

$(e^{r} - d)d/(1 - d^{2}) = q$

$(e^{r} d - d^{2}) = q - d^{2} * q$

$d^2 \times (1 - q) - d \times e^{r} + q = 0$

Solving the quadratic equation; Roots are:

$= - b +/- \text{sqrt}(b2 - 4ac) \; / 4a$

$= e^{r} +/- \text{sqrt}(e^{2r} - 4 * (1 - q) * q) / 4 * (1 - q)$

Substituting values of r and q gives

= (1.072508 + 0.469412)/ 0.74 or (1.072508 - 0.469412)/ 0.74

d = 2.0970 or 0.8202

since 'd' has to be less than 1

d = 0.8202

u = 1.2192

iii) In this case price = 0.63^ 2 x exp (-0.07×2) + 2 × 0.63 × 0.37 x exp (-0.07×2)

= 0.3476 + 0.4042

= 0.7519

Answer 5

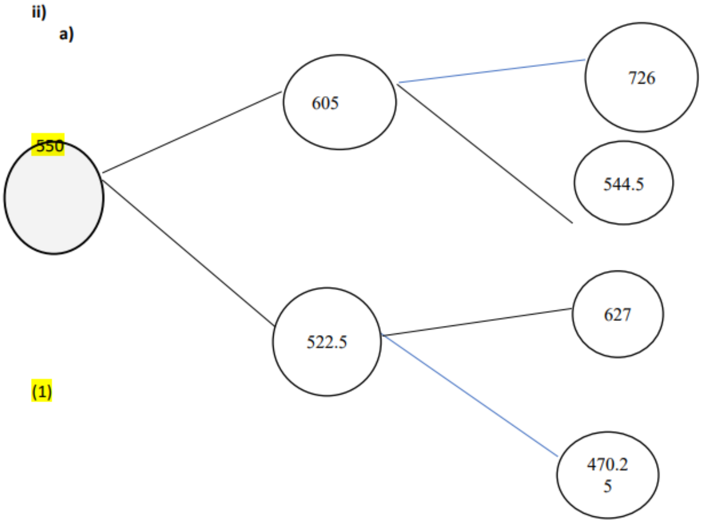
i) The recombining binomial tree is assumed to have the size of up-steps and down-steps to be same under all states and across all time intervals. It means $u_t(j)=u$ and $d_t(t)=d$ for all times t and states j, also $d< e^r<u$

Hence the risk neutral probability 'q' is also constant for all times and in all states.

The main advantage of recombining binomial tree is that it greatly reduces the computational time by reducing the number of states.

ii)

a)



The risk-neutral probability for each state

$q_1 = (e^{0.02} – 0.95)/(1.10 – 0.95) \quad = 0.4680$ (time 1)

$q_2 = (e^{0.03} – 0.90)/(1.20 – 0.90) \quad = 0.4349$ (time 2)

Possible Payoffs after six months are

p_{uu} = Max(500-726,0) = 0

p_{ud} = Max(500-544.5,0) = 0

p_{du} = Max(500-628,0) =0

p_{dd} = Max(500-470.25) = 29.75

The value of the option at first 3 months if upside

$V_1(1) = e^{-0.03} (0.4348 * 0 + 0.5652 * 0)$

$= 0$

For downside

$V_1(2) = e^{-0.03} (0.4348 * 0 + 0.5652 * 29.75)$

$= 16.3178$

At time 0 the value of the put option will be

$V_0 = e^{-0.02}(0.4680 * 0 + 0.5320 * 16.3178)$
$= 8.5092$

(1)
[Max 6]

b) For the first 3 months, the payoff if upside
Max(500-3/6*605,0) = 197.5
And if downside
Max(500 - 3/6*522.5,0) = 238.75

(1)

The value of the option at first 3 months [since the final payoffs are same as ii)a)]
$V_1(1) = 0$ and $V_1(2) = 16.3178$

It is optimal to exercise the American put option early in this case as the payoffs are high compared to the value of put option at first 3 months (1)

Hence, the value of the put option assuming at is exercised at the first 3 months
$V_0 = e^{-0.02}(0.4680 * (197.5-0) + 0.5320 * (238.75-16.3178))$
$= 206.5905$ (2)

At expiry the American put option value is same as the European put option derived in ii) a) which is = 8.5092

Hence, the total value of American put is 206.5905+8.5092 = 215.0997

Answer 6

Let dXt = At dt + Bt dZt,

Where, At = α μ (T-t) , Bt = σ $\sqrt{(T-t)}$ Eq 1

dF = $\frac{\partial f}{\partial x}$ Bt dZt + ($\frac{\partial f}{\partial t}$ + $\frac{\partial f}{\partial x}$ At + $\frac{1}{2}$ $\frac{\partial 2f}{\partial x2}$ Bt) dt *(Ito's lemma)*

$= - f Bt\, dZt + f \frac{\partial m}{\partial t} (T - t)dt \quad - f\ At\ dt + \frac{1}{2}\ f\ Bt^2\ dt$

[Since $\frac{\partial f}{\partial x}$ $= - e^{(m(T-t)-x)}$ and $\frac{\partial f}{\partial t}$ $= \frac{\partial m}{\partial t}\ (T\ -\ t) *\ e^{(m(T-t)-x)}$ (using chain rule) and $\frac{\partial 2f}{\partial x2}$ $=\ e^{(m(T-t)-x)}$]

dF $= f\ (\frac{\partial m}{\partial t} (T - t) - \ At + \frac{1}{2}\ \ Bt^2\ )$ dt - $f Bt\ dZt$

For f to be a martingale, $\frac{\partial m}{\partial t} (T - t) - \ At + \frac{1}{2}\ Bt^2 \quad = 0$

Thus, $\frac{\partial m}{\partial t} (T - t) \ = \ \ At \ - \frac{1}{2}\ Bt^2$

Substituting Eq 1 above gives $\frac{\partial m}{\partial t}$ = α μ - $\frac{1}{2}$ σ²

Answer7

i) Given

$F(t,x) = e^{-t}x^2$

$\frac{df}{dt} = - e^{-t}x^2 \quad = -f$

$\frac{df}{dx} = 2e^{-t}x$

$\frac{df}{dx^2} = 2e^{-t}$

$Y_t = f\ (t,X_t)$

Applying Ito's lemma

$dY_t = \frac{df}{dt}\ dt\ +\ \frac{df}{dx}\ dX_t\ + ½ \frac{d^2f}{dx^2}\ \ \sigma^2\ X_t^2\ dt$

$= -fdt + 2e^{-t}\ X_t dX_{t\ +}\ e^{-t}\sigma^2\ X_t^2\ dt$

$= -Y_t dt\ + 2e^{-t}\ X_t^2 \frac{dXt}{Xt} + e^{-t}\sigma^2\ X_t^2\ dt$

$= -Y_t dt\ + 2Y_t\ [0.25\ dt + \sigma\ dW_t]\ \ + \sigma^2\ Y_t\ dt$

$\frac{dYt}{Yt}\ =\ [\ 2 * (0.25) - 1 + \sigma^2\ ]dt\ + 2\sigma\ dWt$

$= [\sigma^2 - 0.5]\ dt\ + 2\sigma\ dWt$

Therefore

$dYt = [\sigma^2 - 0.5]\ Yt\ dt\ + 2\sigma\ Yt\ dWt$

ii) The process is martingale if drift is zero. This means $\sigma^2 - 0.5 = 0$

i.e. $\sigma^2 = 0.5$

Answer 8

The required probability is the probability of the stock price being greater than Rs. 258 in 6 months' time.

The stock price follows Geometric Brownian motion
i.e. St = S0 exp(μ – σ^2 /2)t + σWt

Therefore Ln (St) follows normal distribution with mean Ln (S0) + (μ – σ^2 /2)t and variance (σ^2)t

Implies Ln (St) follows φ(Ln 254 + (0.16-0.35^2/2)*0.5, 0.35*0.5^(1/2))
= φ (5.59, 0.247)

This means [Ln (St) – St)]/ σt^(1/2) follows standard normal distribution.

Hence the probability that stock price will be higher than the strike price of Rs. 258 in 6 months' time =
1- N(5.55-5.59)/0.247 = 1- N(-0.1364) =0.5542.

The put option is exercised if the stock price is less than Rs. 258 in 6 months' time.
The probability of this = 1- 0.5542= 0.4457

Answer 9

i) The given relationship can be written as:

$S_t = S_0\, e^{\mu t + \sigma B t}$

Since St is a function of standard Brownian motion, Bt, applying Ito's Lemma, the SDE for the underlying stochastic process becomes:

$dBt = 0 \text{ X } dt + 1 \text{ X } dBt$

Let $G(t, B_t) = S_t = S_0\, e^{\mu t + \sigma Bt}$, then

$dG/dt = \mu\, S_0\, e^{\mu t + \sigma Bt} = \mu\, S_t$

$dG/dB_t = \sigma\, S_0\, e^{\mu t + \sigma Bt} = \sigma\, S_t$

$d^2G/dB^2_t = \sigma^2\, S_0\, e^{\mu t + \sigma Bt} = \sigma^2\, S_t$

Hence, using Ito's Lemma from Page 46 in the Tables we have:

$dG = [0 \text{ X } \sigma\, S_t + \frac{1}{2} \text{ X } 1^2 \text{ X } \sigma^2\, S_t + \mu\, S_t]\, dt + 1 \text{ X } \sigma\, S_t\, dB_t$

i.e. $dS_t = (\mu + \frac{1}{2}\sigma^2)\, S_t\, dt + \sigma\, S_t dB_t$

Thus,

$dS_t/S_t = \sigma\, dB_t + (\mu + \frac{1}{2}\sigma^2)\, dt$

So, $c_1 = \sigma$ and $c_2 = \mu + \frac{1}{2}\sigma^2$

ii)

The expected value of S_t is:

$E[S_t] = E\,[S_0\, e^{\mu t + \sigma Bt}] = S_0\, e^{\mu t}\, E[e^{\sigma Bt}]$

Since $B_t \sim N\,(0,1)$, its MGF is $E[e^{\Theta Bt}] = e^{\frac{1}{2}\Theta 2t}$

So, $E[S_t] = S_0\, e^{\mu t} \text{ X } e^{\frac{1}{2}\sigma 2t} = S_0\, e^{\mu t + \frac{1}{2}\sigma 2t}$

The variance of S_t is:

$Var[S_t] = E[S^2_t] - (E[S_t])^2$

$= E[S^2_0\, e^{2\mu t + 2\sigma Bt}] - (S_0\, e^{\mu t + \frac{1}{2}\sigma 2t})^2$

$= S^2_0\, e^{2\mu t}\, E[e^{2\sigma Bt}] - S^2_0\, e^{2\mu t + \sigma 2t}$

$= S^2_0\, e^{2\mu t + 2\sigma 2t} - S^2_0\, e^{2\mu t + \sigma 2t}$

$= S^2_0\, e^{2\mu t}\, (e^{2\sigma 2t} - e^{\sigma 2t})$

iii)

$Cov[S_{t1}, S_{t2}] = E[S_{t1}, S_{t2}] - E[S_{t1}]\, E[S_{t2}]$

From above,

$E[S_{t1}] = S_0\, e^{\mu t1 + ½\, \sigma 2t1}$ and $E[S_{t2}] = S_0\, e^{\mu t2 + ½\, \sigma 2t2}$

The expected value of the product is:

$E[S_{t1}, S_{t2}] = E[S_0 \exp(\mu t_1 + \sigma\, B_{t1})\, S_0 \exp(\mu t_2 + \sigma\, B_{t2})]$

$$= S^2{}_0\, e^{\mu(t1 + t2)}\, E[\exp(\sigma B_{t1} + \sigma B_{t2})]$$

To evaluate this we need to split B_{t2} into two independent components:

$B_{t2} = B_{t1} + (B_{t2} - B_{t1})$ where $B_{t2} - B_{t1} \sim N(0, t_2 – t_1)$

Hence,

$E[S_{t1}, S_{t2}]$

$= S^2{}_0\, e^{\mu(t1 + t2)}\, E[\exp(\sigma B_{t1} + \sigma \{ B_{t1} + (B_{t2} - B_{t1})\})]$

$= S^2{}_0\, e^{\mu(t1 + t2)}\, E[\exp(2\sigma B_{t1} + \sigma \{ B_{t2} - B_{t1}\})]$

$= S^2{}_0\, e^{\mu(t1 + t2)}\, E[\exp(2\sigma B_{t1})]\, E[\exp \{ B_{t2} - B_{t1}\})]$

$= S^2{}_0\, e^{\mu(t1 + t2)} \exp(2\sigma^2 t_1) \exp [\, ½\, \sigma^2 (t_2 – t_1)]$

$= S^2{}_0\, e^{\mu(t1 + t2)} \exp(\frac{3}{2}\sigma^2 t_1 + \frac{1}{2}\sigma^2 t_2)$

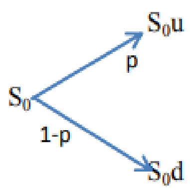
Putting all the equations together:

$$Cov[S_{t1}, S_{t2}] = S^2{}_0\, e^{\mu(t1 + t2)} \exp(\frac{3}{2}\sigma^2 t_1 + \frac{1}{2}\sigma^2 t_2) - S_0\, e^{\mu t1 + ½\, \sigma 2t1}.\ S_0\, e^{\mu t2 + ½\, \sigma 2t2}$$

$$= S^2{}_0\, e^{\mu(t1 + t2)} (\exp(\frac{3}{2}\sigma^2 t_1\,) – \exp(\frac{1}{2}\sigma^2 t_1))\, \exp(\, ½\, \sigma^2 t_2)$$

Answer 10

i) Setting up the commodity tree using u for up move and d for down move, p is up-step probability:



Where p is the up probability and (1-p) the down probability

Then $E(C_t) = S_0[pu+(1-p)d]$, and

$Var(C_t) = E(C_t^2) - E(C_t)^2$

$= S_0^2\ [pu^2+(1-p)d^2] - S_0^2[pu+(1-p)d]^2$

$= S_0^2\ [pu^2+(1-p)d^2-(pu+(1-p)d)^2]$

$= S_0^2\ [p(1-p)u^2+ p(1-p)d^2-2p(1-p)]$ $\quad (\because d= 1/u)$

$= S_0^2\ p(1-p)(u-d)^2$

Equating moments:

$S_0 e^{rt} = S_0[pu+(1-p)d]$ ________________(A)

And $\sigma^2\ S_0^2 t = S_0^2\ p(1-p)(u-d)^2$ ________________(B)

From (A) we get

$p = \frac{e^{rt}-d}{u-d}$ ____________________________(C)

Substituting p into equation (B), we get

$\sigma^2\ t = \frac{e^{rt}-d}{u-d}\left(1 - \frac{e^{rt}-d}{u-d}\right)(u-d)^2$

$= -(e^{rt}-d)(e^{rt}-u) = (u+d)\, e^{rt} - (1+ e^{2rt})$

Putting d = 1/u, and multiplying through by u we get

$u^2 e^{rt} - u\,(1+ e^{2rt} + \sigma^2\ t) + e^{rt} = 0$

This is a quadratic in u which can be solved in the usual way.

ii)

a) $\sigma = 0.15$, $t = 0.25$ => $u = \exp(.15*\sqrt{.25}) = \exp(.075) = 1.077884$, $d = 1/u = .92774$

The tree is

t=0	t=.25	t=.5	t=.75	
			100.186	Node A
		92.947		
	86.231		86.232	Node B
80		80.001		
	74.22		74.22	Node C
		68.857		
			63.882	Node D

b) r=0, we have p = $\frac{e^{rt}-d}{u-d} = \frac{(1-.927744)}{(1.077884-.927744)}$ = .48126

Discounting back the final payoff at t=.75 to t=0 along the tree using p and (1-p), we get

t=0	t=.25	t=.5	t=.75	
			20.186	Node A
		12.948		
	7.787		6.232	Node B
4.496		2.999		
	1.443		0	Node C
		0		
			0	Node D

Hence value of the call option is 4.496.

c) The lookback call pays the difference between the minimum value and the final value.
Notate paths by U for up and D for down, in order

We get the payoffs

UUU	(100.186 - 80)= 20.186	Node A
UDU	(86.232-80)= 6.232	Node B
UUD	(86.232- 80) =6.232	Node B
UDD	(74.22-74.22) = 0	Node C
DUU	(86.232- 74.22) = 12.012	Node B
DUD	(74.22-74.22) = 0	Node C
DDU	(74.22-68.857) = 5.363	Node C
DDD	(63.882-63.882)= 0	Node D

The lookback payoffs are, for each successful path (i.e. with a non- zero result)

Probabilities of arriving at each node are:

Node A= p^3 = .11147
Node B= $p^2(1-p)$ = .12015
Node C= $p(1-p)^2$ = .12950
Node D= $p(1-p)^3$ = .13959

Hence the tree value of lookback option is:

(.11147*20.186)+ (.12015*[6.232+ 6.232+ 12.012]) + (.12950*5.363)

= 5.8854

Answer 11

Given Z(t) is standard Brownian

a. $dU(t) = 2dZ(t) – 0$
$= 0dt + 2dZ(t)$.

Thus, the stochastic process {U(t)} has zero drift.

b. $dV(t) = d[Z(t)]^2 – dt$.
$d[Z(t)]^2 = 2Z(t)dZ(t) + 2/2\, [dZ(t)]^2$
$= 2Z(t)dZ(t) + dt$ by the multiplication rule

Thus, $dV(t) = 2Z(t)dZ(t)$. The stochastic process {V(t)} has zero drift.

c. $dW(t) = d[t^2 Z(t)] – 2t\, Z(t)dt$
Because $d[t^2 Z(t)] = t^2\, dZ(t) + 2tZ(t)dt$, we have
$dW(t) = t^2\, dZ(t)$.

Thus The process {W(t)} has zero drift

Answer 12

i. Let $f = f(S_t,t) = S^k$

$$\frac{\partial f}{\partial S} = kS^{k-1}, \qquad \frac{\partial^2 f}{\partial S^2} = k(k-1)S^{k-2}, \qquad \frac{\partial f}{\partial t} = 0$$

Using Ito calculus, $df = kS^{k-1}dS_t + ½\, k(k-1)S^{k-2}(dS_t)^2$
$= f[k\mu + ½k(k-1)\sigma^2]dt + f[k\sigma]dW_t$

Hence $f = S^k$ follows Geometric Brownian motion, with drift $\mu' = k\mu + ½k\,(k-1)\,\sigma^2$ and volatility $\sigma' = k\sigma$.

Hence $f_t = f_0 e^{(\mu' - ½\sigma'^2)t + \sigma' W_t}$

$This\ means\ \frac{f}{f_0} \sim lognormal\ ((\mu' - ½\sigma'^2)t, \sigma'^2 t)$

The mean and variance of a log normal distribution with parameters μ and σ^2

Is given by $e^{\mu + 1/2\sigma^2}$ and $(e^{\sigma^2} - 1)e^{2\mu + \sigma^2}$

This means $E(f) = f_0 e^{\mu' t}$ and $V(f) = f_0^2 e^{2\mu' t}(\, e^{\sigma'^2 t} - 1)$

ii. Let $f = f(t,S_t) = e^{-rt}S_t$

$$\frac{\partial f}{\partial t} = -re^{-rt}S_t, \qquad \frac{\partial f}{\partial S} = e^{-rt}, \qquad \frac{\partial^2 f}{\partial S^2} = 0$$

By Ito calculus, $df = -re^{-rt}S_t dt + e^{-rt}dS_t$
$= (\mu - r)f\,dt + \sigma f\,dW_t$ (on substituting the expression for dS_t)

which is a martingale if and only if $\mu = r$

iii. Combining results from part (a) and (b), we need, for discounted S^k to be a martingale,
$$k\mu + ½k(k-1)\sigma^2 = r$$
For given values of r, σ and k, one can solve for the value of μ for which discounted S^k will be a martingale.