INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Subject: Financial Engineering 1

Chapter:

Category: Assignment 2 Questions

1. The current price of a non-dividend paying stock is £65 and its volatility is 25% per annum. The continuously compounded risk-free interest rate is 2% per annum.
Consider a European call option on this share with strike price £55 and expiry date in six months' time. Assume that the Black-Scholes model applies.

(i) Calculate the price of the call option.

(ii) Define algebraically the delta of the call option.

(iii) Calculate the value of the delta of the call option.

(iv) Calculate the value of the delta of a European put option written on the same underlying, with the same strike and maturity as above.

2. i) What are 'delta' and 'vega'?

ii) Show that the 'vega's for a European call option and a European put option on the same stock with same strike price K and same time to expiry τ are identical.

iii) Given the following data, find the price of a European call and a European put option:

- Stock price = \$55
- Strike = \$50
- Annualised Volatility = 25%
- Continuously compounded risk-free rate of interest = 5%
- No dividends
- Time to expiry = 1 year

iv) What does it mean for a portfolio to be 'delta-hedged' and 'vega-hedged'?

v) With \$1000, construct such a portfolio using the below instruments only:

- European call option with features as in part (iii)
- European put option with features as in part (iii)
- A 1-year forward on the stock

3. (i) Write down an expression for the price of a derivative in a Black-Scholes market in terms of an expectation under the risk-neutral measure, defining any additional notation that you use.

Consider an option on a non-dividend-paying stock when the stock price is £50, the exercise price is £49, the continuously compounded risk-free rate of interest is 5% per annum, the volatility is 25% per annum, and the time to maturity is six months.

(ii) Calculate the price of the option using the Black-Scholes formula, if the option is a European call.

(iii) Determine the price of the option if it is an American call.

(iv) Calculate the price of the option if it is a European put.

(v) Determine how the prices of the contracts in parts (ii) to (iv) would change in the case of a dividend-paying underlying stock. [Note that you do not have to perform any further calculations.]

4. (i) State the Cameron-Martin-Girsanov theorem.

(ii) State an important property of the discounted value of a security price process under the risk-neutral measure.

5. Consider a non-dividend-paying stock, with price S_t, and a European call option on that stock, whose value can be modelled using the Black-Scholes model.

(i) Write down the formula for the delta of this option under this model.

Suppose that the stock price at time 0 is $S_0 = \$40$ and the continuously compounded risk-free rate is 2% per annum. The call option has strike price \$45.91, term to maturity 5 years and a delta of $\Delta = 0.6179$.

(ii) Determine the implied volatility of the stock to the nearest 1%.

A second stock with price R_t is currently priced at $R_0 = \$30$ and has volatility

$\sigma_R = \sqrt{15\%}$ per annum.

An exotic option pays an amount c at time T if $S_1/S_0 < k_S$ and $R_1/R_0 < k_R$.

(iii) Give a formula for the value of the option at time 0 if the two stocks are independent, defining any additional notation used.

(iv) Explain how the structure of the option could be simplified if the assets were perfectly correlated.

Assume now that the stock prices are independent. The option has term T = 1 year, payoff c = \$50 and strike prices kS = 0.8 and kR = 0.6.

(v) Determine the value of the option at time 0.

6. A share is currently priced at 640p. A writer of 100,000 units of a one year European put option with an exercise price of 630p has delta-hedged the option with a portfolio which holds cash and is short 24,830 shares. The continuously compounded risk-free rate of interest is 3% p.a. and no dividends are payable during the life of the option.
The assumptions of the Black-Scholes model apply.

(i) (a) Write down an expression for the delta of the option.
(b) Calculate its value in this case.

(ii) Prove that the volatility of the share implied by the delta is 7.1% p.a. (assuming it is less than 100%).

(iii) (a) Calculate the price of the option.
(b) Determine the value of the cash holding in the hedging portfolio.

7. (i) Define delta, gamma and vega for an individual derivative.

A bank is considering selling a European call option on a share, and wants to hedge some of its risk. The share is non-dividend paying and has the following properties:
Strike price = \$50
Option price = \$17.91
Underlying share price = \$60
Volatility = 25% p.a.
Time to expiry = 3 years
The continuously compounded risk-free rate of interest is 3% p.a. and the vega for this option is \$29.00.

(ii) Calculate delta for this option.

(iii) Identify a delta-hedged replicating portfolio using the share and the risk-free asset.

Assume that the volatility has instantaneously increased to 27% p.a., with everything else except the option price remaining the same.

(iv) Estimate the new option price.

8. In a Black-Scholes model, the delta of a call option is $\Delta = \Phi(d1)$.
(i) Define delta.

Suppose that the stock price at time zero is $S0$ = \$100, the continuously compounded risk-free rate is 3% and that a European call option written on S with strike price \$109.42 and maturity t = 1 year has a delta of $\Delta = 0.42074$.

(ii) Find the implied volatility of the stock to the nearest 1%.

9. In a Black-Scholes market, let S be the price of a stock and D be the price of a derivative written on S, with maturity T, where $Dt = g(t, St)$ for any $t < T$ and $g(T, x) = f(x)$.

(i) Write down the partial differential equation (PDE) that g must satisfy, including the boundary condition for time T.

Suppose that the derivative pays $St^n/S0^{(n-1)}$ at time T, where n is an integer greater than 1.

(ii) Show, using (i),that the price of the derivative at time t is given by
Dt = $(St^n/S0^{(n-1)})\ e^{\mu(T-t)}$ for some µ which you should determine.

10. (i) State and prove the put-call parity for a stock paying no dividends.

In a Black-Scholes market, a European call option on the dividend-free stock, with strike price $120 and expiry T = 1 year is priced at $10.09. The continuously compounded risk-free rate is 2% p.a. and the stock is currently priced at $110.

(ii) Estimate the implied volatility of the stock to the nearest 1%.

A European put option on the same stock has strike price $121 and the same maturity.
An investor holds a portfolio which is long one call and short one put.

(iii) Sketch a graph of the payoff at maturity of the portfolio against the stock price

(iv) (a) Determine an upper and a lower bound on the value of the portfolio at maturity.
(b) Deduce bounds for the current put price.

(v) Determine the fair price of the put.

11. A one-year European call option on a non-dividend paying stock in Company ABC has a strike of $150. The continuously compounded risk-free rate is 2% p.a. The current stock price is $117.98. Assume that the market follows the assumptions of a Black-Scholes model.
An institutional investor holds a delta-hedged portfolio with 100,000 call options, no cash and short 18,673 shares of Company ABC.

(i) Calculate the delta of the call option.

(ii) Calculate the implied volatility for the underlying.

(iii) Calculate the price of a one-year put on the same stock with a strike of $150.

The investor retains their holding of call options and trades in the put and the stock to achieve a delta and gamma-hedged portfolio.

(iv) Calculate the investor's new holdings of the put and the stock.

12. (i) State the main assumptions underpinning the Black-Scholes model.

Consider a put option on a non-dividend paying stock when the stock price is £8, the exercise price is £9, the continuously compounded risk-free rate of interest is 2% per annum, the volatility is 20% per annum and the time to maturity is three months.

(ii) Calculate the price of the option using the Black-Scholes model.

(iii) Discuss how the price of the contract in part (ii) would change if the rate of interest increases. (There is no need to carry out further calculations.)

13. Describe the similarities and differences between the Vasicek and Hull-White models of interest rate.

14. In some country, the interest rate are modelled as per Cox–Ingersoll–Ross (CIR) model with the following parameters

$\propto = 0.2$, $\mu = 0.08$, $\sigma = 0.1$

The bond price under CIR model is given by:

$B(t,T) = e^{a(\tau)-b(\tau)r(t)}$

Where

$$b(\tau) = \frac{2(e^{\theta\tau}-1)}{(\theta+\alpha)(e^{\theta\tau}-1)+2\theta}$$

$$a(\tau) = \frac{2\alpha\mu}{\sigma^2}\log\left(\frac{2\theta(e^{(\theta+\alpha)\tau/2}}{(\theta+\alpha)(e^{\theta\tau}-1)+2\theta}\right)$$

$$\theta = \sqrt{\alpha^2+2\sigma^2}$$
$$\tau = T-t$$

i) Write the Stochastic Differential Equation (SDE) for the interest rate under Q

Corporate bonds are also traded in above country and an analyst intends to model the corporate bond by allowing for the credit risk by tweaking the SDE for the interest rates. He adds another term φ r(t) dt to the SDE.

ii) Write the revised SDE and rework the value of the parameters for CIR model if φ = 0.06

iii) Find out the corporate bond price at time 5 and 10 years with the parameters determined in (ii) above if the current rate of interest is 7% (7)

15. Black-Derman-Toy model for short rate 'r' is given by $d(\ln(r)) = \theta(t)\,dt + \sigma dW$
Compare and contrast this model with Vasicek Model of short rates.

16. Interest rates 'r' in a particular country are modelled using a Black-Karasinski Model given by:

$(\ln r(t)) = k(t)(\theta(t) - \ln r(t))dt - \sigma(t)dW(t)$ where W is the Weiner Process under risk neutral measure.

i) Compare and contrast the given model with the Vasicek Model.

ii) State the distribution that 'r' follows.