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Subject: Financial Engineering 1

Chapter: Unit 1

Category: Practice Questions

1. CT8 September 2011 Q4

Assume that there is no arbitrage in the market. A forward contract is available on a physical asset. The continuously compounded costs of managing the asset are x% of its value, and it provides an income stream of £y per ton payable at six monthly intervals, a payment has just been made.
Let S_t be the spot price of one ton of the asset at time t and let r be the continuously compounded risk-free rate of interest per annum which is assumed to be constant.

Derive the current price of a forward contract written on one ton of the asset with maturity T years where (6 months < T < 1 year).

2. CT8 April 2011 Q5

Assume that a non-dividend-paying security with price S_t at time t can move to either S_t u or S_t d at time t + 1. The continuously compounded rate of interest is r, and $u > e^r > d$. A financial derivative pays α if $S_{t+1} = S_t u$ and β if $S_{t+1} = S_t d$.
A portfolio of cash (amount x) and the underlying security (value y) at time t exactly replicates the payoff of the derivative at time t + 1.

(i) Derive expressions for x and y in terms of r, u, d, α and β.
(ii) Derive an expression for the risk-neutral probability of the security having value S_t u at time t + 1 in terms of (x + y), r, α and β.

Assume $S_t = 100$, u = 1.25, d = 0.8 and r = 0.
(iii) (a) Calculate the prices of at-the-money call and put options.
(b) Check that the put-call parity holds for this model.

3. CT8 September 2011 Q7

A non-dividend-paying stock, S_t, has a current price of 200p. After 6 months the price of the stock could increase to 230p or decrease to 170p. After a further 6 months, the price could increase from 230p to 250p or decrease from 230p to 200p. From 170p the price could increase to 200p or decrease to 150p. The semi-annually compounded risk-free rate of interest is 6% per annum and the real-world probability that the share price increases at any time step is 0.75. Adopt a binomial tree approach with semi-annual time-steps.

(i) Calculate the state-price deflator after one year.

(ii) Calculate, using the state-price deflator from (i), the price of a non-standard option which pays out max {0, log(S_1 − 180)} one year from now.
(iii) State how the answer to (ii) would change if the real-world probability of a share price increase at each time step was 0.6.

4. CT8 April 2012 Q3

A non-dividend paying stock has a current price of S_0 =150p and trades in a market which is arbitrage free and has a constant effective risk-free rate of interest r. After one year the price of the stock could increase to 280p or decrease to 120p. Over the following year the price could increase from 280p either to 420p or to 322p. If the stock price had decreased to 120p, then over the following year it could increase to 168p or decrease to 112p.

(i) Determine the range of values that the annual risk-free rate of interest could take.

Assume that r takes the value 20% p.a.
(ii) Calculate the price at time 0 of a non-standard derivative which pays off $(S_2 – 100)^2$ at the end of two years.

5. CT8 April 2012 Q4

Let c be the price of a four- month European call option on a dividend paying share. Assume the strike price is $30, the underlying is currently valued at $28 and a dividend of $0.50 is expected in 2 months. The continuously compounded risk-free rate is constant and equal to 5% p.a.
(i) Derive upper and lower bounds on the price c of this call option, taking into account the dividend.

The price of a put option with the same underlying, the same strike price and the same maturity is $3.
(ii) Calculate the price c of the call option exactly

6. CT8 September 2012 Q2

A non-dividend-paying stock has a current price of S_0 = 400p . Over each of the next three years its price could increase by 20% (so $S_{t+1} = 1.2S_t$), or decrease by 20% (so $S_{t+1} = S_t /1.2$). The continuously compounded risk-free rate is 6% p.a. The stock price move in each year is independent of the move in other years.

A non-standard derivative pays off $\sqrt{S_3}$ after three years, provided that at some point over three years the stock price has moved up in one year and then immediately down in the following year. Otherwise, the derivative pays zero.

Calculate the current price of this non-standard derivative.

7. CT8 September 2012 Q4

A non-dividend paying stock is currently priced at $S_0 = £80$. Over each of the next two three-month periods it is expected to go up by 6% or down by 5% on each period. The continuously compounded risk-free interest rate is 5% p.a.

(i) Calculate the value of a six-month European call option with a strike price of £82.

(ii) Calculate the value of a six-month European put option with a strike price of £82.

(a) Directly.

(b) Using put-call parity.

(iii) Explain whether, if the put option were American, it would ever be optimal to exercise early.

8. CT8 April 2013 Q7

A non-dividend-paying stock in an arbitrage-free market has a current price of 150p. Over each of the next two years its price will either be multiplied by a factor of 1.2 or divided by 1.2. The continuously compounded risk-free rate is 1% p.a. The value of an option on the stock is 50p.

Denote by P_{uu} the value of the payoff if both stock price moves are up, P_{ud} for the value of the payoff if one move is up and one is down (this is the same whichever order the price moves occur), and P_{dd} for the value of the payoff if both stock price moves are down. The price of the stock is to be modelled using a binomial tree approach with annual time steps.

(i) Derive, and simplify an equation for P_{uu} in terms of P_{ud} and P_{dd}

(ii) Calculate, using your answer to part (i), or otherwise, the range of values that P_{uu} could take.

(iii) Determine the value of the option in each of the two cases below, assuming that P_{uu} takes its maximum possible value:

(a) If the first stock price move is up.

(b) If the first stock price move is down.

9. CT8 September 2013 Q7

The continuously compounded risk-free rate of interest is r, and a stock, with maturity T, pays dividends continuously at rate q.

(i) Determine the forward price at time 0 for a forward contract on the stock.

(ii) Show that there exists a portfolio that earns the risk-free rate r, containing:

- the stock
- a European call option on the stock
- and a European put option on the stock

10. CT8 April 2014 Q4

Consider the following long position in European and American call options written on a stock, with strikes and times to expiry as set out in the table below.

Option	*European/American*	*Strike price*	*Time to expiry*
A	American	400	3 years
B	American	400	2 years
C	American	420	3 years
D	European	400	3 years
E	European	400	2 years

Rank these options in order of value to the extent that this is possible.

11. CT8 September 2014 Q4

A non-dividend paying stock currently trades at \$65. Every two years the stock price either increases by a multiplicative factor 1.3 or decreases by a multiplicative factor 0.8. The effective risk-free rate is 4% p.a.

Calculate the price of an American put option written on the stock with strike price \$70 and maturity four years, using a two-period binomial model.

12. CT8 September 2014 Q5

Let S be the price of a non-dividend paying share, and let r be the continuously compounded risk-free rate.
(i) Derive the forward price at time zero for the forward contract on S with maturity T.

Assume that, at time zero, the share price is 500, and that the forward contract has maturity of two years. The share pays a dividend of 5% of the share price every six months with the next dividend due in two months, and the continuously compounded risk-free rate is 3% p.a.

(ii) Determine the forward price for this contract.

(iii) Comment on whether the high dividend yield relative to the risk-free rate offers an arbitrage opportunity

13. CT8 April 2015 Q6

Consider a non-dividend paying share with price S_t at time t.
(i) State and prove the put-call parity relationship for this share.

Two options written on this share have the following characteristics:
1. a European call option maturing in two years, strike price \$10.15, option price \$3.87
2. a European put option maturing in two years, strike price \$10.15, option price \$0.44

The continuously compounded risk-free rate of interest is 4% p.a.
(ii) Calculate the share price implied by the option prices.

14. CT8 September 2015 Q6

Consider a non-dividend paying share with price S_t at time t in a market with continuously compounded risk-free rate of interest r.
(i) Show that the fair price of a forward contract on the share maturing at time T is $K = S_0 e^{rT}$.

A share is currently worth S_0 = £5. The continuously compounded risk-free rate of interest is 3% p.a. for $0 \leq t < 1$, 5% p.a. for $1 \leq t < 2$ and 2% p.a. for $2 \leq t \leq 4$.
(ii) Calculate the fair price at t = 0 of a forward contract written on the share with delivery at t = 4.

An investor enters into the above forward contract at time t = 0. At time t = 1 the share price has increased to £6.

(iii) Calculate the value to the investor of the forward contract at t = 1.

15. CT8 April 2016 Q7

Consider a non-dividend-paying share with price S_t at time t (in years) in a market with continuously compounded risk-free rate of interest r.

(i) Show that the fair price at t = 0 of a forward contract on the share maturing at time T is $K = S_0e^{rT}$.

A share is currently worth S_0 = €20. The continuously compounded risk-free rate of interest is 1% per annum.

(ii) Calculate the fair price at t = 0 of a forward contract written on the share with delivery at t = 2.

(iii) Give an expression for the value to the investor of the forward contract in part (ii) at time $t \leq 2$, in terms of S_t , t and r.

An investor enters into the above forward contract at time t = 0. At time t = 1 the risk-free rate of interest has increased to 4% per annum. The share price has not changed.

(iv) Calculate the value to the investor of the forward contract at t = 1.

16. CT8 April 2016 Q8

Consider a three-period binomial tree model for the stock price process S_t.

Let S_0 = 100 and let the price rise by 10% or fall by 5% at each time step.

Assume also that the risk-free rate is 4% per time period, continuously compounded.

(i) (a) State the conditions under which the market is arbitrage free.

(b) Verify that there is no arbitrage in the given market.

(ii) Calculate the price of a European call option on this stock, with maturity at the end of the third period and a strike price of 103.

A special option, called a European "Paylater" call option, has the following payoff at maturity T:

$$(S_T\text{-}K\text{-}c) \text{ if } S_T>K$$

and zero otherwise. K is the strike price, and c is the premium paid for the option.

The premium is paid at maturity and is only paid if the option expires in-the-money.

Further, the option premium is set such that the value of the option at time t = 0 is zero.

Assume that K = 103 and the maturity of the contract is at time t = 3.
(iii) Determine the premium c of this contract.

17. CT8 September 2016 Q7

Consider a binomial tree model for the stock price S_t . Let S_0 = 50 and let the price rise by 10% or fall by 5% each month for the next three months. Assume also that the risk-free rate is 5% per annum continuously compounded.

(i) State the conditions under which the market is arbitrage free.
(ii) Calculate the price at time t = 0 of a European call option on this stock, which expires in three months and is struck at-the-money (i.e. strike price K = 50).

A special option, called a knock-out barrier option, goes out of existence (i.e. expires without any payoff or value) if the underlying asset reaches a pre-specified barrier b > 0 either from above (down-and-out) or from below (up-and-out). The down-and-out call has the following payoff at time T:
$\max(S_T - K, 0) \quad if\ min_{0 \le t \le T}\, S_t \ge b,$
0 otherwise
Assume this special option is written on the given stock, has the same strike price and maturity as the European call option described in part (ii) and the barrier b is fixed at 48.

(iii) Calculate the price of this contract using the binomial tree model and risk neutral valuation.
(iv) Determine the price of the down-and-out contract when b = 40, without performing any further calculations.

18. CT8 April 2017 Q5

Consider a three-period binomial tree model for a stock price process S_t , under which the stock price either rises by 18% or falls by 15% each month. No dividends are payable.
The continuously compounded risk-free rate is 0.25% per month.
Let S_0 = \$85.
Consider a European put option on this stock, with maturity in three months (i.e. at time t = 3) and strike price \$90.

(i) Calculate the price of this put option at time t = 0.

(ii) Calculate the risk-neutral probability that the put option expires out-of-the money.

(iii) Assess whether the probability calculated in part (ii) would be higher or lower under the real-world probability measure. [No further calculation is required.]

19. CT8 September 2017 Q3

Consider a European call option with price c_t written on an underlying non-dividend paying security with price S_t at current time t.

(i) State whether each of the following changes in underlying factors would increase or reduce the price of this option:

(a) a fall in the price of the underlying security

(b) an increase in the strike price of the option

(c) an increase in the volatility of the underlying security price

(d) a fall in the risk-free rate of interest

[You should assume that each change occurs on a standalone basis, i.e. all other factors are unchanged.]

(ii) Explain each of your statements in part (i).

Consider a European put option with price p_t written on the same underlying security, with the same strike price K and the same maturity T as the call option described above.

The continuously compounded risk-free rate of interest is r.

(iii) Write down a formula that relates the values of c_t and p_t .

The call option has a value of £0.50 at time t = 0, and the put option has a value of £1.00. Both options are written on a security with a current value S_0 = £5, and both options have strike price £6.00 and maturity T = 3 years.

(iv) Determine the continuously compounded risk-free rate r.

(v) Suggest, with justification, how the formula in part (iii) can be rewritten as an inequality if both options are American options.

20. CT8 September 2017 Q4

Consider a one-period binomial tree model for the stock price process S_t .
Let S_0 = \$100 and assume that in three months' time the stock price is either \$125 or \$105. No dividends are payable on this stock.
Assume also that the continuously compounded risk-free rate is 5% per annum.

(i) Verify that this market is not arbitrage-free by considering the relationship between the risk-free rate and the stock price movements.
(ii) (a) Identify a portfolio which would generate an arbitrage profit.
(b) Calculate this profit.

Now assume that the continuously compounded risk-free rate is 20% per annum.
Consider a European put option on this stock, expiring in three months' time and with strike price K = \$120.
(iii) Calculate the current price of this put option.

21. CT8 April 2018 Q6

Consider a three-period binomial tree model for the non-dividend paying stock price process S_t, in which the stock price either rises by u% or falls by d% each period till maturity. Let r denote the continuously compounded risk-free rate of interest.
(i) State the conditions under which this market is arbitrage free.

Let S_0 = £95 and assume this price either rises or falls by 20% each year for the next three years. Assume also that the risk-free rate is 5% per annum continuously compounded.

(ii) Calculate the price of a vanilla European put option with maturity in three years and strike price 110.

Assume a change in market conditions such that the same share price now either rises or falls by 5% each year for the next three years.
(iii) Determine how this change would impact on the option price.

22. CT8 September 2018 Q6

Consider a call option c_t and a put option p_t written on a non-dividend paying stock S_t.
(i) Prove the put-call parity relationship by constructing two portfolios that produce the same value at maturity.

A stock market includes four options set out below. All the options are for a term of 10 years and relate to a single non-dividend paying stock, currently priced at $5. The continuously compounded risk-free rate is 3% per annum.

	Type	*Strike price*	*Option price*
Option A	European Call	$8	$0.32
Option B	European Put	$8	?
Option C	European Put	$10	?
Option D	American Put	$10	?

(ii) Calculate the price of Option B.
(iii) Determine lower and upper bounds for the price of option C.
(iv) Determine lower and upper bounds for the price of option D.

23. CT8 September 2018 Q8

Consider a binomial tree model for the non-dividend paying stock with price S_t . Assume this price either rises by 30% or falls by 20% each quarter (3 months) for the next three quarters. Assume also that the risk-free rate is 2% per annum continuously compounded. Let S_0 = £60.

(i) Calculate the price of a vanilla European call option with maturity in nine months' time and a strike price of £55.
(ii) Calculate the price of a vanilla European put option with the same maturity and strike price as the contract in part (i).

Assume the investor has a portfolio formed by a short position in the call option given in part (i) and a long position in the put option given in part (ii).
(iii) Determine how the value of the portfolio would differ if the possible change in the stock price was a fall of 30% instead of 20%.

24. CM2A April 2019 Q6

Consider a share with price S_t at time t. The continuously compounded risk-free rate is r per annum.
(i) Show that the fair price of a forward contract on S_t maturing at time T is $K = S_0e^{rT}$.

A share S_0 is currently worth £12. The continuously compounded risk-free rate is 5% per annum.
(ii) Calculate the fair price of a forward contract written on the share at time t = 0 with expiry at time t = 5.

An investor takes a long position in the forward contract at time 0. At time 1 the share price fell to £10.
(iii) Calculate the value to the investor of the forward contract at time t = 1.

At time t = 2 the share unexpectedly pays a one-off dividend.
(iv) Explain, with reasons, how the forward price might change as a result of the one-off dividend.

25. CM2A September 2019 Q5

A non-dividend paying share is currently priced at $80. Each year the share price will either increase by 10% or fall by 10% with equal probability. A call option is written on the share with a strike price of $75 and expiry in two years. The risk-free force of interest is 5% per annum.

(i) Calculate the risk-neutral price of the option using a binomial tree.

26. CM2A September 2019 Q6

Let p_t denote the value at time t of a European put option on a non-dividend paying share S_t with maturity at time T and a strike price K. The risk-free rate of interest is r.
(i) Derive the lower bound for p_t in terms of S_t and K.
(ii) Explain how the lower bound would change if p_t were an American put option.

The put option p_t has the following characteristics:

- Strike price = £100
- Time to expiry 6 months.

The risk-free rate of interest is 4% per annum.
(iii) Calculate an upper bound for the value of the option p_t.
(iv) Explain the conditions necessary for the option price to approach the upper bound in part (iii).

27. CM2A April 2021 Q2

A dividend-paying share is currently priced at \$120. Every 3 months, the share price will either increase by 25% or decrease by 20%.
An insurer is considering purchasing a 6-month European call option on the share with a strike price of \$120. Each share will pay a dividend of \$4 just before the option expires.
The risk-free force of interest is 0% per annum.

(i) State the difference between American and European options.
(ii) Calculate the risk neutral price of the option, using a binomial tree.
(iii) Calculate the volatility of the share implied by the option price.
(iv) State the benefits of using a risk-neutral derivative pricing approach.

Now consider an American option on the same underlying share with all other parameters unchanged.
(vi) Explain why you may expect the price of the American option to be different to your answer in part (ii).
(vii) Calculate the price of the American put option.

28. CM2A September 2021 Q4

A non-dividend paying share, with price S_t at time t, has a European call option written on it with value c_t at time t. The call matures at time T and has a strike price of K. The continuously compounded risk-free rate is r.
(i) State an upper bound for the value of the call option, c_t.

Consider a portfolio containing one call option and $Ke^{-(T-t)\,r}$ cash.
(ii) Demonstrate that, at time T, the value of the portfolio will always be greater than or equal to the value of the share, S_T.
(iii) Determine, using the result in part (ii), a lower bound for the value of the call option:

$$c_t \geq S_t - Ke^{-(T-t)r}.$$

An investor holds an American call option on the same share. Assume that r > 0.
(iv) Explain, using the result in part (iii), why it would never be optimal for the investor to exercise this option before its maturity date.
(v) Discuss how your answer to part (iv) would change if the share paid dividends.

29. CM2A September 2021 Q7

A one-period binomial tree has been constructed. In it, a stock with initial value S_0 can evolve over a single time period to be worth either S_0u or S_0d, where u > d. The continuously compounded risk-free rate of return is r.

(i) Demonstrate that, to avoid arbitrage, the relationship $d < e^r < u$ must hold.

(ii) State the formula for the risk-neutral probability, q, of an up movement.

(iii) Demonstrate that the relationship in part (i) is equivalent to $0 < q < 1$.

30. CM2A April 2022 Q6

Consider a share with price S_0 at time t = 0. The share will pay out a dividend of X at time t = 1 and again at time t = 2. The continuously compounded risk-free rate is r per unit of time.
Assume that the dividend payments are reinvested at the risk-free rate.

(i) Demonstrate that the fair price of a forward contract on S_t maturing at time T > 2 is $K = (S_0 - I)e^{rT}$, where I is the present value of the two dividends. [4]

A share is worth \$100 at time t = 0. It will pay a dividend of \$5 at time t = 1 and again at time t = 2. The continuously compounded risk-free rate is 5% per unit of time.

(ii) Calculate the fair price of a forward contract on the share maturing at time T = 3.

An investor takes a long position in the forward contract in part (ii) at time t = 0. Immediately afterwards, the proposed dividends on the underlying share are cancelled.

(iii) Discuss the implications of this for the investor.

31. CM2A September 2022 Q5

A non-dividend paying stock has a price at time t = 0 of \$8. In any unit of time (t, t + 1), the price of the stock either increases by 25% or decreases by 20%, and \$1 held in cash at time t receives interest to become \$1.04 at time t + 1. The stock price after t time units is denoted by S_t.
A derivative contract is written on the stock with expiry date t = 2, which pays \$10 if and only if S_2 is not \$8 (and otherwise pays \$0).

(i) Explain what is meant by a risk-neutral probability measure.

(ii) Calculate the up-step and down-step probabilities under the risk-neutral probability measure for this model.

(iii) Calculate the price (at $t = 0$) of the derivative contract. [4]