INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Subject: Financial Engineering 1

Chapter: Unit 2

Category: Practice Questions

1. CT8 April 2010 Q1

Let $(X_t; t \geq 0)$; be a stochastic process satisfying:

$$X_t = X_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s$$

where W is a standard Brownian motion. Let f :$\Re \times \Re \rightarrow \Re$ be a function, twice partially differentiable with respect to x, once with respect to t.

(i) State the stochastic differential equation for f (t, X_t).

Let $dX_t = -\gamma X_t dt + \sigma dW_t$

(ii) Prove that the solution of this stochastic differential equation is given by:

$$X_t = X_0 \exp(-\gamma t) + \sigma \int_0^t \exp(\gamma(s-t))dW_s$$

2. CT8 April 2010 Q7

(i) State the Cameron-Martin-Girsanov Theorem.
(ii) Derive the value of a which makes $\exp(\sigma B_t - at)$ a martingale when B is a standard Brownian Motion.

In a Black-Scholes market, the stock price is given by:
$S_t = S_0 \exp(0.2B_t + 0.2t)$, where B is a standard Brownian Motion under the real-world measure.

A derivative security written on this stock in the same market has price:
$D_t = 2\exp(0.6B_t + 0.39t)$ at time t.

(iii) (a) Calculate the value of c such that $B_t + ct$ is a standard Brownian Motion under the Equivalent Martingale Measure.
(b) Calculate the risk-free rate of interest.

3. CT8 September 2010 Q6

Under the real-world measure P, W is a standard Brownian motion and the price of a stock, S is given by $S_t = S_0 \exp(\sigma W_t + (\mu - ½\, \sigma^2)t)$.
The continuously compounded risk-free rate of interest is r and a zero coupon bond with maturity T has price $B_t = e^{-r(T-t)}$.
Suppose that in the market any contract which pays f(S_t) at time T is valued at:
$p_t = E[e^{-r(T-t)} f(S_t)\Delta_T | F_t]$

Where $\Delta_T = \exp(mW_t - \frac{1}{2} m^2 t)$ for t ≤ T for some real number m.

i) a) Prove using Ito's formula that, Δ_T is a martingale.
b) Show that $E[\exp(mW_t)] = \exp(\frac{1}{2} m^2 t)$

ii) a) Derive an expression for p_0 when f(x) = x.
b) Show that there is an arbitrage in the market unless m = (r – μ)/σ.

4. CT8 September 2011 Q3

An investor wishes to save for a retirement fund of £100,000 in 10 years' time. The instantaneous, constant continuously compounded risk-free rate of interest is 4% per annum. The investor can purchase shares on a non-dividend paying security with price S_t governed by the Stochastic Differential Equation (SDE):

$$dS_t = S_t (\mu dt + \sigma dZ_t)$$

where:

- Z_t is a standard Brownian motion
- μ = 12%
- σ = 25%
- t is the time from now measured in years; and
- $S_0 = 1$

(i) (a) Derive the distribution of S_t.

(b) Calculate the amount, A, that the investor would need to invest in shares to give a 50:50 probability of building up a retirement fund of £100,000 in 10 years' time.

5. CT8 September 2011 Q6

Under the real-world probability measure, P, the price of a zero-coupon bond with maturity T is given by:

$$B(t,T) = \exp\left\{-(T-t)r_t + \frac{\sigma^2}{6}(T-t)^3\right\}$$

where r_t is the short rate of interest at time t and satisfies the following stochastic differential equation under the real-world measure P:

$$dr_t = \mu r_t\, dt + \sigma dZ_t$$

where $\mu > 0$ and Z_t is a standard Brownian motion under P.

(i) Derive a formula for the instantaneous forward rate f(t, T), based on this model.

(ii) Derive an expression for the market price of risk.

(iii) Deduce the stochastic differential equation for r_t under the risk-neutral measure Q defining all terms used.

6. CT8 April 2013 Q5

(i) State the five key features of a standard Brownian motion B_t.

Consider a stochastic differential equation

$$dX_t = Y_t dB_t + A_t\, dt\,,$$

Where A_t is a deterministic process & Y_t is a process adapted to the natural filtration of B_t

(ii) Write down Ito's lemma for $f(t, X_t)$, where f is a suitable function

(iii) Determine $df(t, X_t)$ Where $f(t, X_t) = e^{2tX_t}$.

7. CT8 April 2013 Q6

Suppose that at time t we hold the portfolio (a_t, b_t, c_t) where a_t, b_t and c_t represent the number of units held at time t of securities with respective price processes A_t, B_t and C_t. Assume (a_t, b_t, c_t) are previsible. Let V_t be the value of this portfolio at time t.

(i) Explain what it means for (a_t, b_t, c_t) to be previsible.

(ii) Write down an equation for the instantaneous change in the value of the portfolio, including cash inflows and outflows, at time t.

(iii) Give the condition for this portfolio to be self-financing.

(iv) Define a replicating strategy for a derivative with payoff X at a future time U, contingent on the path taken by a_t, b_t and c_t

(v) Describe how the no-arbitrage condition and a self-financing strategy can be used to value the derivative in (iv) at time 0.

(vi) Give a condition for the market to be complete.

8. CT8 September 2013 Q5

The share price in Santa Insurance Co, S_t, is currently 97p and can be modelled by the stochastic differential equation:

$$dS_t = 0.4S_t\,dt + 0.5S_t\,dB_t$$

where B_t is a standard Brownian motion

(i) (a) Determine $d\log S_t$, using Ito's Lemma.

(b) Calculate the expectation and variance of the Santa Insurance Co share price in two years' time.

The share price in Rudolf financial services plc. R_t is also currently at 97p & can be modelled by the stochastic differential equation

$$dR_t = -0.4R_t\,dt + 0.5dB_t$$

Let $U_t = e^{0.4t} R_t$

(ii) (a) Calculate dU_t.

(b) Calculate the expectation and variance of the Rudolf Financial Services plc share price in two years' time.

9. CT8 September 2013 Q6

A non-dividend-paying stock has a current price of 300p. Over each of the next two three-month periods its price will either go up by 30p or down by 30p. Price movements for each period are independent of each other. An investment in a cash account returns 2% per quarter.

A European call option on the stock pays out in six months based on a strike price of 290p. The price of the stock is to be modelled using a binomial tree approach with three-month time steps.

(i) Calculate the value of the call option today using a risk-neutral pricing approach.

Assume that the real-world probability of the stock price moving up in each of the next three month periods is 0.7

(ii) (a) Calculate the values of the state price deflator after six months

(b) Calculate and the value of the call option today using your answers to part (ii)(a).

(c) Compare this to your answer to part (i).

Assume that the real-world probability has now dropped from 0.7 to 0.6.

(iii) (a) Explain, without performing any further calculations, how the state price deflator would change in value.

(b) Comment on the impact that this would have on the option price.

10. CT8 April 2014 Q3

Let W be a standard Brownian motion.

(i) State the continuous-time log-normal model of a security price S, defining all the terms used.

Let f be a function of t and W_t^2 .

(ii) (a) Find a function f such that f (t, W_t^2) is a F_t -martingale, with F the Brownian filtration.

$$\text{Hint: } E(W_t^2|\mathcal{F}_s) = W_s^2 + t - s \text{ for all } t \geq s.$$

(b) Use Ito's lemma to show that f (t, W_t^2) is a process with zero drift.

Let X be the process defined as $X_t = t^{\propto} W_{t^{\beta}}$

(iii) Derive the values of α β and for which Xt defines a standard Brownian motion.

11. CT8 September 2014 Q3

Let (Z_t: t ≥ 0) be a standard Brownian motion.

(i) Calculate the probability of the event that $Z_1 > 0$ and $Z_2 < 0$.

Hint: Write $Z_1 = W$, $Z_2 = W + X$, where W and X are both independent, identically distributed $N(0,1)$ random variables.

(ii) State the model for geometric Brownian motion.

(iii) Explain why the standard Brownian motion is less suitable than the geometric Brownian motion as a model of stock prices.

12. CT8 April 2015 Q5

Let $(X_t; t \geq 0)$ be a stochastic process satisfying $dX_t = \mu_t dt + \sigma_t \, dW_t$ where W_t is a standard Brownian motion.

Let $f(t,x)$ be a function, twice partially differentiable with respect to x, once with respect to t.

(i) State the stochastic differential equation for $f(t, X_t)$.

Let $dX_t = \lambda X_t dt + \sigma dW_t$.

(ii) Solve this differential equation, by considering $X_t = U_t e^{\lambda t}$ or otherwise.

13. CT8 September 2015 Q5

An actuary plans to retire in five years' time, and hopes to celebrate retirement with a round-the-world cruise. The cruise will cost €20,000. The actuary chooses to save for the cruise by buying non-dividend paying shares with price S_t governed by the Stochastic Differential Equation:

$$dS_t = S_t(\mu dt + \sigma dZ_t)$$

where:

- Z_t is a standard Brownian motion.
- μ = 10%.
- σ = 20%.
- t is the time from now measured in years; and
- $S_0 = 1$.

The instantaneous, constant, continuously compounded risk-free rate of interest is 4% p.a.
Derive the distribution of S_t.

14. CT8 April 2016 Q6

Suppose that at time t a portfolio (ϕ_t, ψ_t) is held, where ϕ_t represents the number of units of a stock, with price S_t , held at time t and ψ_t is the number of units of a cash bond, with price B_t , held at time t. The processes ϕ and ψ are previsible.

Let $V(t) = \phi_t S_t + \psi_t B_t$ be the value of the portfolio at time t.

(i) Explain what it means for this portfolio to be self-financing.

Consider a stock paying a continuous dividend at a rate δ and denote its price at any time t by S_t.
Let C_t and P_t be the price at time t of a European call option and European put option respectively, written on the stock S, each with strike price K and maturity $T \geq t$.
The instantaneous risk-free rate is denoted by r.

(ii) Prove put-call parity in this context by constructing two self-financing portfolios whose value must be equal by the principle of no arbitrage.

15. CT8 April 2017 Q3

Consider a non-dividend-paying security with price S_t at time t. The security price follows the stochastic differential equation:

$$dS_t = S_t(\mu dt + \sigma dZ_t)$$

where:
- Z_t is a standard Brownian motion
- μ = 16% per annum
- σ = 25% per annum
- t is the time from now measured in years
- $S_0 = 1$

Derive the distribution of S_t .

16. CT8 April 2017 Q6

The market price S_t of a traded security satisfies the following stochastic differential equation:

$$dS_t = (\mu - \lambda\sigma)S_t dt + \sigma S_t dW_t,$$

where W_t is a standard Brownian motion under the probability measure P*.

(i) Determine the value of λ such that the discounted asset price process $\tilde{S}_t = e^{-rt}S_t$ is a martingale under the given probability measure.

(ii) Explain whether the probability measure P* is the real-world or risk-neutral measure, for the value of λ obtained in part (i).

17. CT8 September 2017 Q5

(i) State the Cameron-Martin-Girsanov theorem.

(ii) State an important property of the discounted value of a security price process under the risk-neutral measure.

The price process S_t of a traded security satisfies the following stochastic differential equation:

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

where W_t is a standard Brownian motion under the real-world probability measure, and μ and σ are constants, with σ > 0.
Let r > 0 be the continuously compounded risk-free rate of interest.

(iii) Show, using parts (i) and (ii), that $W_t + \lambda t$ is a Brownian motion under the risk-neutral probability measure, if λ =(μ – r)/ σ

(iv) Calculate the value of λ in the case in which μ = 0.04 + r and σ = 0.4.

Another traded asset has a price process satisfying the stochastic differential equation

$$dA_t = (0.06 + r)A_t dt + \gamma A_t dW_t.$$

(v) Determine the value of the volatility coefficient γ, using your result from part (iv).

18. CT8 April 2018 Q3

The value of an investment asset follows the equation A(t) = exp(B_t), where B_t follows standard Brownian motion.
(i) State the five defining properties that apply to B_t as a standard Brownian motion.

An actuarial student invests $1,000 in the asset at time 0.

(ii) Calculate the expected value of this investment at time 5.

(iii) Calculate the probability that the value of the student's investment is less than $10,000 at time 5

19. CT8 April 2018 Q7

(i) Define a complete market.

The price process of a traded security satisfies the following stochastic differential equation
$dS_t = \mu S_t dt + \sigma S_t dW_t$,
where W_t is a Brownian motion under the real-world probability measure P. Let r > 0 be the continuously compounded risk-free rate of interest, with $r \neq \mu$.

(ii) Show that the discounted stock price $e^{-rt}S_t$ is not a martingale under the real-world probability measure P.

(iii) Demonstrate how the discounted asset price $e^{-rt}S_t$ can be a martingale under an equivalent martingale measure Q.

20. CT8 September 2018 Q9

The price process of a non-dividend paying stock S_t satisfies the following stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

where W_t is a Brownian motion under the real-world probability measure P. Let V(t) be the value at t of a self-financing portfolio, consisting of Φ_t stocks and ψ_t cash bond.
(i) Show that $d(e^{-rt}V(t)) = \Phi_t d(e^{-rt}S_t)$.

(ii) Determine the conditions under which the discounted value $e^{-rt}V(t)$ is a martingale.

21. CM2A September 2019 Q2

(i) State the four key features of a standard Brownian motion B_t .

Consider a stochastic differential equation

$$dX_t = Y_t\, dB_t + A_t\, dt$$

where A_t is a deterministic process and Y_t is a stochastic process adapted to the natural filtration of B_t.

(ii) Write down Ito's lemma for $f(t,X_t)$, where f is a suitable function.

(iii) Determine $df(t,X_t)$ where $f(t,X_t) = \exp(4t^2 X_t)$.

22. CM2A September 2020 Q7

Consider the price, D_t, at time t of a call option on an underlying stock S_t. The call option has a strike price of K and matures at time T. Let C_t be a cash account at time t ($t < T$) and r be the continuously compounded risk-free rate. T is measured in years.

(i) Define a self-financing strategy (a_t, b_t).

Consider the two portfolios below:

A: a_t units of S_t and b_t units of C_t

B: 1 unit of D_t

(ii) Construct two equations that must be satisfied so that Portfolio A is self-financing and replicates the price of Portfolio B.

Consider now the scenario where:

S_0 = £20, T = 2, K = £20, r = 5%

In the period from t = 0 to t = 1, the price can either increase by 50%, or decrease by 20%. In the period from t = 1 to t = 2, the price can either increase by 40%, or decrease by 30%.

(iii) Calculate the price of the call option at t = 0 using a 2-period binomial tree.

(iv) Derive the portfolio of stocks and cash at t = 0 that replicates the option value at t = 1.

(v) Show that the replicating portfolio from part (iv) matches the option price at t = 1, for the two possible share prices at t = 1.

(vi) Comment on the limitations of using a binomial tree to set up and maintain a replicating portfolio for this option in the real world.

23. CM2A April 2022 Q4

Suppose that under the unique equivalent measure martingale measure, Q, for a term structure model, the Stochastic Differential Equation satisfied by the instantaneous interest rate r is:

$$dr_t = \alpha(\mu - r_t)dt + \sigma dZ_t$$

where α > 0, μ and σ are fixed parameters and under Q, Z is a standard Brownian Motion.

The process X is defined by:

$$X_t = r_t b(T-t) + \int_0^t r_s \, ds$$

where the function b is given by b(s) = $\frac{(1-e^{-\alpha s})}{\alpha}$

The function f is given by f(x,t) = exp(a(T-t)-x) where a is a differentiable function.

(i) Apply Ito's formula to f(X_t, t).

[Hint: You may use, without proof, the fact that $dX_t = A_t dt + B_t dZ_t$ where $A_t = \alpha\mu b(T-t)$, and $B_t = \sigma b(T-t)$.]

(ii) Find a differential equation that the function a must satisfy, in order for f(X_t, t) to be a martingale.

(iii) Determine an additional condition on a that is necessary for a bond with unit payoff at time T to have a price given by the formula:

$$B(t,T) = f(X_t, t)\exp\left(\int_0^t r_s ds\right)$$

24. CM2A September 2023 Q7

Consider the process A_t defined by the following integral:

$$\int_0^t W_s ds$$

where W_s is a standard Brownian motion.

(i) Explain why this process is not a martingale.

(ii) Show that A_t is Normally distributed with mean 0 and variance $t^3/3$.

[**Hint**: Note that $tW_t = \int_0^t tdW_s$.]