INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Subject: Financial Engineering 1

Chapter: Unit 4

Category: Practice Questions

1. CT8 April 2010 Q4

Consider the following stochastic differential equation for the instantaneous risk-free rate (also referred to as the short-rate):

$$dr(t) = a\left(b - r(t)\right)dt + \sigma dW_t$$

Its solution is given by:

$$r(t) = r_0 \exp(-at) + b\left(1 - \exp(-at)\right) + \sigma \exp(-at)\int_0^t \exp(as)\,dW_s$$

You may also use the fact that for T > t:

$$\int_t^T r(u)du = b(T-t) + \left(r(t) - b\right)\frac{1 - \exp\left(-a(T-t)\right)}{a} + \frac{\sigma}{a}\int_t^T \left(1 - \exp\left(-a(T-s)\right)\right)dW_s$$

(i) Derive the price at time t of a zero-coupon bond with maturity T.

(ii) (a) State the main drawback of such a model for the short-rate.
(b) State the name and stochastic differential equation of an alternative model for the short-rate that is not subject to the drawback.

2. CT8 September 2010 Q3

Discuss whether one-factor models are good models for the short-rate of interest (instantaneous risk-free rate.
Include discussion of extensions that may be considered to improve the model. Illustrate your discussion by defining and referring to particular models.

3. CT8 April 2011 Q10

Let B(t,T) be the price at time t of a zero-coupon bond paying £1 at time T, r_t be the short rate of interest, P be the real world probability measure and Q the risk neutral probability measure.
(i) Write down two equations for the price of a zero-coupon bond, one of which uses the risk-neutral approach to pricing and the other of which uses the state price-deflator approach to pricing.

(ii) State the Stochastic Differential Equation (SDE) of the short rate r_t under Q for the Vasicek model and the general type of process this SDE represents.

(iii) Solve the SDE for the short rate r_t from (ii).

(iv) Deduce the form of the distribution of the zero-coupon bond price under this model.

4. CT8 September 2011 Q5

(i) List the desirable characteristics of a model for the term structure of interest rates.

(ii) Write down the stochastic differential equation for the short rate r_t under Q in the Hull-White model.

(iii) Indicate whether or not the Hull-White model shows the characteristics listed in (i).

5. CT8 September 2011 Q6

Under the real-world probability measure, P, the price of a zero-coupon bond with maturity T is given by:

$$B(t,T) = \exp\left\{-(T-t)r_t + \frac{\sigma^2}{6}(T-t)^3\right\}$$

where r_t is the short rate of interest at time t and satisfies the following stochastic differential equation under the real-world measure P:

$$dr_t = \mu r_t dt + \sigma dZ_t,$$

where $\mu > 0$ and Z_t is a standard Brownian motion under P.

(i) Derive a formula for the instantaneous forward rate f(t, T), based on this model.

(ii) Derive an expression for the market price of risk.

(iii) Deduce the stochastic differential equation for rt under the risk-neutral measure Q defining all terms used.

6. CT8 April 2012 Q6

(i) Write down a stochastic differential equation for the short rate r (t) for the Vasicek model.

(ii) State the type of process of which the Vasicek model is a particular example.

(iii) Solve the stochastic differential equation in (i).

(iv) State the distribution of r(t) for t given.

(v) Derive the expected value and the second moment of r(t) for t given.

(vi) Outline the main drawback of the Vasicek model.

7. CT8 September 2012 Q6

(i) State the Stochastic Differential Equations for the short rate r(t) in the Vasicek model and the Cox-Ingersoll-Ross model.

(ii) Explain the impact of a movement in the short rate on the volatility term in both models.

8. CT8 April 2013 Q8

(i) Describe three limitations of one-factor term structure models.

(ii) Write down, defining all terms and notation used, the two-factor Vasicek model.

9. CT8 September 2013 Q8

(i) Write down a stochastic differential equation for the short rate r in the Vasicek model defining any notation used.

(ii) List the desirable and undesirable features of this model for the term structure of interest rates.

(iii) (a) Solve the stochastic differential equation from your answer to part (i).
(b) Comment on the statistical properties of r_T , T > t.

10. CT8 April 2014 Q5

Consider the following model for the short-rate r:

$$dr_t = \mu r_t dt + \sigma dZ_t$$

where μ and σ are fixed parameters and Z is a standard Brownian motion.

(i) Comment on the suitability of this model for the short-rate.

An alternative model for the short-rate is the Vasicek model:

$$dr_t = a(\mu - r_t)dt + \sigma dZ_t.$$

(ii) Derive an expression for $\int_t^T r(u)du$.

(iii) State the distribution of $\int_t^T r(u)du$.

11. CT8 September 2014 Q6

Consider a one-factor model for the short- rate r.
(i) Explain why a tradeable asset has to be introduced in order to build an arbitrage-free model.

Consider a specific bond with maturity T_1, suppose its price satisfies the following Stochastic Differential Equation (SDE) under the real-world probability measure P:

$$dB(t,T_1) = B(t,T_1)[m(t,T_1)dt + S(t,T_1)dW(t)]$$

where W is a standard Brownian Motion, $m(t, T_1)$ is the drift and $S(t, T_1)$ is the volatility.

(ii) (a) State the market price of risk.
(b) Explain what it represents.
(c) Show how it can be used in transforming the SDE above from the real-world probability measure P to a risk-neutral probability measure Q.

(iii) Show how the above results would be used in calculating zero-coupon bond prices.

(iv) Explain how this is typically done in practice.

12. CT8 April 2015 Q10

There are two risk-free zero-coupon bonds trading in a market, Bond X and Bond Y.

The short rate of interest, r_t , follows a Vasicek model:

$$dr_t = \alpha(\mu - r_t)dt + \sigma dW_t$$

where W_t is a standard Brownian motion.
(i) Write down the formula for the price of a risk-free zero-coupon bond at time t, with bond maturity at time T, under the Vasicek model.

In this market the parameters for the Vasicek model are α = 0.5, μ = 4% and σ= 10%. The short-rate at time 0, r(0), is 2% p.a. Bond X matures at time 1, and Bond Y matures at time 3. Both bonds are for a nominal value of $100.

(ii) Calculate the fair price of Bond X.

Bond Y has a fair price at time 0 of $90.
(iii) Derive the market-implied risk-free spot rate of interest with maturity 3 years.
(iv) Derive the market-implied risk-free forward rate of interest from time 1 to time 3.

13. CT8 April 2016 Q10

In the Vasicek model, the short rate of interest under the risk-neutral probability measure is given by:

$$r_t = \theta + e^{-kt}(r_0 - \theta) + \sigma \int_0^t e^{-k(t-u)} dW_u$$

where k, θ, σ > 0 and W is a standard Brownian motion.
Consider the related process:

$$R_t = \int_0^t r_s ds$$

where r_t is the short rate defined above.

(i) Show that R_t has a normal distribution with mean and variance given by:

$$E(R_t) = \theta t + (r_0 - \theta)\frac{1-e^{-kt}}{k} \text{ and}$$

$$\text{Var}(R_t) = \frac{\sigma^2}{k^2}\left(t - \frac{2(1-e^{-kt})}{k} + \frac{1-e^{-2kt}}{2k}\right).$$

Let P(0,t) be the price at time 0 of a zero-coupon bond with redemption date t > 0.

(ii) Show that, under the Vasicek model:

$$P(0,t) = e^{-E(R_t) + \frac{\text{Var}(R_t)}{2}}$$

(iii) Show, by using the results from parts (i) and (ii), that:

$$P(0,t) = A(t)e^{-B(t)r_0}$$

where $B(t) = \dfrac{1-e^{-kt}}{k}$

and $A(t) = \exp\left[(B(t)-t)\left(\theta - \dfrac{\sigma^2}{2k^2}\right) - \dfrac{\sigma^2}{4k}B(t)^2\right].$

(iv) State the main drawback of the above model for the term structure of interest rates.

14. CT8 September 2016 Q9

(i) Write down the properties of the following two models for interest rates:

(a) the one-factor Vasicek model

(b) the Cox-Ingersoll-Ross model

[You are not required to give any formulae for the models.]

The Vasicek term structure model is described by the following stochastic differential equation:

$$dr_t = a(b - r_t)dt + \sigma dW_t,$$

with initial value r_0 and a, b, σ > 0.

(ii) Show, by solving the Vasicek stochastic differential equation, that:

$$r_t = r_0 e^{-at} + b\left(1 - e^{-at}\right) + \sigma \int_0^t e^{-a(t-s)} dW_s\ .$$

(iii) Determine the expectation, the variance and the distribution of the short rate r_t

15. CT8 April 2017 Q8

(i) List the desirable characteristics of a term structure model.

Let B(t,T) be the price at time t > 0 of a zero-coupon bond which pays a value of 1 when it matures at time T.

Let F (t,S ,T) be the forward rate at time t for a deposit starting at time S > t and expiring at time T > S.

Consider the following two investment strategies implemented at time t:

A	At time t: Purchase one zero-coupon bond maturing at time T. Continue to hold the bond to time T.
B	At time t: Purchase $\alpha = e^{-F(t,S,T)(T-S)}$ zero-coupon bonds maturing at time $S < T$. At time S: Invest the redemption amount from the bond at the forward rate $F(t,S,T)$ and continue to hold this deposit to time T.

(ii) Show that:

$$B(t,T) = e^{-F(t,S,T)(T-S)}B(t,S).$$

(iii) Derive an expression for B(t,T) in terms of the instantaneous forward rate, using the result from part (ii).

16. CT8 September 2017 Q7

(i) State the main potential drawback of the Vasicek model.

(ii) Discuss the extent to which this drawback may be a problem.

(iii) Explain how the Cox-Ingersoll-Ross model avoids this drawback.

The Vasicek term structure model is described by the following stochastic differential equation:

$$dr_t = a(b - r_t)dt + \sigma dW_t,$$

and a, b, $\sigma > 0$.
Under this model, the short rate r_t follows a Normal distribution with mean

$$E(r_t) = r_0 e^{-at} + b\,(1 - e^{-at})$$

and variance $\text{Var}(r_t) = \frac{\sigma^2}{2a}(1 - e^{-2at})$.

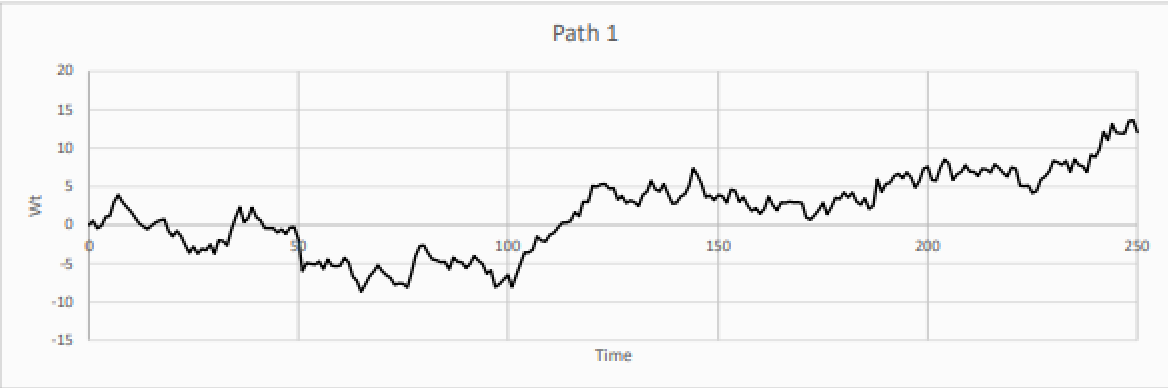
(iv) Assess, using the information provided above, whether the model generates interest rates that are mean reverting and, if so, the value to which they revert.

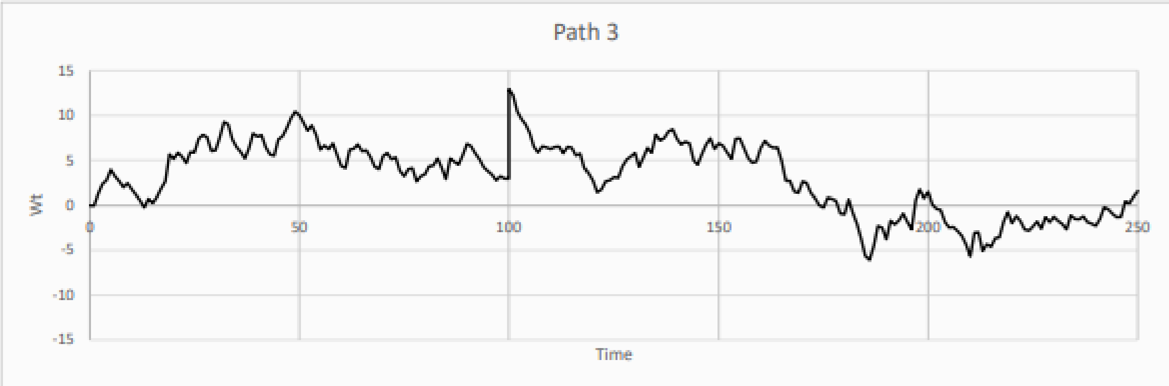
(v) Assess, using the information provided above, the relevance of the parameter a to any mean reversion.

17. CM2A April 2023 Q2

(i) State three potential uses for stochastic security price models. For each use, explain why the model should be stochastic.

The charts below show sample paths for three different stochastic processes.

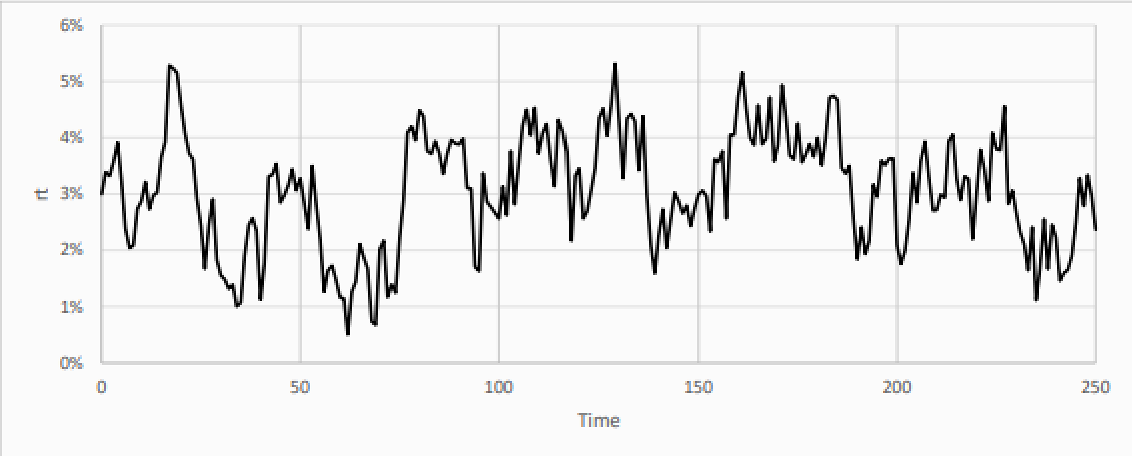




(ii) For each chart explain if, and why, the path is likely to be a Wiener process.

The chart below shows a single sample path of a Vasicek model represented by the following stochastic differential equation:

$$dr_t = \kappa(\theta - r_t)dt + \sigma d\widehat{W}_t$$



(iii) Describe the likely impact on the chart of the following parameter changes:

(a) An increase in σ

(b) An increase in θ

(c) A reduction in κ.

18. CM2A September 2023 Q5

An analyst at a bank wants to model interest rates and is considering using either the Vasicek or Cox–Ingersoll–Ross models.

(i) State two similarities and two differences between these models.

The analyst is looking to calibrate their model to the yield curves of a single country. In the period modelled, the short-term interest rates in this country were consistently negative over a period of multiple years.

(ii) Explain why the Cox–Ingersoll–Ross model will not be appropriate for the analyst.

(iii) Explain how the analyst could adjust the Cox–Ingersoll–Ross model to reflect the points you have identified in part (ii).