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Subject: Financial Engineering 2

Chapter:

Category: Assignment Solutions

1.

(i) The price of a zero-coupon bond can be written as

$$B(t,T)=E\left[\exp\left(-\int_t^T r(s)ds\right)\Big|F_t\right].$$

Since $\int_t^T r(u)du$ is a Gaussian random variable, we can compute explicitly the price of the zero-coupon bond in terms of the expected value and variance (conditional) of $\int_t^T r(u)du$:

$$B(t,T)=\exp\left[-E\left[\int_t^T r(s)ds|F_t\right]+\frac{1}{2}V\left[\int_t^T r(s)ds|F_t\right]\right]$$

where $E\left[\int_t^T r(s)ds|F_t\right]=b(T-t)+\left(r(t)-b\right)\left(\frac{1-\exp\left(-a(T-t)\right)}{a}\right)$ and

$$V\left[\int_t^T r(s)ds|F_t\right]=\frac{\sigma^2}{a^2}(T-t)-\frac{\sigma^2}{2a^3}\left(\exp\left(-2a(T-t)\right)-1\right)+\frac{2\sigma^2}{a^3}\left(\exp\left(-a(T-t)\right)-1\right).$$

(ii) Main issue: possibility to have negative interest rates when using the Vasicek model. An alternative is the CIR model:

$$dr(t)=a\left(b-r(t)\right)dt+\sigma\sqrt{r(t)}dW_t.$$

2.

The model should be arbitrage free.

Interest rates should be positive.

The short rate and other interest rates should exhibit some form of mean-reverting behaviour.

It should be straightforward to calculate the prices of bonds and certain derivative contracts.

The model should produce realistic dynamics.

The model should be able to be calibrated easily to current market data.

The model should be flexible enough to cope properly with a range of derivative contracts.

The model should provide a satisfactory fit to historical data.

3.

(i) Interest rates may not be positive.

Interest rates do not display mean reversion.

This model is computationally tractable.

This model won't give a realistic range of yield curves.

It won't fit historical data well.

It cannot be calibrated to current market data.

It is not very flexible (single factor model).

It is arbitrage-free.

(ii) Since the Vasicek model is an Ornstein-Uhlenbeck process we can solve the SDE for the short rate to get:

$$r(u) = r(t)e^{-a(u-t)} + \mu\left(1 - e^{-a(u-t)}\right) + \sigma e^{-au}\int_t^u e^{as}\,dZ_s.$$

Hence

$$\int_t^T r(u)\,du = r(t)\int_t^T e^{-a(u-t)}du + \mu\int_t^T \left[1 - e^{-a(u-t)}\right]du + \sigma\int_t^T e^{-au}\int_t^u e^{as}\,dZ_s\,du$$

and so, carrying out the deterministic integrals, we find:

$$\int_t^T r(u)\,du = \mu(t-t) + \left[r(t)-\mu\right]\frac{1-e^{-a(T-t)}}{a} + \sigma\int_t^T \frac{1-e^{-a(T-s)}}{a}\,dZ_s$$

So, $\int_t^T r(u)\,du$ is a Gaussian random variable.

4. (i)

(a) It incorporates mean reversion [½]
It is time homogenous, i.e. the future dynamics of $r(t)$ only depend upon the current value of $r(t)$ rather than what the present time t actually is. [1]
It is arbitrage free. [½]
It allows negative interest rates. [½]
It is easy to implement since the characteristic functions of all related quantities are available. [1]
It has constant volatility [½]
[Max 2]

(b) It incorporates mean reversion.... [½]
... is arbitrage free... [½]
... and time homogenous. [½]
Volatility depends on the level of the rates: it is high/low when rates are high/low. [1]
It does not allow negative interest rates. [½]
However it is more involving to implement than Vasicek model [½] as it is linked to the chi-squared distribution. [½]
It is a one factor model [½]
[Max 2]

(ii) Use Itô's lemma on the auxiliary process $X_t = e^{at} r_t$: [1]

$$\frac{dX}{dr} = e^{at}, \frac{d^2X}{dr^2} = 0, \frac{dX}{dt} = ae^{at} r_t. \quad [1]$$

And so Itô gives:

$$dX_t = [e^{at} a(b - r_t)dt + ae^{at} r_t]dt + e^{at} \sigma dW_t . \quad [1]$$

And hence:

$$dX_t = de^{at} r_t = abe^{at} dt + \sigma e^{at} dW_t . \quad [½]$$

By direct integration from 0 to t, it follows that:

$$e^{at} r_t = r_0 + b\left(e^{at} - 1\right) + \sigma \int_0^t e^{as} dW_s \quad [1]$$

and hence, as required, $r_t = r_0 e^{-at} + b\left(1 - e^{-at}\right) + \sigma \int_0^t e^{-a(t-s)} dW_s$. [½]

[Max 4]

(iii) From Result 3.2 of the Core Reading, r_t follows a Normal distribution with mean: [1]

$$Er_t = r_0 e^{-at} + b\left(1 - e^{-at}\right)$$ [1]

and variance

$$\text{Var}(r_t) = E(r_t - Er_t)^2 = E\left[\left(\sigma\int_0^t e^{-a(t-s)}dW_s\right)^2\right]$$ [1]

$$= \sigma^2 \int_0^t e^{-2a(t-s)}ds$$ [1]

$$= \frac{\sigma^2}{2a}\left(1 - e^{-2at}\right).$$ [½]

[Max 3]
[Total 11]

5.

i) Using continuous compounding.

[Note to markers: please accept any correct attempt using different compounding convention]

a. $-\frac{1}{0.25} ln\frac{97.5}{100} = 10.13\%$ [1]

b. $-\frac{1}{0.5} ln\frac{94.9}{100} = 10.47\%$ [1]

c. $0.04 \times (94.9 + 90) + 104 \times e^{-0.1068\times1.5} = 96$ [1]

d. $\frac{(0.1054\times1 - 0.1047\times0.5)}{1-0.5} = 10.60\%$ [1]

e. $\frac{(0.1068\times1.5 - 0.1054\times1)}{1.5-1} = 10.97\%$ [1]

ii) Two standard models for the short rate of interest are the Vasicek model and the CIR model.

The corresponding SDEs are respectively

$$dr_t = k(\theta - r_t)dt + \sigma dW_t$$
$$dr_t = k(\theta - r_t)dt + \sigma\sqrt{r_t}dW_t$$

[1 mark each]

Alternatively: another standard model is the Hull and White model which extends the Vasicek model to allow for time-inhomogeneity, therefore the parameters in the SDE are time dependent.

6. (i) The three types of credit risk model are:
Structural models: these are explicit models of a corporate entity issuing both debt and equity. They aim to link default events explicitly to the fortunes of the issuer.

Reduced-form models: these are statistical models which use market statistics (such as credit ratings) rather than specific data relating to the issuer, and give statistical models for their movement.

Intensity-based models: these model the factors influencing the credit events which lead to default and typically do not consider what triggers these events.

(ii) In the Merton model, the company is modelled as having a fixed debt, 40 with term 10 years and variable assets S_t. The equity holders can be regarded as holding a European call on the assets with a strike of 40.

In the current question the value of the option is 20.

Using Black Scholes formula, with $(T - t) = 10$, $K = 40$, $S_0 = 60$, $r = 0.05$, solve for σ, the implied volatility.
[Candidates need not actually do this calculation]

The assets of the company therefore follow a geometric Brownian motion under the risk neutral measure with drift $r = 0.05$ and volatility σ.

Therefore $\log(S_{10}/ S_0)$ follows a normal distribution with mean $10*(0.05 - \sigma_2/2)$ and variance $10* \sigma^2$.

The risk neutral probability of default is obtained by calculating the probability that $\log(S_{10}/ S_0)$ is less than $\log(40/60)$.

(iii) In the two state model for credit rating with deterministic transition intensity, the formula for the Zero Coupon Bond price is

$$B(t, T) = e^{-r(T-t)} \left(1 - (1 - \delta)\left(1 - e^{-\int_t^T \tilde{\lambda}(s)ds}\right)\right).$$

It follows that the risk-neutral default intensity is given by

$$\tilde{\lambda}(s) = s/2.$$

7. Default risk is the risk of loss arising from the outright failure of a counterparty to perform on its liabilities and contractual obligations.

Default Correlation denotes a measure of Default Dependency between different borrowers when considered as part of a Credit Portfolio. It measures the likelihood of

Joint Default within the period of consideration. Default correlation measures whether credit risky assets are more likely to default together or separately.

Default correlation determines the credit risk of a portfolio and the economic capital required to support that portfolio, along with default probability and loss in the event of default.

8 i) . **Default risk is the risk that a lender takes on in the chance that a borrower will be unable to make the required payments on their debt obligation.**

Whenever a lender extends credit to a borrower, there is a chance that the loan amount will not be paid back. The measurement that looks at this probability is the default risk. Default risk does not only apply to individuals who borrow money, but also to companies that issue bonds and due to financial constraints, are not able to make interest payments on those bonds.

Whenever a lender extends credit, calculating the default risk of a borrower is crucial as part of its risk management strategy. Whenever an investor is evaluating an investment, determining the financial health of a company is crucial in gauging investment risk.

ii) Recovery risk refers to a company's exposure to loss as a result of damage to its ability to conduct day-to-day operations.

Analysis of recovery risk involves categorizing threats according to short-, medium- and long-term impact. Short-term threats may include damage to computer systems or workers' inability to reach the job site due to natural disasters. Medium-term impact threats may include infrastructure failure or loss of staff. Long-term impact threats may include extensive property damage.

iii) Exposure risk refers to the risk inherent in an investment, indicating the amount of money an investor stands to lose.

Knowing and understanding financial exposure, which is an alternative name for risk, is a crucial part of the investment process.

The simplest way to minimize financial exposure is to put money into principal-protected investments with little to no risk.

Another way to reduce financial exposure is to diversify among many investments and asset classes. To build a less volatile portfolio, an investor should have a combination of stocks, bonds, real estate, and other various asset classes. Within the equities, there should be further diversification among market capitalizations and exposure to domestic and international markets.

9. Credit events, which might result in a failure to meet an obligation (defined for the purposes of credit derivatives).

Their types include:

i. actions that are associated with bankruptcy or insolvency laws: ie the bond issuer becomes insolvent.
ii. downgrade by 'Nationally Recognised Statistical Rating Organisations', (NRSROs such as Moody's, S&P and Fitch) This is of particular concern when a bond is issued with a guaranteed minimum credit rating.
iii. failure to pay: ie either a coupon or the capital amount is not paid in full and on time.
iv. repudiation / moratorium: ie the validity of the contract is disputed or a temporary suspension of activity is imposed on the issuer.
v. restructuring – when the terms of the obligation are altered so as to make the new terms less attractive to the debt holder, such as a reduction in the interest rate, rescheduling, change in principal, change in the level of seniority.