INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Class: TY BSc

Subject : Financial Engineering - 2

Chapter: Unit 2 Chapter 1

Chapter Name: Black Model

We previously learned

- Exotic options are options contracts that differ from traditional options in their payment structures, expiration dates, and strike prices.
- Exotic options usually trade in the over-the-counter (OTC) market.
- Exotic options have unique underlying conditions that make them a good fit for high-level active portfolio management and situation-specific solutions.
- A **gap option** is an option where there's a **trigger price** that's different from the strike price.

- A **forward start option** is an option that **starts** at some time **in the future.**

- A **compound option** is an **option** on an **option.**

- A **chooser option** (or **"as-you-like-it"** option) is an option where the holder has the right to **choose** whether it is a **call** or a **put** option at some point during it's life.

- A **barrier option** is an option where the payoff depends on whether the price of the underlying asset reaches a certain level (**barrier**) during a certain period of time.

- A **binary** (or **digital**) **option** is an option where the payoff is **"all-or-nothing"** (and so they have a discontinuous payoff)

Continued

- A **lookback** (or **hindsight**) **option** allows the holder the advantage of **knowing history** when determining when to exercise their option.

- A **shout option** is an European option where the holder can "shout" at various points of the duration of the option. At expiry, the holder receives the greater of the payoff from the **European option** and the **intrinsic values** at shouts.

- An **asian** (or **average**) **option** is an option where the **payoff** depends on the **average price** of the underlying during at least some part of the life of the option.

- An **exchange option** is an option to exchange one asset for another.

- A **rainbow option** is an option where the payoff is dependent on **two or more** underlying variables.

- A **basket option** is an option where the payoff is based on the value of a **basket** (ie portfolio) of assets.

Agenda

1. Black Model
    1. Introduction
    2. Assumptions
    3. Black Scholes Formula – SDE
    4. Black's Model for valuing Futures options

2. Interest Rate Options

1. Swaptions

1.1 Introduction

- In the early 1970s, Fischer Black, Myron Scholes, and Robert Merton achieved a major breakthrough in the pricing of European stock options.

- This was the development of what has become known as the Black–Scholes–Merton (or Black–Scholes) model.

- The model has had a huge influence on the way that traders price and hedge derivatives.

- In 1997, the importance of the model was recognized when Robert Merton and Myron Scholes were awarded the Nobel prize for economics. Sadly, Fischer Black died in 1995; otherwise he too would undoubtedly have been one of the recipients of this prize.

The Black-Scholes analysis of option prices is underpinned by a number of key assumptions. We discuss these first in Section 1 and consider how realistic they are in practice. Even though the assumptions do not all hold in practice, this does not prevent the Black-Scholes model providing a good approximation to reality. The approach offers valuable insight into option pricing and is widely used in practice.

1.2 Assumptions

The assumptions underlying the Black-Scholes model are as follows:

1. The price of the underlying share follows a Geometric Brownian Motion.
2. There are no risk-free arbitrage opportunities.
3. The risk-free rate of interest is constant, the same for all maturities and the same for borrowing or lending.
4. Unlimited short selling (that is, negative holdings) is allowed.
5. There are no taxes or transaction costs.
6. The underlying asset can be traded continuously and in infinitesimally small numbers of units.

The key general implication of the underlying assumptions is that the market in the underlying share is complete: that is, all derivative securities have payoffs which can be replicated.

This consequence is at odds with the real world and implies problems with the underlying assumptions.

1.2 Unrealistic Assumptions

It is clear that each of these assumptions is unrealistic to some degree, for example:

- **Share prices can jump.** This invalidates assumption 1. since Geometric Brownian Motion has continuous sample paths. However, hedging strategies can still be constructed which substantially reduce the level of risk.
- **The risk-free rate of interest does vary and in an unpredictable way.** However, over the short term of a typical derivative, the assumption of a constant risk-free rate of interest is not far from reality.
- **Distributions of share returns tend to have fatter tails** than suggested by the lognormal model, invalidating assumption 1.
- **Unlimited short selling may not be allowed**, except perhaps at penal rates of interest. These problems can be mitigated by holding mixtures of derivatives which reduce the need for short selling. This is part of a suitable risk management strategy as discussed in Section 5.2.
- **Shares can normally only be dealt in integer multiples of one unit**, not continuously, and dealings attract transaction costs: invalidating assumptions 4, 5 and 6. Again we are still able to construct suitable hedging strategies which substantially reduce risk.

Despite all of the potential flaws in the model assumptions, analyses of market derivative prices indicate that the Black-Scholes model does give a very good approximation to the market.

1.3 Underlying SDE

Suppose that we have a European call option on a non-dividend-paying share S_t which is governed by the stochastic differential equation (SDE):

$$dS_t = S_t\,(\mu dt + \sigma dZ_t)$$

where Z_t is a standard Brownian motion.

Investors are allowed to invest positive or negative amounts in this share. Investors can also have holdings in a risk-free cash bond with price B_t at time t.

This is governed by the ordinary differential equation: $dB_t = rB_t dt$

where r is the (assumed-to-be) constant risk-free rate of interest. Hence:

$$S_t = S_0 \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma Z_t\right]$$

$$B_t = B_0 \exp(rt)$$

To check this solution, we define the function $g(t, Z_t) = S_0 \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma Z_t\right]$ so that

$S_t = g(t, Z_t)$. Note that this is a function of t and Z_t.

Replacing $g = g(t, Z_t)$ with S_t , we have the original SDE and the check is complete:

$dS_t = \mu S_t dt + \sigma S_t dZ_t$

Since Z_t is normally distributed, S_t is lognormally distributed with all of the usual properties of that distribution.

This price process is sometimes called a lognormal process, geometric Brownian motion or exponential Brownian motion.

1.3 Black-Scholes Formula

Let f(t, s) be the price at time t of a call option given:

- The current share price is S_T = s
- The time of maturity is T > t
- The exercise price is K.

Proposition: (The Black-Scholes formula)

For such a call option

$$f(t, S_t) = S_t\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2)$$

Where

$$d_1 = \frac{\log\frac{S_t}{k} + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}}$$

$$d_2 = d_1 - \sigma\sqrt{(T-t)}$$

And $\Phi(z)$ is the cumulative distribution function of the standard normal distribution.

For a put option we also have $f(t, S_t) = Ke^{-r(T-t)}\Phi(-d_2) - S_t\Phi(-d_1)$
where d_1 and d_2 are as defined above.

1.4 Black's Model for valuing futures options

According to the Black Scholes model, the fair prices at time 0 of a European call option maturing at time T on a share that pays continuous dividend is:

$$c_0 = S_0 e^{-qT}\Phi(d_1) - Ke^{-rT}\Phi(d_2)$$

Where $d_1, d_2 = \frac{log\frac{S_0}{K} + (r - q \pm \sigma^2)T)}{\sigma\sqrt{T}}$

If we let $F_0 = S_0 e^{-(r-q)T}$, Which equals the forward price for the share at times 0 we can write:

$$c_0 = e^{-rT}[S_0 e^{-(r-q)T}\Phi(d_1) - K\Phi(d_2) = e^{-rT}[F_0\Phi(d_1) - K\Phi(d_2)]$$

And $d_1, d_2 = \frac{log\frac{S_0 e^{(r-q)T}}{K} \pm \left(\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} = \frac{log\frac{F_0}{K} \pm \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$

So we can express the price of a European call option and also put option in terms of Forward price of the underlying asset. This idea forms the basis of Black's model.

1.4 Description of Black's Model

BLACK'S MODEL

The value of a T-year option with strike price K on an underlying variable V, whose forward volatility is σ and whose current T-year forward price is F_0, is:

$$c_0 = e^{-rT}[F_0\Phi(d_1) - K\Phi(d_2)]$$

$$p_0 = e^{-rT}[K\Phi(-d_2) - F_0\Phi(-d_1)]$$

Where $d_1 = \frac{log\frac{F_0}{K}+\frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$, d2 $= \frac{log\frac{F_0}{K}-\frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$

- Black's model can be considered to be a generalized form of the Black-Scholes and Garman-Kohlhagen formulae. It has many applications, including pricing options on bonds and futures.
- It can be used to value a European call or put option on any variable that can be assumed to have a lognormal distribution at the expiry date T .
- Options on a physical asset (whose price = V) are valued via the corresponding current forward price (= F0). For a share, this reproduces the Garman-Kohlhagen formula

1.4 Description of Black's Model

- Black's model can also be used to value futures options, ie second-order derivatives where the underlying "asset" for the option is the futures price of some security.
- The P(0,T) factor is the price at time 0 of a T-year risk-free zero-coupon bond, which is used as a discount factor. This corresponds to the factor e^{-rT} when interest rates are constant.
- The formula for the call option corresponds to $e^{-rT}\{E_Q[V_T]\ \Phi(d_1)$-K $\Phi(d_2)\}$,where $E_Q[V_T]$ is the risk-neutral expectation of the underlying variable at time T.
- Note that a variable can be lognormally distributed at time T without conforming to Brownian motion before that time. For example, its log could follow a (mean-reverting) Ornstein-Uhlenbeck process.

2 Interest Rate Options

Description

- An interest rate cap / floor pays the excess / shortfall of the market rate (eg 3-month LIBOR) over an agreed fixed rate on each interest date over a specified period.
- The individual payments are called caplets / floorlets and are paid in arrears. The dates (typically quarterly) on which new values for the market interest rate take effect are called reset dates.
- No payment is made on the first reset date because the payoff on this date would be completely predictable at the outset and a non-zero payoff would simply push the initial option price up or down by a known amount.
- A typical caplet can be viewed as a European call option on the interest rate variable R_k or a put option on a notional zero-coupon bond. (See below.)
- A typical floorlet can be viewed as a put option on the interest rate variable R_k or a call option on a notional zero-coupon bond.
- Interest rate options typically have terms ranging from 1 to 7 years.
- These are OTC contracts.

2 Interest Rate Options

Payoffs from an interest rate options

For the time period between t_k and t_{k+1} the payoffs at time t_{k+1} (k=1, 2, 3,...) are:

Caplet: $L\delta_k \max(R_k - R_X, 0)$

Floorlet: $L\delta_k \max(R_X - R_k, 0)$

Where

- $\delta_k = t_{k+1} - t_k$ (typically =1/4)
- R_k is the quoted LIBOR rate at time t_k (relating to the period from t_k to t_{k+1})
- L is the principal
- R_X is the cap rate.

2 Interest Rate Options

Uses of interest rate options (caps and floors)

- An interest rate cap gives a borrower protection against rising interest rates.
- An interest rate floor gives a lender protection against falling interest rates.

Special features

- An interest rate collar is a long cap + short floor combination. It guarantees that the interest rate paid will lie between the two strike rates. Because a collar involves buying one option and selling another, the strike rates can usually be chosen so that there is no initial premium payable.
- The total payoff from a long cap and a short floor with the same strike rate is:

$$L\delta_k \max(R_k - R_X, 0) - L\delta_k \max(R_X - R_k, 0) = L\delta_k (R_k - R_X)$$

So: Cap - Floor = Pay - fixed swap

(An adjustment must be made for the first reset date.)

2 Interest Rate Options

Pricing / valuation

The value of a cap / floor can be found by valuing each caplet / floorlet individually and then summing.

Each caplet / floorlet can be valued individually as a European call / put option on the forward interest rate for the period (t_k, t_{k+1}) using Black's model (which assumes that the value of the interest rate has a lognormal distribution).

Valuing a Caplet/ floorlet

- $V_{caplet} = L\delta_k P(0, t_{k+1})[F_k\Phi(\mathrm{d}_1) - \mathrm{R_X}\Phi(\mathrm{d}_2)]$
- $V_{floorlet} = L\delta_k P(0, t_{k+1})[\mathrm{R_X}\Phi(-\mathrm{d}_2) - F_k\Phi(-\mathrm{d}_1)]$

Here, F_k is the forward interest rate at time 0 for the period between time t_k and t_{k+1}, and σ_k is the volatility of this forward interest rate.

So, alternatively, each caplet / floorlet can be valued as a European put / call option on the forward price of a zero-coupon bond using Black's model (which assumes the zero-coupon bond prices have lognormal distributions at time t_{k+1}).

Note the distinction between the use of spot volatilities, where a different volatility is assumed for valuing each caplet / floorlet, and flat volatilities, where a constant volatility (representing an average value over the life of the option) is used.

3 Swaption

Description

- As option is an OTC option on a swap. It gives the purchaser the option (with no obligation) to enter into a swap at a rate that is agreed now (rather than at prevailing swap rate)
- As an option, it confers additional rights and so an initial premium is payable example: 0.5 % of the notional principal.
- Swaptions may be European American or Bermudan style. (with American or Bermudan swaptions, the swap starts when the option is exercised – you don't have to wait till time T.)
- Suppose that an organization purchase an n-year European swaption exercisable at time T which strike rate $R_x =$ 5% on a pay-fixed swap. At time T, the organization will compare the prevailing n-year swap rate R with the 5%. If R> R_x, eg= 6%, the organization will exercise the option, since entering the swap will allow them to pay 5% fixed interest, which is below the market rate of 6%. In this case, they have made a profit of 1% each year for n years. If R< R_x, they will not exercise.
- If the principal is L and interest payments are made m times per year, the payoff for a swaption consists of an annuity of mn equal payments payable during the n-year period.

3 Swaption

Payoffs from a swaption

Option on a pay-fixed swap (a payer swaption)	$\frac{L}{m}\max(R - R_x, 0)$
Option on pay-floating swap (a receiver swaption)	$\frac{L}{m}\max(R_x - R, 0)$

Uses of swaptions

- A swaption can be used to hedge against adverse future swap rates (resulting from interest rate movements).
- Purchasing a swaption ensures that borrowing will be possible at an acceptable rate.
- Life insurance can use swaptions to hedge interest rate guarantees. Example guaranteed annuity rates.
- American swaptions give a borrower greater flexibility than European swaptions over the timing of the swap and they impose fewer constraints on other related arrangements (example a loan that is linked to a swap).
- Bermudan swaptions (example ones where you can exercise on the last day of each calendar month) have the desirable features of American options, but are easier to administer.

3 Swaption

Special features

- The strike rate is usually chosen to be similar to current swap rates (which, in turn, are similar to current interest rates for the same term).

Pricing/ Valuation

- A European swaption can be valued using Black's formula. The underlying variable is the n-year forward swap rate, whose current value is F_0.

Valuing a European Swaption

Option on a pay-fixed swap: $V_{swaption} = L\,A[F_0\Phi(d_1) - R_X\,\Phi(d_2)]$

Option on a pay-floating swap: $V_{swaption} = L\,A[R_X\,\Phi(-d_2)] - F_0\Phi(-d_1)]$

Where A=$\frac{1}{m}\sum_{i=1}^{mn} P(0,t_i)$

- Here, A denotes the current value of an annuity of 1 unit per annum on each interest payment date and $P(0,t_i)$ is the market discount factor for a payment at time t_i, F_0 is the forward swap rate and R_X is the Strike price.

Recap

- Fischer Black, Myron Scholes, and Robert Merton achieved a major breakthrough in the pricing of European stock options.
- The assumptions underlying the Black-Scholes model are as follows:
  i. The price of the underlying share follows a Geometric Brownian Motion.
  ii. There are no risk-free arbitrage opportunities.
  iii. The risk-free rate of interest is constant, the same for all maturities and the same for borrowing or lending.
  iv. Unlimited short selling (that is, negative holdings) is allowed.
  v. There are no taxes or transaction costs.
  vi. The underlying asset can be traded continuously and in infinitesimally small numbers of units.
- The assumptions are unrealistic as :
  i. Share prices can jump.
  ii. The risk-free rate of interest does vary and in an unpredictable way.
  iii. Distributions of share returns tend to have fatter tails.
  iv. Unlimited short-selling may not be allowed.
  v. Shares can normally only be dealt in integer multiples of one unit.
- Using Black-Scholes Model, For a Call option, $f(t, S_t) = S_t\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2)$
- For a Put option, $f(t, S_t) = Ke^{-r(T-t)}\Phi(-d_2) - S_t\Phi(-d_1)$
- Where, $d_1 = \frac{\log\frac{S_t}{k} + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}}$, $d_2 = d_1 - \sigma\sqrt{(T-t)}$

Continued

- The value of a T-year option with strike price K on an underlying variable V, whose forward volatility is σ and whose current T-year forward price is F_0, is: $c_0 = e^{-rT}[F_0\Phi(d_1) - K\Phi(d_2)]$ & $p_0 = e^{-rT}[K\Phi(-d_2) - F_0\Phi(-d_1)]$
- where $d_1 = \frac{log\frac{F_0}{K}+\frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$, d2 $= \frac{log\frac{F_0}{K}-\frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$
- Black's model can be considered to be a generalized form of the Black-Scholes and Garman-Kohlhagen formulae. It has many applications, including pricing options on bonds and futures.
- For the time period between t_k and t_{k+1} the payoffs at time t_{k+1} (k=1, 2, 3,…) are:
- Caplet: $L\delta_k \max(R_k - R_X, 0)$ & Floorlet: $L\delta_k \max(R_X - R_k, 0)$
- Where $\delta_k = t_{k+1} - t_k$ (typically =1/4), $\mathrm{R_k}$ is the quoted LIBOR rate at time $\mathrm{t_k}$ (relating to the period from $\mathrm{t_k}$ to $\mathrm{t_{k+1}}$)
- $V_{caplet} = L\delta_k P(0, t_{k+1})[F_k\Phi(\mathrm{d_1}) - \mathrm{R_X}\Phi(\mathrm{d_2})]$ & $V_{floorlet} = L\delta_k P(0, t_{k+1})[\mathrm{R_X}\Phi(-\mathrm{d_2}) - F_k\Phi(-\mathrm{d_1})]$
- Payoffs from a swaption : Option on a pay-fixed swap (a payer swaption) = L/m max(R−R_x,0)
- Option on pay-floating swap (a receiver swaption) = L/m max(R_x−R,0)
- Valuing a European Swaption : Option on a pay-fixed swap: $V_{swaption} = L\ A[F_0\Phi(d_1) - R_X\ \Phi(d_2)]$
- Option on a pay-floating swap: $V_{swaption} = L\ A[R_X\ \Phi(-d_2)] - F_0\Phi(-d_1)]$,Where A=$\frac{1}{m}\sum_{i=1}^{mn} P(0, t_i)$