



Class: TY BSc

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Chapter: Unit 2 Chapter 2

Chapter Name: Interest rate derivatives and hedging - 2

1 Introduction

- For the derivatives, we need to calculate Greeks for Risk Management and hedging purposes.
- However, because of the term structure of interest rates, there is no longer a single underlying price or volatility parameter to differentiate with respect to.
- So the definitions of the Greeks are no longer clear cut

2 Delta

- Delta could be defined as the price change corresponding to a One basis point ($=0.01\%$) parallel shift in the yield curve (i.e. a uniform increase across all term).
- However, practitioners prefer to define it as a set of partial derivatives with respect to the quoted price of each of the financial instruments from which the yield curve is derived, which they consider to be more fundamental as measures of exposure.

2 Delta

- Delta risk is the risk associated with a shift in the zero curve. Because there are many ways in which the zero curve can shift, many deltas can be calculated. Some alternatives are:
 - i. Calculate the impact of a 1-basis-point parallel shift in the zero curve. This is sometimes termed a DV01.
 - ii. Calculate the impact of small changes in the quotes for each of the instruments used to construct the zero curve.
 - iii. Divide the zero curve (or the forward curve) into a number of sections (or buckets). Calculate the impact of shifting the rates in one bucket by 1 basis point, keeping the rest of the initial term structure unchanged.
 - iv. Carry out a principal components analysis as outlined in the upcoming example. Calculate a delta with respect to the changes in each of the first few factors. The first delta then measures the impact of a small, approximately parallel, shift in the zero curve; the second delta measures the impact of a small twist in the zero curve; and so on.

2 Delta

- In practice, traders tend to prefer the second approach. They argue that the only way the zero curve can change is if the quote for one of the instruments used to compute the zero curve changes. They therefore feel that it makes sense to focus on the exposures arising from changes in the prices of these instruments.

2 Delta

ALM Delta Hedging

- The asset–liability management (ALM) committees of banks now monitor their exposure to interest rates very carefully. Matching the durations of assets and liabilities is sometimes a first step, but this does not protect a bank against nonparallel shifts in the yield curve.
- A popular approach is known as GAP management. This involves dividing the zero-coupon yield curve into segments, known as buckets. The first bucket might be 0 to 1 month, the second 1 to 3 months, and so on. The ALM committee then investigates the effect on the value of the bank's portfolio of the zero rates corresponding to one bucket changing while those corresponding to all other buckets stay the same.
- If there is a mismatch, corrective action is usually taken. This can involve changing deposit and lending rates in the way described in Section 4.12. Alternatively, tools such as swaps, FRAs, bond futures, Eurodollar futures, and other interest rate derivatives can be used.

3 Gamma

- Gamma could be defined as a second partial derivative corresponding to a parallel shift in the yield curve.
- Alternatively, if the yield curve is derived from n instruments, with prices $x_1, x_2, \dots, x_n$, we could calculate n possible gammas of the form $\frac{\partial^2 f}{\partial x_i^2}$.
- If we want to explore the interactions between all the underlying instruments, there are a total of $1/2n(n-1)$ possible second derivatives of the form $\frac{\partial^2 f}{\partial x_i \partial x_j}$ ($i \neq j$).

3 Gamma

- When several delta measures are calculated, there are many possible gamma measures. Suppose that 10 instruments are used to compute the zero curve and that deltas are calculated by considering the impact of changes in the quotes for each of these.
- Gamma is a second partial derivative of the form $\partial^2 \Pi / \partial x_i \partial x_j$, where Π is the portfolio value.
- There are 10 choices for x_i and 10 choices for x_j and a total of 55 different gamma measures. This may be “information overload”.
- One approach is ignore cross-gammas and focus on the 10 partial derivatives where $i = j$.
- Another is to calculate a single gamma measure as the second partial derivative of the value of the portfolio with respect to a parallel shift in the zero curve.
- A further possibility is to calculate gammas with respect to the first two factors in a principal components analysis.

4 Vega

- Vega could be defined as a rate of change of the theoretical portfolio value when the volatility parameter in the Black formulae used to value interest rate options is increased by small amount.
- However, the parameters value will normally be different for different options and different payment dates.
- Alternatively, we can apply principal component analysis to determine the predominant volatility factors driving the prices (which will effectively be weighted averages of the individual volatilities).
- We can then calculate vegas as Partial derivatives with respect to these factors

4 Vega

- The vega of a portfolio of interest rate derivatives measures its exposure to volatility changes.
- One approach is to calculate the impact on the portfolio of making the same small change to the Black volatilities of all caps and European swap options. However, this assumes that one factor drives all volatilities and may be too simplistic.
- A better idea is to carry out a principal components analysis on the volatilities of caps and swap options and calculate vega measures corresponding to the first 2 or 3 factors.

5 Principal Component Analysis (PCA)

- One approach to handling the risk arising from groups of highly correlated market variables, like interest rates, is principal components analysis (PCA). This is a standard statistical tool with many applications in risk management. It takes historical data on movements in the market variables and attempts to define a set of components or factors that explain the movements.
- The approach is best illustrated with an example. The market variables we will consider are swap rates with maturities 1 year, 2 years 3 years, 4 years, 5 years, 7 years, 10 years, and 30 years.

5 Principal Component Analysis (PCA)

The yield curve is a line that plots the various interest rates of bonds with equal credit quality and different maturities.

According to the expectations theory of interest rates, the yield curve is made up of two aspects:

1. *An average* of market expectations concerning future short-term interest rates.
 2. *The term premium* — the extra compensation an investor receives for holding a longer-term bond
- We can model these aspects of the yield curve using principal components decomposition. Data has two main properties: noise and signal.
 - Principal components analysis aims to extract the signal and reduce the dimensionality of a dataset; by finding the **least amount of variables that explain the largest proportion of the data**. It does this by transforming the data from a correlation/covariance matrix onto a subspace with fewer dimensions, where all explanatory variables are orthogonal (perpendicular) to each other, i.e there is no multicollinearity.

5 Principal Component Analysis (PCA)

- When finding the principal components of the yield curve, the main theory held by econometricians is that:
 PC1 = constant $\approx$ long term interest rate $\approx R^*$
 PC2 = slope $\approx$ term premium
 PC3 = curvature
 There is a function in scikit-learn to perform PCA.
- The next step is to standardize the data into z-scores, assuming a mean of 0 and a variance of 1.
- We then form a covariance matrix from the standardized data using numpy.
- We can then create the eigenvalues and eigenvectors from the matrix. This performs eigendecomposition on our standardized data. We can use eigenvalues to find the proportion of the total variance that each principal component explains
- To form a time series for the principal components, we simply need to calculate the dot product between the eigenvectors and the standardized data.

5 Principal Component Analysis (PCA)

- The next two tables show results produced for these market variables using 2,780 daily observations between 2000 and 2011. The first column in The following table shows the maturities of the rates that were considered. The remaining eight columns in the table show the eight factors (or principal components) describing the rate moves.

Factor loadings for swap data.

	<i>PC1</i>	<i>PC2</i>	<i>PC3</i>	<i>PC4</i>	<i>PC5</i>	<i>PC6</i>	<i>PC7</i>	<i>PC8</i>
1y	0.216	−0.501	0.627	−0.487	0.122	0.237	0.011	−0.034
2y	0.331	−0.429	0.129	0.354	−0.212	−0.674	−0.100	0.236
3y	0.372	−0.267	−0.157	0.414	−0.096	0.311	0.413	−0.564
4y	0.392	−0.110	−0.256	0.174	−0.019	0.551	−0.416	0.512
5y	0.404	0.019	−0.355	−0.269	0.595	−0.278	−0.316	−0.327
7y	0.394	0.194	−0.195	−0.336	0.007	−0.100	0.685	0.422
10y	0.376	0.371	0.068	−0.305	−0.684	−0.039	−0.278	−0.279
30y	0.305	0.554	0.575	0.398	0.331	0.022	0.007	0.032

The factor loadings have the property that the sum of their squares for each factor is 1.0. Also, note that a factor is not changed if the signs of all its factor loadings are reversed.

5 Principal Component Analysis (PCA)

- The first factor, shown in the column labeled PC1, corresponds to a roughly parallel shift in the yield curve. When we have one unit of that factor, the 1-year rate increases by 0.216 basis points, the 2-year rate increases by 0.331 basis points, and so on.
- The second factor is shown in the column labeled PC2. It corresponds to a “twist” or change of slope of the yield curve. Rates between 1 year and 4 years move in one direction; rates between 5 years and 30 years move in the other direction. The third factor corresponds to a “bowing” of the yield curve. Relatively short rates (1 year and 2 year) and relatively long rates (10 year and 30 year) move in one direction; the intermediate rates move in the other direction. The interest rate move for a particular factor is known as factor loading. In our example, the first factor’s loading for the 1-year rate is 0.216.
- Because there are eight rates and eight factors, the interest rate changes observed on any given day can always be expressed as a linear sum of the factors by solving a set of eight simultaneous equations. The quantity of a particular factor in the interest rate changes on a particular day is known as the factor score for that day.

5 Principal Component Analysis (PCA)

- The importance of a factor is measured by the standard deviation of its factor score. The standard deviations of the factor scores in our example are shown in the table below and the factors are listed in order of their importance. The numbers in the table are measured in basis points. A quantity of the first factor equal to 1 standard deviation, therefore, corresponds to the 1-year rate moving by $0.216 \times 17.55 = 3.78$ basis points,
- the 2-year rate moving by $0.331 \times 17.55 = 5.81$ basis points, and so on.

Standard deviation of factor scores (basis points).

<i>PC1</i>	<i>PC2</i>	<i>PC3</i>	<i>PC4</i>	<i>PC5</i>	<i>PC6</i>	<i>PC7</i>	<i>PC8</i>
17.55	4.77	2.08	1.29	0.91	0.73	0.56	0.53

5 Principal Component Analysis (PCA)

- The factors have the property that the factor scores are uncorrelated across the data. For instance, in our example, the first factor score (amount of parallel shift) is uncorrelated with the second factor score (amount of twist) across the 2,780 days. The variances of the factor scores have the property that they add up to the total variance of the data. From the last table, the total variance of the original data (that is, sum of the variance of the observations on the 1-year rate, the variance of the observations on the 2-year rate, and so on) is

$$17.55^2 + 4.77^2 + 2.08^2 + \dots + 0.53^2 = 338.8$$

- From this it can be seen that the first factor accounts for $\frac{17.55^2}{338.8} = 90.9\%$ of the variance in the original data; the first two factors account for $\frac{17.55^2 + 4.77^2}{338.8} = 97.7\%$ of the variance in the data; the third factor accounts for a further 1.3% of the variance. This shows that most of the risk in interest rate moves is accounted for by the first two or three factors. It suggests that we can relate the risks in a portfolio of interest rate dependent instruments to movements in these factors instead of considering all eight interest rates.

Example

- To illustrate how a principal components analysis can be used to hedge, consider a portfolio with the exposures to interest rate moves shown in table below. A 1-basis-point change in the 3-year rate causes the portfolio value to increase by \$10 million, a 1-basispoint change in the 4-year rate causes it to increase by \$4 million, and so on.

Change in portfolio value for a 1-basis-point
rate move (\$ millions).

<i>3-year rate</i>	<i>4-year rate</i>	<i>5-year rate</i>	<i>7-year rate</i>	<i>10-year rate</i>
+10	+4	-8	-7	+2

Example

- Suppose the first two factors are used to model rate moves. (As mentioned above, this captures 97.7% of the variance in rate moves.) Using the data in 1st PCA Table, the exposure to the first factor (measured in millions of dollars per factor score basis point) is

$$10 \times 0.372 + 4 \times 0.392 - 8 \times 0.404 - 7 \times 0.394 + 2 \times 0.376 = +0.05$$
- and the exposure to the second factor is

$$10 \times (-0.267) + 4 \times (0.110) - 8 \times 0.019 - 7 \times 0.194 + 2 \times 0.371 = -3.88$$
- Suppose that f_1 and f_2 are the factor scores (measured in basis points). The change in the portfolio value is, to a good approximation, given by

$$\Delta P = +0.05f_1 - 3.88f_2$$
- The factor scores are uncorrelated and have the standard deviations given in 2nd PCA table. The standard deviation of ΔP is therefore

$$\sqrt{0.05^2 \times 17.55^2 + 3.88^2 \times 4.77^2} = 18.48$$

Example

- The delta was calculated for the pair of factor 1 & 2.
- The first delta then measures the impact of a small, approximately parallel, shift in the zero curve; the second delta measures the impact of a small twist in the zero curve; and so on.

6 Recall back - The Black Model (Swaption)

Special features

- The strike rate is usually chosen to be similar to current swap rates (which, in turn, are similar to current interest rates for the same term).

Pricing/ Valuation

- A European swaption can be valued using Black's formula. The underlying variable is the n-year forward swap rate, whose current value is F_0 .

Valuing a European Swaption

$$\text{Option on a pay-fixed swap: } V_{\text{swaption}} = L A [F_0 \Phi(d_1) - R_X \Phi(d_2)]$$

$$\text{Option on a pay-floating swap: } V_{\text{swaption}} = L A [R_X \Phi(-d_2)] - F_0 \Phi(-d_1)]$$

$$\text{Where } A = \frac{1}{m} \sum_{i=1}^{mn} P(0, t_i)$$

Here, A denotes the current value of an annuity of 1 unit per annum on each interest payment date and $P(0, t_i)$ is the market discount factor for a payment at time t_i , F_0 is the forward swap rate and R_X is the Strike price.

7 Interest Rate Swaption Hedging

Pricing European Swaption

- The cash flow made to the buyer of a payer swaption at time T amounts to

$$\sum_{i=1}^n N * e^{-(t_i-T)} * (i_F - i_S) * (t_i - t_{i-1}),$$

Where, $F_0 = i_F$ = forward swap rate and $R_X = i_S$ = Strike price

- if $i_F > i_S$ and 0 otherwise. It's value today therefore equals

$$e^{-iT} * \sum_{i=1}^n N * e^{-i(t_i-T)} * (i_F - i_S) * (t_i - t_{i-1}) = N * \sum_{i=1}^n e^{-it_i} * (i_F - i_S) * (t_i - t_{i-1})$$

- Now, the price of a European payer swaption is determined using the Black model. The i^{th} term:

$$N(t_i - t_{i-1}) * (i_F - i_S) +$$

- of the cash flow corresponds to the price of a European call option with expiry t_i . According to the Black model the price of this option at time 0 is

$$N * e^{-it_i} * (t_i - t_{i-1}) * [i_F \Phi(d_1) - i_S \Phi(d_2)],$$

- Where. $d_1 = (\ln \frac{i_F}{i_S} + \frac{1}{2} \sigma_F^2 T) / \sigma_F \sqrt{T}$, $d_2 = (\ln \frac{i_F}{i_S} - \frac{1}{2} \sigma_F^2 T) / (\sigma_F) * \sqrt{T} = d_1 - \sigma_F \sqrt{T}$

7 Interest Rate Swaption Hedging

Pricing European Swaption

- The forward swap rate i_F is computed. The value of the payer swaption P_{PS} itself is obtained by summing up all individual call options:

$$P_{PS} = NA[i_F\Phi(d_1) - i_S\Phi(d_2)],$$

- Whereby, $A = \sum_{i=1}^n e^{-it_i}(t_i - t_{i-1})$
- By analogy, in the case of a European receiver swaption, we obtain through,

$$P_{RS} = NA[i_S\Phi(-d_2) - i_F\Phi(-d_1)]$$

Note this formula is same as the one in the previous chapter with slight change in the notations.

7 Interest Rate Swaption Hedging

- Partial derivatives lay the practical foundation for the application of special trading and hedging techniques on options and derivative securities. They quantify the influence (risk) of changes in market factors on the option price. In this respect and in many instances, partial derivatives are as important as the theoretically determined price of the option, as they tell the user in a short and accurate manner which direction to go in the current investment (assets, liabilities): buy, sell or maintain.
- The most important risk parameters the ("Greeks") are presented with respect to swaptions. This is followed by some trading strategies in brief.
- We shall denote the price of a general derivative security with D and the price of its underlying security with B . Concretely, in the case of swaptions (under consideration here) this means: D stands for the price P_{PS} of the Payer swaption and B stands for the corresponding swap price i_F .

7 Interest Rate Swaption Hedging

Delta

- Given a specific yield curve (interest rate structure) plus swap as underlying, the price of a swaption depends on the expiry date T and the strike price S (see, e. g., [11]). Therefore the impact of a shift in the swap price (strike rate i_S kept constant) on the swaption price is dependent on the variables T and i_F .
- Generally, the parameter delta describes the rate of change of the price of the derivative security with respect to the asset (price of the underlying):

$$\Delta = \frac{\partial D}{\partial B}$$

- In the case under consideration, our underlying is the fixed forward swap rate i_F . By analogy with the delta of an equity call, the following holds for the delta Δ_{PS} of a European payer swaption

$$\Delta_{PS} = \frac{\partial P_{PS}}{\partial i_F} = NA\Phi(d_1),$$

- whereby $A = \sum_{i=1}^n e^{-it_i}(t_i - t_{i-1})$
- By analogy, we obtain for the delta of a European receiver swaption the following value

$$\Delta_{RS} = NA(\Phi(d_1) - 1)$$

7 Interest Rate Swaption Hedging

Gamma

- The gamma of a derivative product (e.g., swaption or portfolio thereof) is the second derivative of the price of the derivative with respect to the underlying:

$$\Gamma = \frac{\partial^2 D}{\partial B^2} = \frac{\partial \Delta}{\partial B}$$

- Like in the case of delta, a closed form formula can be derived for gamma through differentiation:

$$\Gamma_{PS} = \frac{NA\varphi(d_1)}{\sqrt{T}i_F\sigma_F}$$

- whereby $\varphi(d_1) = \Phi'(d_1)$ stands for the probability density function of the gaussian distribution.
- According to the definition, gamma is the sensitivity of the delta to the underlying security. Therefore, it measures how much and how often Gamma must be reheded in order to maintain a delta-neutral portfolio.

7 Interest Rate Swaption Hedging

Theta

- The theta of a portfolio of derivative products is the rate of change of the portfolio price with respect to time to maturity T (time left for option to expire):

$$\Theta = \frac{\partial D}{\partial T}$$

- In the special case of a payer swaption one obtains

$$\Theta = \frac{\partial P_{PS}}{\partial T} = -i_F P_{PS} + \frac{N A i_F \sigma_F \varphi(d_1)}{2\sqrt{T}}$$

7 Interest Rate Swaption Hedging

Vega

- Vega is the sensitivity of the derivative (or portfolio of financial derivative products) price to volatility σ

$$V = \frac{\partial D}{\partial \sigma}$$

- In the special case of swaptions one obtains the following equation through differentiation:

$$V_{PS} = V_{RS} = N A i_F \sqrt{T} \varphi(d_1)$$

- The implicit quantity σ_F is the estimate for the expected forward swap rate. A vega-hedged portfolio is protected against fluctuations of volatility.

7 Interest Rate Swaption Hedging

Delta-Hedging

- The so called delta-hedging is a dynamic hedging strategy. Here, it is sought, price changes of the swap to be compensated with price changes of the swaption. This is achieved by setting up a portfolio by holding (or shorting) the derivative (swaption) and shorting (or holding) a quantity Δ of the underlying (swap); this is referred to as hedge portfolio.
- In this way, within the portfolio, price increases of the swap are compensated by price drops of the swaption and vice-versa. Risks caused by fluctuations of the underlying security are practically eliminated. As can be verified, this portfolio has a delta of zero (let P_{port} be the price of the portfolio):

$$\Delta_{Port} = \frac{\partial P_{Port}}{\partial B} = \Delta * \frac{\partial B}{\partial B} - \frac{\partial D}{\partial B} = \Delta * 1 - \Delta = 0$$

7 Interest Rate Swaption Hedging

Delta-Hedging

- Therefore, by way of delta-hedging, one can eliminate (at least theoretically and to a great extent practically) the risk. The proportion of the underlying security in the portfolio must be continuously changed since the quantity Δ depends on both the price of the underlying and the remaining period to maturity of the swaption. This process is called dynamic hedging (or rebalancing) of the portfolio. Therefore (theoretically), one continuously has to buy and sell swaps. However, in the case of a discrete model, rebalancing of delta is done at discrete time intervals Δt .

7 Interest Rate Swaption Hedging

Delta-Gamma-Hedging

- A little value for gamma indicates that by definition, the rate of change of delta is little. This means rebalancing of the hedge-portfolio may be carried out in larger intervals of time. Conversely, larger gamma values are an indication that delta is very sensitive with respect to shifts in the underlying, resulting in the increase in risk inherent in a shift in portfolio value.
- Because of the cost of frequent hedging, it is natural to try to minimize the need to rebalance the portfolio too frequently. The corresponding hedging procedure is called a gamma-neutral strategy . To achieve this objective, we have to buy and sell more swaptions, not just the swap. By simple differentiation, you can check that a position in the underlying asset has zero gamma:

$$\frac{\partial^2 B}{\partial B^2} = 0, \text{ particularly, } \frac{\partial^2 i_F}{\partial i_F^2} = 0$$

7 Interest Rate Swaption Hedging

Delta-Gamma-Hedging

- Thus, we cannot change the gamma of our position by adding the underlying. However, we can add another swaption in quantity, which will make the portfolio gamma-neutral. By holding two different swaptions we can make the portfolio both delta- and gamma-neutral. Note that a delta-neutral portfolio $\Delta_{port} = 0$ has gamma equal to Γ and a traded swaption has gamma equal to Γ_0 . If the number of traded swaptions added to the portfolio is w_0 , the gamma of the portfolio is

$$\widetilde{\Gamma}_{port} = \Gamma + w_0 \Gamma_0$$

- Hence, the portfolio becomes gamma-neutral, if our position in the traded swaption is equal to $w_0 = -\Gamma/\Gamma_0$. Of course, as we add the traded swaption, the delta of the portfolio changes. So the position in the underlying (swap) then has to be changed to maintain delta-neutrality. Due to

$$\widetilde{\Delta}_{port} = 0 + w_0 \Delta_0 = 0 - \left(\frac{\Gamma}{\Gamma_0} \right) \Delta_0$$

- the quantity $\widetilde{\Delta}_{port}$ of the underlying (swap) has to be added for hedging.

7 Interest Rate Swaption Hedging

Practical Approach

- In practice, portfolio rebalancing to achieve delta-, gamma-, vega-neutrality etc. is not a continuous process. If it were, transaction costs would render it extremely expensive. Instead, the individual risks are analyzed to find out if they are worth taking or not. The aforementioned risk parameters play the role of quantifying various aspects of portfolio risk. If the risk is acceptable, no action is taken. Otherwise rebalancing is carried out as outlined above.

8 SABR Model

- The stochastic alpha beta rho model is a stochastic volatility model for forward prices commonly used in the modelling of interest rate derivatives. The alpha, beta and rho in the name are parameters to be calibrated. Alpha describes the magnitude of the volatility in the price of the underlying asset; beta describes the sensitivity of forward price movements to the spot price; and rho describes the correlation between movements in the forward price and movements in the volatility of the price of the underlying asset. The model first appeared in a 2002 paper by Patrick Hagan, Deep Kumar, Andrew Lesniewski and Diana Woodward.
- SABR Model has the distinctive advantage of being useful for IRD modelling even when the interest rates are negative.
- They are used for hedging widely in the industry.