Lecture

INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Class: TY BSc

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Chapter: Unit 3 Chapter 1

Chapter Name: Models for Term Structure of Interest Rates

1.1 Interest Rate Modelling

Introduction

- Interest rate modelling is the most important topic in derivative pricing. Interest rate derivatives account for around 80% of the value of derivative contracts outstanding, mainly swaps and credit derivatives used to support the securitization of debt portfolios.
- More philosophically, is the fact that at any one time there will be a multitude of contracts (bonds) written on the same underlying (an interest rate), and derivative pricing is principally concerned with pricing derivatives coherently.
- The multitude of traded instruments leads to the first challenge in interest rate modelling: the multitude of definitions of interest rates.

1.2 Zero Coupon Bonds

- The most basic interest rate contract is an agreement to pay some money now in exchange for a promise of receiving a (usually) larger sum later. In general, the worth of such a contract will depend on factors other than just the time value of money, such as the credibility of the promisor and the perceived legality of the promise.
- Matters such as creditworthiness and the like are not our concern here, and it is for the bond market, not the interest rate market, to price them. We are solely concerned with the time value of money for default-free borrowing.
- This basic contract only requires two numbers to describe it - its length, or maturity, which records when we are to receive the later payment, and the ratio of the size of that payment to our initial payment. We can call the maturity date on which we are paid T, and the fraction of the final payment which is the initial, $P(0,T)$. In other words, one dollar at time T can be bought at time zero for $P(0, T)$ dollars.

1.2 Zero Coupon Bonds

- The basic debt instrument is the discount bond (or, equivalently, the zero-coupon bond). This is an asset that will pay one unit of currency at time T and is traded at time $t < T$. If the interest rate, R, is constant between t and T then we can say that the price of the discount bond purchased at t and maturing at T is given by $P(t, T)$ where:

$$\boldsymbol{P(t,T) = \frac{1}{\left(1 + R(t,T)\right)^{T-t}}}$$

- Observe that $P(T, T) = 1$ and for all $t < T$, $P(t, T) < P(T, T) = 1$. We define $\tau := T - t$ in what follows.
- The discrete bond yield calculated from discount bond prices is:

$$\boldsymbol{R(t, t+\tau) = \frac{1}{P(t, t+\tau)^{\frac{1}{t}}} - 1}$$

- If a 'spot' rate is paid m times a year, then:

$$\boldsymbol{\frac{1}{P(0,n)} = \left(1 + \frac{R}{m}\right)^{nm}}$$

1.2 Zero Coupon Bonds

- The limit as $m \to \infty$ is a continuously compounded rate, *r(t, T)* ('force of interest'), such that:

$$e^{-r(t,T)\tau} = P(t,T) = \frac{1}{\left(1 + R(t,T)\right)^{t}}$$

- The continuously compounded bond yield is calculated as:

$$r(t,T) = \frac{lnP(t,T)}{T - t}$$

- The spot rate *r(t,T)* is the continuously compounded rate of interest applicable over the period from time *t* to time *T* that is implied by the market prices at time *t* .

2 Yield Curves

- Fixing $t = 0$ and plotting yield, $R(0,T)$ or $r\,(0,T)$, against maturity, T, gives the yield curve which gives information on the term structure, how interest rates for different maturities are related. Typically, the yield curve increases with maturity, reflecting uncertainty about far-future rates. However, if current rates are unusually high, the yield curve can be downward sloping, and is inverted.

2 Yield Curves

Theories

- There are various theories explaining the shape of the yield curve:

- The **expectations theory** argues that the long-term rate is determined purely by current and future expected short-term rates, so that the expected final value of investing in a sequence of short-term bonds equals the final value of wealth from investing in long-term bonds.
- The **market segmentation theory** argues that different agents in the market have different objectives: pension funds determine longer-term rates, market makers determine short-term rates, and businesses determine medium-term rates, which are all determined by the supply and demand of debt for these different market segments.
- The **liquidity preference theory** argues that lenders want to lend short term while borrowers wish to borrow long term, and so forward rates are higher than expected future zero rates (and yield curves are upward sloping).

- Note : *Zero rate* is another name for the spot rate.

3 Forward Rates and Short Rates

Short Rate

- The short or instantaneous rate, *r(t)* , is the interest rate charged today for a very short period (ie overnight). This is defined (equivalently) as:

$$r(t) = r(t, t+\delta) \approx R(t, t+\delta)$$

- where δ is a small positive quantity. So the short rate *r(t)* is the force of interest that applies in the market at time *t* for an infinitesimally small period of time δ . Using the relationship developed in the opening section we have:

$$r(t) = -\frac{\partial}{\partial \delta} lnP(t, t+\delta)$$

- The short rate is often the basis of some interest rate models; however, it will not generate, on its own, discount bond prices.

3 Forward Rates and Short Rates

Forward Rate

- The forward rate $-F(0; t, T)$ if discretely compounded and $f(0; t, T)$ if continuously compounded – relates to a loan starting at time $t > 0$, for the fixed forward rate repaid at maturity, $T > t > 0$. It involves three times, the time at which the forward rate agreement is entered into (typically 0), the start time of the forward rate, t and the maturity of the forward rate agreement, T. The law of one-price/the no-arbitrage principle, implies:

$$F(0; t, T) = \left(\frac{P(0, t)}{P(0, T)}\right)^{\frac{1}{T-t}} - 1$$

- or, for continuously compounded forward rates:

$$f(0; t, T) = \frac{r(0, T)T - r(0, t)t}{T - t} = r(0, t) + \frac{r(0, T) - r(0, t)}{T - t}T$$

- In the limit $t \to T$, we get the instantaneous forward rate ~ (not there are two, rather than three, parameters because $(0; T, T + \delta)$ is abbreviated to $(0,T)$):

$$f(0, T) = r(0, T) + T\frac{(\partial r(0, T))}{\partial T} = -\frac{\partial}{\partial T}\ln P(0, T)$$

- Given this definition, the fundamental theorem of calculus tells us that:

$$P(t, T) = \exp\{-\int_t^T f(s, u)du\}$$

4 Types of Models

- Interest Rate Models can be classified as Equilibrium Models and No-Arbitrage Models:

- **Equilibrium models** start with a theory about the economy, such that interest rates revert to some long-run average, are positive or their volatility is constant or geometric. Based on the model for (typically) the short rate, the implications for the pricing of assets is worked out. Examples of equilibrium models are Rendleman and Bartter, Vasicek and Cox- Ingersoll-Ross.
- Being based on 'economic fundamentals', equilibrium models rarely reproduce observed term structures. This is unsatisfactory.

- **No-arbitrage models** use the term structure as an input and are formulated to adhere to the no-arbitrage principle. An example of a no-arbitrage model is the Hull-White (one- and two factor).

5 Desirable features of a term-structure model

Some of the desirable features are:

- **The model should be arbitrage free :**
  - In very limited circumstances this is not essential, but in the majority of modern actuarial applications, it is essential. Most obviously, anything involving dynamic hedging would immediately identify and exploit any arbitrage opportunities.
  - The markets for government bonds and interest rate derivatives are generally assumed to be pretty much arbitrage-free in practice.

- **Interest rates should ideally be positive :**
  - Banks have to offer investors a positive return to prevent them from withdrawing paper cash and putting it ‘under the bed’. This might be impractical for a large life office or pension fund but, nevertheless, it typically holds in practice.
  - Some term structure models do allow interest rates to go negative. One such example is the Vasicek model we will see further on.
  - Whether or not this is a problem depends on the probability of negative interest rates within the timescale of the problem in hand and their likely magnitude if they can go negative. It also depends on the economy being modelled, as negative interest rates have been seen in some countries.

5 Desirable features of a term-structure model

- ***r(t)* and other interest rates should exhibit some form of mean-reverting behaviour :**
  - Again this is because the empirical evidence suggests that interest rates do tend to mean revert in practice.
  - This might not be particularly strong mean reversion but it is essential for many actuarial applications where the time horizon of a problem might be very long.

- **How easy is it to calculate the prices of bonds and certain derivative contracts?**
  - This is a computational issue. It is no good in a modelling exercise to have a wonderful model if it is impossible to perform pricing or hedging calculations within a reasonable amount of time.
  - This is because we need to act quickly to identify any potential arbitrage opportunities or to rebalance a hedged position.
  - Thus we aim for models that either give rise to simple formulae for bond and option prices, or that make it straightforward to compute prices using numerical techniques.

- **Does the model produce realistic dynamics?**
  - For example, can it reproduce features that are similar to what we have seen in the past with reasonable probability?
  - Does it give rise to a full range of plausible yield curves, ie upward-sloping, downward-sloping and humped?

5 Desirable features of a term-structure model

- **Does the model, with appropriate parameter estimates, fit historical interest rate data adequately?**

- **Can the model be calibrated easily to current market data?**
  - If so, is this calibration perfect or just a good approximation? This is an important point when we are attempting to establish the fair value of liabilities. If the model cannot fit observed yield curves accurately then it has no chance of providing us with a reliable fair value for a set of liabilities.

- **Is the model flexible enough to cope properly with a range of derivative contracts?**

6 Risk-neutral approach to pricing

- We will assume that the short rate is driven by an Ito diffusion:

$$\boldsymbol{dr_t = \mu(t, r_t)dt + \sigma(t, r_t)d\widetilde{W}_t}$$

- where: $\mu(t, r_t)$ is the drift parameter; $\sigma(t, r_t)$ is the volatility parameter and $\widetilde{W}_t$ is a Wiener process under the martingale measure
- If we are to have a model that is arbitrage-free then we need to consider the prices of tradeable assets, with the most natural of these being the zero-coupon bond prices $P(t, T)$.
- Modelling the short rate $r(t)$ does not tell us directly about the prices of the assets traded in the market. To see whether arbitrage opportunities exist or not, we need to examine these prices.
- We can use an argument similar to the derivation of the Black-Scholes model using the martingale approach to demonstrate that:

$$\boldsymbol{P(t, T) = E_Q[\exp\left(-\int_t^T r_u du\right)|r_t]}$$

- where Q is called the risk-neutral measure.

6 Risk-neutral approach to pricing

- Risk neutral measures give investors a mathematical interpretation of the overall market’s risk averseness to a particular asset, which must be taken into account in order to estimate the correct price for that asset.

- A risk neutral measure is also known as an equilibrium measure or equivalent martingale measure.

7 Vasicek Model (1977)

Vasicek assumes that:

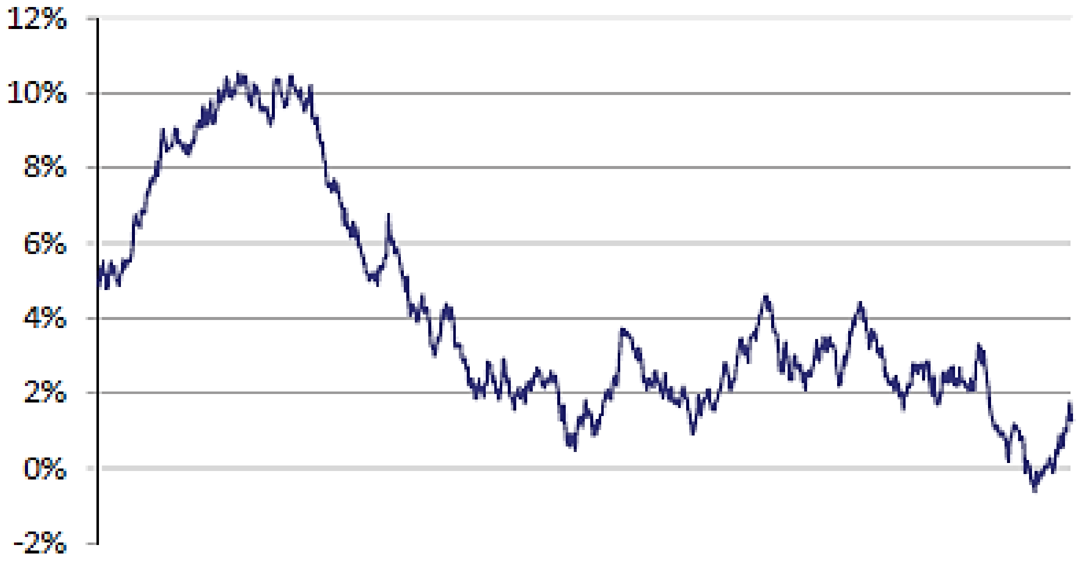
$$dr_t = \alpha\,(\mu - r_t)dt + \sigma\, d\,\widetilde{W}_t$$

for constants $\alpha > 0$, μ and, σ .

Here μ represents the ‘mean’ level of the short rate. If the short rate grows (driven by the stochastic term) the drift becomes negative, pulling the rate back to μ. The speed of the ‘reversion’ is determined by α . If α is high, the reversion will be very quick.

7 Vasicek Model

The graph below show a simulation of this process based on the parameter values $\alpha = 0.1$, $\mu = 0.06$ and $\sigma = 0.02$.



Example simulation of short rate from the Vasicek model

8 The Cox-Ingersoll-Ross (CIR) model (1985)

In Vasicek's model (and Hull-White, below) interest rates are not strictly positive. This assumption is not ideal for a short-rate model. CIR use the Feller, or square root mean reverting process which is positive (it can instantaneously touch 0 but immediately rebounds):

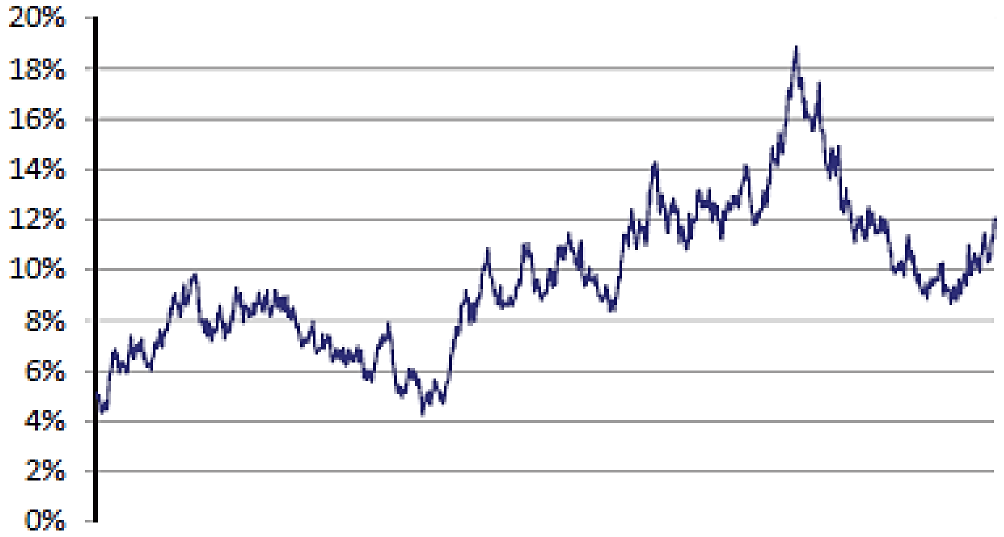
$$dr_t = \alpha\,(\mu - r_t)dt + \sigma\,\sqrt{r_t}\,d\widetilde{W}_t$$

for constants $\alpha > 0$, $\mu > 0$ and, σ .

The volatility coefficient in this model is not constant, but varies (randomly) with the current value of the short rate. As the short rate decreases, so does the volatility, which makes it "harder" for the process to reach zero. The square root is the critical power that "just" prevents the process going negative.

8 The Cox-Ingersoll-Ross (CIR) model (1985)

The graph below shows a simulation of this process based on the parameter values $\alpha = 0.1$, $\mu = 0.06$ and $\sigma = 0.1$



Simulation from Cox-Ingersoll-Ross model

9 The Hull-White model (1990)

The Hull-White model is an extension of Vasicek where the mean-reversion level, ▪ , is a deterministic function of time:

$$dr_t = \alpha\,(\mu(t) - r_t)dt + \sigma\, d\widetilde{W}_t$$

for constants $\alpha > 0$ and σ.

The key feature of this model is that the mean reversion level $\mu(t)$ is not assumed to be constant. Instead it is a deterministic function that is chosen so that the model exactly reproduces the current yield curve (and hence all current bond prices).

This model is also called the extended Vasicek model.

10 Comparison of Short rate Models

Vasicek

+ Simplest model

+ Derivative pricing formulae are based on a normal distribution

- Allows r(t) to take negative values.

- Results don't match with current yield curve exactly.

CIR

+ Prevents r(t) from taking negative values.

- Derivative pricing formulae are based on non-central chi-square distribution.

- In preventing negative values, it tends to impose lower limit on r(t).

- Results don't match with current yield curve exactly.

Hull-White

+ Matches observed yield curve, so better for pricing derivatives

- Allows r(t) to take negative values.

- More complicated because it is time – inhomogeneous.

11 Limitations of One-factor Models

- First, if we look at historical interest rate data we can see that changes in the prices of bonds with different terms to maturity are not perfectly correlated as one would expect to see if a one-factor model was correct. Recent research has suggested that around three factors, rather than one, are required to capture most of the randomness in bonds of different durations.

- Second, if we look at the long run of historical data we find that there have been sustained periods of both high and low interest rates with periods of both high and low volatility.

- Again, these are features which are difficult to capture without introducing more random factors into a model.

- This issue is especially important for two types of problem in insurance: the pricing and hedging of long-dated insurance contracts with interest rate guarantees; and asset-liability modelling and long-term risk management.

- One-factor models do, nevertheless have their place as tools for the valuation of simple liabilities with no option characteristics; or short-term, straightforward derivatives contracts.
- For other problems it is appropriate to make use of models which have more than one source of randomness: so-called multi-factor models.

Summary

The following table summarises the characteristics of the Vasicek, Cox-Ingersoll-Ross (CIR) and Hull-White models.

	Vasicek	**Cox-Ingersoll-Ross (CIR)**	**Hull-White**
Arbitrage-free	Yes	Yes	Yes
Positive interest rates	No	Yes	No
Mean-reverting interest rates	Yes	Yes	Yes
Easy to price bonds and derivatives	Yes	Yes	Yes
Realistic dynamics	No	No	No
Adequate fit to historical data	No	No	Yes
Easy to calibrate to current data	No	No	Yes
Can price a wide range of derivatives	No	No	No