INSTITUTE OF ACTUARIAL
& QUANTITATIVE STUDIES

Class: TY BSc

Subject : Financial Engineering 2

Chapter: Unit 3 Chapter 1 (part 2)

Chapter Name: Models for Term Structure of Interest Rates

6 Risk-neutral approach to pricing

- We will assume that the short rate is driven by an Ito diffusion:

$$\boldsymbol{dr_t = \mu(t, r_t)dt + \sigma(t, r_t)d\widetilde{W}_t}$$

- where: $\mu(t, r_t)$ is the drift parameter; $\sigma(t, r_t)$ is the volatility parameter and $\widetilde{W}_t$ is a Wiener process under the martingale measure
- If we are to have a model that is arbitrage-free then we need to consider the prices of tradeable assets, with the most natural of these being the zero-coupon bond prices $P(t, T)$.
- Modelling the short rate $r(t)$ does not tell us directly about the prices of the assets traded in the market. To see whether arbitrage opportunities exist or not, we need to examine these prices.
- We can use an argument similar to the derivation of the Black-Scholes model using the martingale approach to demonstrate that:

$$\boldsymbol{P(t, T) = E_Q[\exp\left(-\int_t^T r_u du\right)|r_t]}$$

- where Q is called the risk-neutral measure.

6 Risk-neutral approach to pricing

- Risk neutral measures give investors a mathematical interpretation of the overall market's risk averseness to a particular asset, which must be taken into account in order to estimate the correct price for that asset.

- A risk neutral measure is also known as an equilibrium measure or equivalent martingale measure.

7 Vasicek Model (1977)

Vasicek assumes that:

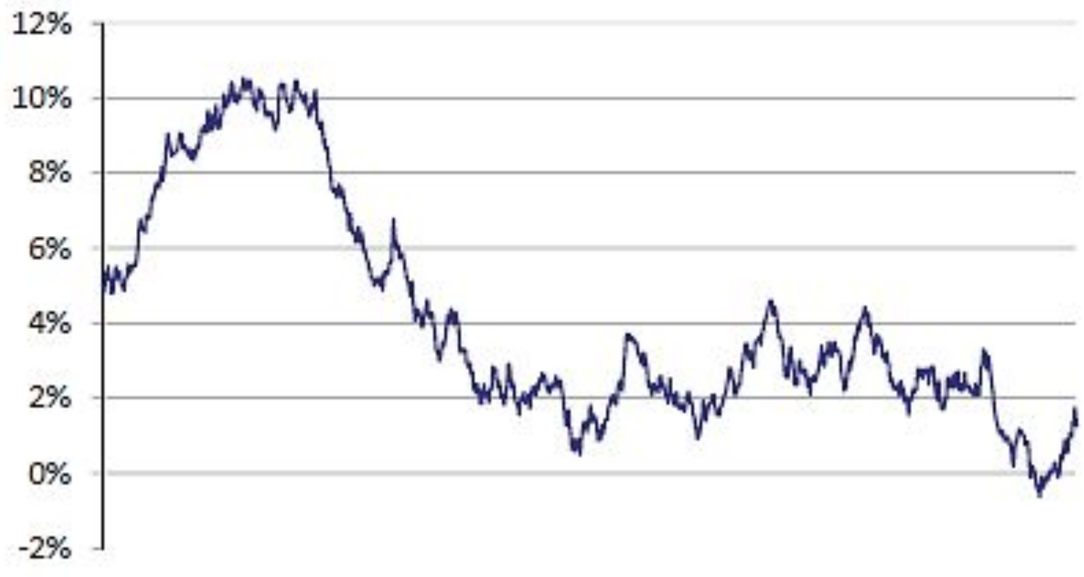
$$dr_t = \alpha\,(\mu - r_t)dt + \sigma\, d\widetilde{W}_t$$

for constants $\alpha > 0$, μ and, σ .

Here μ represents the 'mean' level of the short rate. If the short rate grows (driven by the stochastic term) the drift becomes negative, pulling the rate back to μ. The speed of the 'reversion' is determined by α . If α is high, the reversion will be very quick.

7 Vasicek Model

The graph below show a simulation of this process based on the parameter values $\alpha = 0.1$, $\mu = 0.06$ and $\sigma = 0.02$.



Example simulation of short rate from the Vasicek model

8 The Cox-Ingersoll-Ross (CIR) model (1985)

In Vasicek's model (and Hull-White, below) interest rates are not strictly positive. This assumption is not ideal for a short-rate model. CIR use the Feller, or square root mean reverting process which is positive (it can instantaneously touch 0 but immediately rebounds):

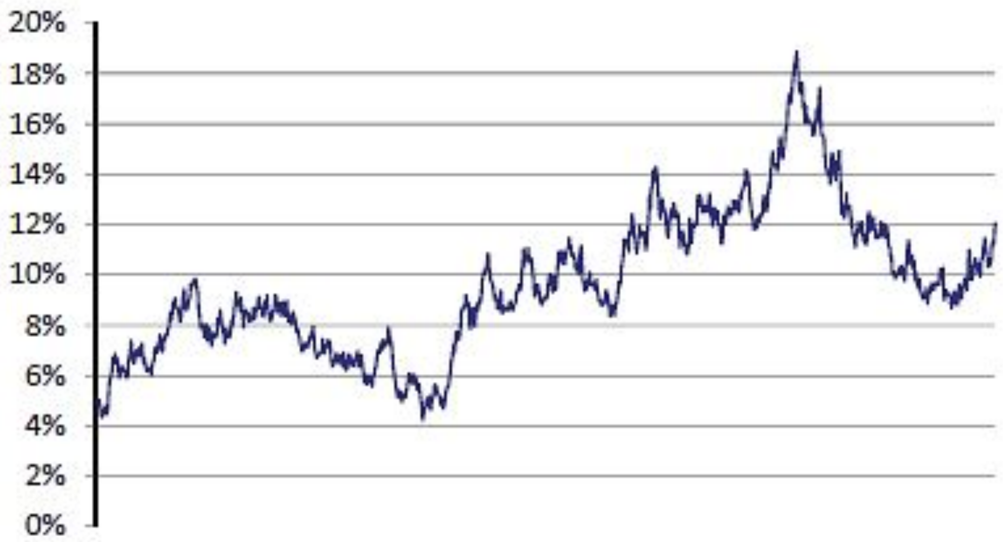
$$dr_t = \alpha (\mu - r_t) dt + \sigma \sqrt{r_t}\, d\widetilde{W}_t$$

for constants $\alpha > 0$, $\mu > 0$ and, σ .

The volatility coefficient in this model is not constant, but varies (randomly) with the current value of the short rate. As the short rate decreases, so does the volatility, which makes it "harder" for the process to reach zero. The square root is the critical power that "just" prevents the process going negative.

8 The Cox-Ingersoll-Ross (CIR) model (1985)

The graph below shows a simulation of this process based on the parameter values $\alpha = 0.1$, $\mu = 0.06$ and $\sigma = 0.1$



Simulation from Cox-Ingersoll-Ross model

9 The Hull-White model (1990)

The Hull-White model is an extension of Vasicek where the mean-reversion level, □ , is a deterministic function of time:

$$dr_t = \alpha\,(\mu(t) - r_t)\,dt + \sigma\, d\widetilde{W}_t$$

for constants $\alpha > 0$ and σ.

The key feature of this model is that the mean reversion level $\mu(t)$ is not assumed to be constant. Instead it is a deterministic function that is chosen so that the model exactly reproduces the current yield curve (and hence all current bond prices).

This model is also called the extended Vasicek model.

10 Comparison of Short rate Models

Vasicek

+ Simplest model
+ Derivative pricing formulae are based on a normal distribution

- Allows r(t) to take negative values.
– Results don't match with current yield curve exactly.

CIR

+ Prevents r(t) from taking negative values.

- Derivative pricing formulae are based on non-central chi-square distribution.
– In preventing negative values, it tends to impose lower limit on r(t).
– Results don't match with current yield curve exactly.

Hull-White

+ Matches observed yield curve, so better for pricing derivatives

– Allows r(t) to take negative values.
– More complicated because it is time – inhomogeneous.

11 Limitations of One-factor Models

- First, if we look at historical interest rate data we can see that changes in the prices of bonds with different terms to maturity are not perfectly correlated as one would expect to see if a one-factor model was correct. Recent research has suggested that around three factors, rather than one, are required to capture most of the randomness in bonds of different durations.

- Second, if we look at the long run of historical data we find that there have been sustained periods of both high and low interest rates with periods of both high and low volatility.

- Again, these are features which are difficult to capture without introducing more random factors into a model.

- This issue is especially important for two types of problem in insurance: the pricing and hedging of long-dated insurance contracts with interest rate guarantees; and asset-liability modelling and long-term risk management.

- One-factor models do, nevertheless have their place as tools for the valuation of simple liabilities with no option characteristics; or short-term, straightforward derivatives contracts.
- For other problems it is appropriate to make use of models which have more than one source of randomness: so-called multi-factor models.

Summary

The following table summarises the characteristics of the Vasicek, Cox-Ingersoll-Ross (CIR) and Hull-White models.

	Vasicek	Cox-Ingersoll-Ross (CIR)	Hull-White
Arbitrage-free	Yes	Yes	Yes
Positive interest rates	No	Yes	No
Mean-reverting interest rates	Yes	Yes	Yes
Easy to price bonds and derivatives	Yes	Yes	Yes
Realistic dynamics	No	No	No
Adequate fit to historical data	No	No	Yes
Easy to calibrate to current data	No	No	Yes
Can price a wide range of derivatives	No	No	No