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Subject: Financial Engineering II

Chapter: Unit 3

Category: Practice Questions

1.

Main issue: possibility to have negative interest rates when using the Vasicek model. An alternative is the CIR model:

$$dr(t) = a\left(b - r(t)\right)dt + \sigma\sqrt{r(t)}dW_t \,.$$

2.

One-factor models
All are arbitrage-free.
Vasicek: easy to implement but problem of possible negative interest rates

CIR: more tricky to implement but positive rates (for suitable choice of parameter values).

HW: more flexible as time-inhomogeneous, so better fit to market data (in particular option prices)., but negative rates are possible

Limitations:

1) historical data shows changes in the prices of bonds with different terms to maturity are not perfectly correlated

2) there have been sustained periods of both high and low interest rates with periods of both high and low volatility

3) we need more complex models to deal effectively with more complex derivative contracts e.g. any contract which makes reference to more than one interest rate should allow these rates to be less than perfectly correlated

Multiple-factor models: to capture more features of market data, better for pricing exotic derivatives.

There is no perfect model. A good model depends on the data available and the use of the model (basic assets, plain vanilla derivatives, more exotic derivatives, short or long maturities…).

Fit to historical data; realistic dynamics

3.

(i) Arbitrage free.
Positive interest rates.
Mean reversion of rates.
Ease of calculation of bonds and certain derivative contracts.
Realistic dynamics.
Goodness of fit to historical data.
Ease of calibration to current market data.
Flexible enough to cope with a range of derivative contracts.

(ii) $dr_t = \alpha(\mu_t - r_t)dt + \sigma dZ_t$ or alternatively $dr = \left[\theta(t) - ar\right]dt + \sigma dz$
Where in both cases Z is a Brownian motion under $\mathbb{Q}$.

(iii) Arbitrage free. Yes
Positive interest rates. No
Mean reversion of rates. Yes
Ease of calculation of bonds and certain derivative contracts. Yes
Realistic dynamics. No
Goodness of fit to historical data. Yes.
Ease of calibration to current market data. Yes
Flexible enough to cope with a range of derivative contracts. No.

4.

(i)

t	$F(t-1,t)$	$B(0,t)$	$R(0,t)$	$C(t)$
0	-	-	-	£100.00
1	2%	£98.02	2.0%	£102.02
2	4%	£94.18	3.0%	£106.18
3	3%	£91.39	3.0%	£109.42
4	5%	£86.94	3.5%	£115.03

i.e. (a) = 5%
(b) = £98.02
(c) = 3.0%
(d) = £109.42

(ii)

t	$F(t-1,t)$	$B(0,t)$
0	-	-
1	-	-
2	5%	95.12
3	4%	91.39
4	6%	86.07

The investor bought 10 bonds maturing at $t = 2$ and 20 bonds maturing at $t = 4$, at a total cost of 10 * 94.18 + 20 * 86.94 = £2,680.60.

The bonds are now worth 10 * 95.12 + 20 * 86.07 = £2,672.60.
Profit is 2,672.60 – 2,680.60 = –£8.00 (i.e. a loss of £8.00).

(iii) Any portfolio consisting only of risk-free assets will return the risk-free rate of interest if rates remain unchanged.

The investor would therefore need to invest in other risky assets, or assets linked to another interest rate, in order to recoup her loss.

5.

(i) The model should be arbitrage-free. [½]

Interest rates should ideally be positive. [½]

Interest rates should exhibit some element of mean reversion. [½]

The model should be computationally tractable / produce simple formulae for bond and option prices. [½]

It should produce realistic dynamics. [½]

It should give a full range of possible yield curves. [½]

It should fit historical data. [½]

Can be calibrated easily to current market data. [½]

Flexible to cope with a range of derivatives. [½]
[Max 3]

(ii) Both strategies pay a value of 1 at time T [1]

By the no arbitrage principle… [1]

… if they have the same value at time T then they must have the same value at time t [1]
Hence $B(t,T) = \alpha\, B(t,S)$ [1]

and so the requested relationship follows, with $\alpha = e^{-F(t,S,T)(T-S)}$. [½]
[Max 4]

(iii) From the expression in part (ii), it follows that:

$$F(t,S,T) = -\frac{\log B(t,T) - \log B(t,S)}{T-S}.$$ [1]

The instantaneous forward rate is defined as $f(t,T) = \lim_{S \to T} F(t,S,T)$,

i.e.

$$f(t,T) = -\lim_{S \to T} \frac{\log B(t,T) - \log B(t,S)}{T-S} = -\frac{\partial \log B(t,T)}{\partial T}.$$

Solving the last equality with respect to the ZCB price, we obtain:

$$B(t,T) = e^{-\int_t^T f(t,s)ds}$$

6.

(i) It allows negative interest rates. [1]

(ii) The extent of the problem depends on the probability of negative interest rates… [½]

… within the timescale of the problem in hand (or, for example, less of an issue if the time horizon is short)… [1]

… and their likely magnitude if they can go negative. [1]

It also depends on the economy being modelled, as negative interest rates have been seen in some countries. [1]

[Max 3]

(iii) The CIR model does not allow interest rates to go negative. [1]

This is because the volatility under the CIR increases in line with the square root of $r(t)$. [1]

Since this reduces to zero as $r(t)$ approaches zero… [½]

… and provided the volatility parameter is not too large… [½]

… $r(t)$ will never actually reach zero. [½]

… provided $\sigma^2 \leq 2\alpha\mu$

[Max 3]

(iv) Letting $t \to \infty$, we note that the mean converges to b. [1]

Hence interest rates under the model are mean reverting [½]

To the long-run mean b [½]

(v) Letting $t \to \infty$, we note that the variance converges to $\sigma^2 / 2a$ [1]

Hence, the variance of the short rate is inversely proportional to a [½]

This implies that the convergence of the rate to the long run mean b is faster the bigger a [1]

So a controls the speed of the mean convergence [1]

7.

i) Using continuous compounding.
[Note to markers: please accept any correct attempt using different compounding convention]

a. $-\frac{1}{0.25} ln \frac{97.5}{100} = 10.13\%$ [1]

b. $-\frac{1}{0.5} ln \frac{94.9}{100} = 10.47\%$ [1]

c. $0.04 \times (94.9 + 90) + 104 \times e^{-0.1068 \times 1.5} = 96$ [1]

d. $\frac{(0.1054 \times 1 - 0.1047 \times 0.5)}{1-0.5} = 10.60\%$ [1]

e. $\frac{(0.1068 \times 1.5 - 0.1054 \times 1)}{1.5-1} = 10.97\%$ [1]

8.

The floating payments can be valued in currency A by (i) assuming that the forward rates are realized, and (ii) discounting the resulting cash flows at appropriate currency A discount rates. Suppose that the value is V_A. The fixed payments can be valued in currency B by discounting them at the appropriate currency B discount rates. Suppose that the value is V_B. If Q is the current exchange rate (number of units of currency A per unit of currency B), the value of the swap in currency A is $V_A - QV_B$. Alternatively, it is $V_A/Q - V_B$ in currency B.

9.

Company A has a comparative advantage in the Canadian dollar fixed-rate market. Company B has a comparative advantage in the U.S. dollar floating-rate market. (This may be because of their tax positions.) However, company A wants to borrow in the U.S. dollar floating-rate market and company B wants to borrow in the Canadian dollar fixed-rate market. This gives rise to the swap opportunity.

The differential between the U.S. dollar floating rates is 0.5% per annum, and the differential between the Canadian dollar fixed rates is 1.5% per annum. The difference between the differentials is 1% per annum. The total potential gain to all parties from the swap is therefore 1% per annum, or 100 basis points. If the financial intermediary requires 50 basis points, each of A and B can be made 25 basis points better off. Thus a swap can be designed so that it provides A with U.S. dollars at LIBOR + 0.25% per annum, and B with Canadian dollars at 6.25% per annum. The swap is shown in Figure S7.4.

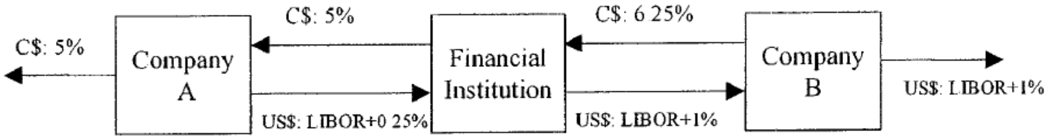

Figure S7.4 Swap for Problem 7.11

Principal payments flow in the opposite direction to the arrows at the start of the life of the swap and in the same direction as the arrows at the end of the life of the swap. The financial institution would be exposed to some foreign exchange risk which could be hedged using forward contracts.

10.

The swap involves exchanging the sterling interest of $20 \times 0.10 = 2.0$ million for the dollar interest of $30 \times 0.06 = \$1.8$ million. The principal amounts are also exchanged at the end of the life of the swap. The value of the sterling bond underlying the swap is

$$\frac{2}{(1.07)^{1/4}} + \frac{22}{(1.07)^{5/4}} = 22.182 \text{ million pounds}$$

The value of the dollar bond underlying the swap is

$$\frac{1.8}{(1.04)^{1/4}} + \frac{31.8}{(1.04)^{5/4}} = \$32.061 \text{ million}$$

The value of the swap to the party paying sterling is therefore

$$32.061 - (22.182 \times 1.85) = -\$8.976 \text{ million}$$

The value of the swap to the party paying dollars is $+\$8.976$ million. The results can also be obtained by viewing the swap as a portfolio of forward contracts. The continuously compounded interest rates in sterling and dollars are 6.766% per annum and 3.922% per annum. The 3-month and 15-month forward exchange rates are $1.85e^{(0.03922-0.06766)\times 0.25} = 1.8369$ and $1.85e^{(0.03922-0.06766)\times 1.25} = 1.7854$. The values of the two forward contracts corresponding to the exchange of interest for the party paying sterling are therefore

$$(1.8 - 2 \times 1.8369)e^{-0.03922\times 0.25} = -\$1.855 \text{ million}$$

$$(1.8 - 2 \times 1.7854)e^{-0.03922\times 1.25} = -\$1.686 \text{ million}$$

The value of the forward contract corresponding to the exchange of principals is

$$(30 - 20 \times 1.7854)e^{-0.03922 \times 1.25} = -\$5.435 \text{ million}$$

The total value of the swap is $-\$1.855 - \$1.686 - \$5.435 = -\$8.976$ million.

11.

The rate is not truly fixed because, if the company's credit rating declines, it will not be able to roll over its floating rate borrowings at LIBOR plus 150 basis points. The effective fixed borrowing rate then increases. Suppose for example that the treasurer's spread over LIBOR increases from 150 basis points to 200 basis points. The borrowing rate increases from 5.2% to 5.7%.