INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Subject: Fixed Income products

Chapter: Unit 3 & 4

Category: Assignment 2 solutions

Answer 1:

The yield-to-maturity per semiannual period is 0.0257 (= 0.0514/2). The coupon payment per period is 1.875 (= 3.75/2). At the beginning of the period, there are 27 years (54 semiannual periods) to maturity. The fraction of the period that has passed is 61/184. Th e full price at that yield-to-maturity is 80.501507 per 100 of par value.

$$PV_0 = \left[\frac{1.875}{(1.0257)^1} + \frac{1.875}{(1.0257)^2} + \cdots + \frac{101.875}{(1.0257)^{54}}\right] \times (1.0257)^{61/184} = 80.501507$$

Raise the yield-to-maturity from 5.14% to 5.19%—therefore, from 2.57% to 2.595% per semiannual period—and the price becomes 79.886293 per 100 of par value.

$$PV_+ = \left[\frac{1.875}{(1.02595)^1} + \frac{1.875}{(1.02595)^2} + \cdots + \frac{101.875}{(1.02595)^{54}}\right] \times (1.02595)^{61/184}$$
$$= 79.886293$$

Lower the yield-to-maturity from 5.14% to 5.09%—therefore, from 2.57% to 2.545% per semiannual period—and the price becomes 81.123441 per 100 of par value.

$$PV_- = \left[\frac{1.875}{(1.02545)^1} + \frac{1.875}{(1.02545)^2} + \cdots + \frac{101.875}{(1.02545)^{54}}\right] \times (1.02545)^{61/184}$$
$$= 81.123441$$

The approximate annualized modified duration for the Treasury bond is 15.368.

$$\text{ApproxModDur} = \frac{81.123441 - 79.886293}{2 \times 0.0005 \times 80.501507} = 15.368$$

The approximate annualized Macaulay duration is 15.763.

ApproxMacDur = 15.368 × 1.0257 = 15.763

Therefore, from these statistics, the investor knows that the weighted average time to receipt of interest and principal payments is 15.763 years (the Macaulay duration) and that the estimated loss in the bond's market value is 15.368% (the modified duration) if the market discount rate were to suddenly go up by 1% from 5.14% to 6.14%.

Answer 2:

PV_0 = 926.1, PV_+ = 871.8, PV_- = 973.5, and Δ Curve = 0.0100. The effective duration of the pension scheme's liabilities is 5.49.

$$\text{EffDur} = \frac{973.5 - 871.8}{2 \times 0.0100 \times 926.1} = 5.49$$

This effective duration statistic for the pension scheme's liabilities might be used in asset allocation decisions to decide the mix of equity, fixed income, and alternative assets.

Answer 3:

1. Bond A:

$$PV_0 = 58.075279$$

$$PV_+ = 58.047598$$

$$\frac{10}{(1.2001)^1}+\frac{10}{(1.2001)^2}+\cdots+\frac{110}{(1.2001)^{10}} = 58.047598$$

$$PV_- = 58.102981$$

$$\frac{10}{(1.1999)^1}+\frac{10}{(1.1999)^2}+\cdots+\frac{110}{(1.1999)^{10}} = 58.102981$$

The approximate modified duration of Bond A is 4.768.

$$\text{ApproxModDur} = \frac{58.102981-58.047598}{2\times 0.0001\times 58.075279} = 4.768$$

Bond B:

PV_0= 51.304203 PV_+= 51.277694

$$\frac{10}{(1.2001)^1}+\frac{10}{(1.2001)^2}+\cdots+\frac{110}{(1.2001)^{20}} = 51.277694$$

$$PV_- = 51.330737$$

$$\frac{10}{(1.1999)^1}+\frac{10}{(1.1999)^2}+\cdots+\frac{110}{(1.1999)^{20}} = 51.330737$$

The approximate modified duration of Bond B is 5.169.

$$\text{ApproxModDur} = \frac{51.330737-51.277694}{2\times 0.0001\times 51.304203} = 5.169$$

Bond C:

$$PV_0 = 50.210636$$

$$PV_+ = 50.185228$$

$$\frac{10}{(1.2001)^1}+\frac{10}{(1.2001)^2}+\cdots+\frac{110}{(1.2001)^{30}} = 50.185228$$

$$PV_- = 50.236070$$

$$\frac{10}{(1.1999)^1}+\frac{10}{(1.1999)^2}+\cdots+\frac{110}{(1.1999)^{30}} = 50.236070$$

The approximate modified duration of Bond C is 5.063.

$$\text{ApproxModDur} = \frac{50.236070-50.185228}{2\times 0.0001\times 50.210636} = 5.063$$

2. Despite the significant differences in times-to-maturity (10, 20, and 30 years), the approximate modified durations on the three bonds are fairly similar (4.768, 5.169, and 5.063). Because the yields-to-maturity are so high, the additional time to receipt of interest and principal payments on the 20- and 30-year bonds have low weight. Nevertheless, Bond B, with 20 years to maturity, has the highest modified duration. If the yield-to-maturity on each is decreased by the same amount—for instance, by 10 bps, from 20% to 19.90%—Bond B would be expected to have the highest percentage price increase because it has the highest modified duration. This example illustrates the relationship between the Macaulay duration and the time-to-maturity on discount bonds in Exhibit 7 . The 20-year bond has a higher duration than the 30-year bond.

Answer 4:

1. The average (annual) modified duration for the portfolio is 6.0495.

$$\left(\frac{4.761}{1+\frac{0.0910}{2}}\times\frac{24{,}886{,}343}{96{,}437{,}017}\right)+\left(\frac{5.633}{1+\frac{0.0938}{2}}\times\frac{27{,}243{,}887}{96{,}437{,}017}\right)+$$
$$\left(\frac{7.652}{1+\frac{0.0962}{2}}\times\frac{44{,}306{,}787}{96{,}437{,}017}\right)=6.0495$$

Note that the annual modified duration for each bond is the annual Macaulay duration, which is given, divided by one plus the yield-to-maturity per semiannual period.

2. The estimated decline in market value if each yield rises by 20 bps is 1.21%: −6.0495 × 0.0020 = −0.0121.

Answer 5:

There are 15 years from the beginning of the current period on 4 April 2014 to maturity on 4 April 2029.

1) The full price of the bond is 99.956780 per 100 of par value.

$$PV_0=\left[\frac{7.25}{(1.0744)^1}+\cdots+\frac{107.25}{(1.0744)^{15}}\right]\times(1.0744)^{83/360}=99.956780$$

2) PV_+ = 99.869964, and PV_- = 100.043703.

$$PV_+=\left[\frac{7.25}{(1.0745)^1}+\cdots+\frac{107.25}{(1.0745)^{15}}\right]\times(1.0745)^{83/360}=99.869964$$

$$PV_-=\left[\frac{7.25}{(1.0743)^1}+\cdots+\frac{107.25}{(1.0743)^{15}}\right]\times(1.0743)^{83/360}=100.043703$$

The approximate modified duration is 8.6907.

$$\text{ApproxModDur}=\frac{100.043703-99.869964}{2\times0.0001\times99.956780}=8.6907$$

The approximate convexity is 107.046.

$$\text{ApproxCon}=\frac{100.043703+99.869964-(2\times99.956780)}{(0.0001)^2\times99.956780}=107.046$$

3) The convexity-adjusted percentage price drop resulting from a 100 bp increase in the yield-to-maturity is estimated to be 8.1555%. Modified duration alone estimates the percentage drop to be 8.6907%. The convexity adjustment adds 53.52 bps.

$$\begin{aligned}\%\Delta PV^{Full}&\approx(-8.6907\times0.0100)+\left[\frac{1}{2}\times107.046\times(0.0100)^2\right]\\&=-0.086907+0.005352\\&=-0.081555\end{aligned}$$

4) The new full price if the yield-to-maturity goes from 7.44% to 8.44% on that settlement date is 91.780921.

$$PV^{Full} = \left[\frac{7.25}{(1.0844)^1} + \cdots + \frac{107.25}{(1.0844)^{15}}\right] \times (1.0844)^{83/360} = 91.780921$$

$$\%\Delta PV^{Full} = \frac{91.780921 - 99.956780}{99.956780} = -0.081794$$

The actual percentage change in the bond price is –8.1794%. The convexity-adjusted estimate is –8.1555%, whereas the estimated change using modified duration alone is –8.6907%.

Answer 6:

Calculate the estimated percentage price change for each bond

Bond A:

$$(-3.72 \times 0.0025) + \left[\frac{1}{2} \times 12.1 \times (0.0025)^2\right] = -0.009262$$

Bond B:

$$(-5.81 \times 0.0015) + \left[\frac{1}{2} \times 40.7 \times (0.0015)^2\right] = -0.008669$$

Bond C:

$$(-12.39 \times 0.0010) + \left[\frac{1}{2} \times 158.0 \times (0.0010)^2\right] = -0.012311$$

Based on these assumed changes in the yield-to-maturity and the modified duration and convexity risk measures, Bond C has the highest degree of interest rate risk (a potential loss of 1.2311%), followed by Bond A (a potential loss of 0.9262%) and Bond B (a potential loss of 0.8669%).

Answer 7:

(i) Need $V_A(i) = V_L(i)$ with $i = 0.08$

$$V_L(i) = 6v^8 + 11v^{15}$$
$$V_A(i) = Xv^5 + Yv^{20}$$

Need $V'_A(i) = V'_L(i)$ with $i = 0.08$

$$V'_L = -48v^9 - 165v^{16}$$
$$V'_A = -5Xv^6 - 20Yv^{21}$$

Thus we have to solve simultaneous equations:

(a) $6v^8 + 11v^{15} = Xv^5 + Yv^{20}$

(b) $-48v^9 - 165v^{16} = -5Xv^6 - 20Yv^{21}$

Taking 5 times (a) + (1+*i*) times (b) we get

$$-18v^8 - 110v^{15} = -15Yv^{20}$$

$$\Rightarrow Y = \frac{18(1+i)^{12} + 110(1+i)^5}{15}$$

$$\Rightarrow Y = 13.79688$$

Substitute back in (a) to get X = 5.50877

Hence the values of the zero-coupon bonds are £5.50877 million and £13.79688 million.

(ii) We need to check that the third condition is satisfied:

$$\begin{aligned} V_A' &= -5Xv^6 - 20Yv^{21} \\ \Rightarrow V_A'' &= 30Xv^7 + 420Yv^{22} \\ \Rightarrow V_A''(0.08) &= 30\times 5.50877\times 1.08^{-7} + 420\times 13.79688\times 1.08^{-22} \\ &= 1162.31 \end{aligned}$$

$$\begin{aligned} V_L' &= -48v^9 - 165v^{16} \\ \Rightarrow V_L'' &= 432v^{10} + 2640v^{17} \\ \Rightarrow V_L''(0.08) &= 432\times 1.08^{-10} + 2640\times 1.08^{-17} \\ &= 913.61 \end{aligned}$$

Therefore $V_A''(0.08) > V_L''(0.08)$

Thus the third condition is satisfied.

[Or note that since the assets have terms of 5 years and 20 years and the liabilities have terms of 8 years and 15 years, the spread of assets around the mean term is greater than that of the liabilities. Hence, the convexity of assets is greater than the convexity of liabilities].

Answer 8:

(i) Working in 000's

$$\begin{aligned} PV_L &= 200a_{\overline{20}|} + 300v^{15} \quad @\ 8\% \\ &= 200\times 9.818147 + 300\times 0.315242 \\ &= 2058.20199 \end{aligned}$$

i.e. £2,058,201.99

(ii)

$$\begin{aligned} DMT_L &= \frac{200v + 200\times 2v^2 + 200\times 3v^3 + \cdots + 200\times 20v^{20} + 300\times 15v^{15}}{200a_{\overline{20}|} + 300v^{15}} \\ &= \frac{200(Ia)_{\overline{20}|} + 300\times 15v^{15}}{2058.20199} \quad @\ 8\% \\ &= \frac{200\times 78.9079 + 300\times 15\times 0.31524}{2058.20199} \\ &= \frac{17200.175}{2058.20199} = 8.3569 \text{ years} \end{aligned}$$

(iii) Redington's first two conditions are:

$$\Rightarrow PV_L = PV_A$$
$$\Rightarrow DMT_L = DMT_A$$

Let the nominal amount in securities A and B be X and Y respectively.

$$PV_A = PV_L \Rightarrow X\left(0.09a_{\overline{12}|} + v^{12}\right) + Y\left(0.04a_{\overline{30}|} + v^{30}\right) = 2058201.99 \ @\ 8\%$$
$$\Rightarrow X(0.09\times 7.5361 + 0.39711) + Y(0.04\times 11.2578 + 0.09938)$$
$$\Rightarrow 1.075361X + 0.549689Y = 2058201.99$$
$$\Rightarrow X = \frac{2058201.99 - 0.549689Y}{1.075361}$$

$$DMT_A = DMT_L \Rightarrow \frac{X\left(0.09(Ia)_{\overline{12|}} + 12v^{12}\right) + Y\left(0.04(Ia)_{\overline{30|}} + 30v^{30}\right)}{2058201.99} = 8.3569$$

$$\Rightarrow X\left(0.09(Ia)_{\overline{12|}} + 12v^{12}\right) + Y\left(0.04(Ia)_{\overline{30|}} + 30v^{30}\right) = 17200175 \text{ @ } 8\%$$

$$\Rightarrow X(0.09 \times 42.17 + 12 \times 0.39711) + Y(0.04 \times 114.7136 + 30 \times 0.09938) = 17200175$$

$$\Rightarrow 8.56066X + 7.56986Y = 17200175$$

$$\Rightarrow 8.56066 \times \frac{(2058201.99 - 0.549689Y)}{1.075361} + 7.56986Y = 17200175$$

$$\Rightarrow 3.19394Y = 815370.9$$

$$\Rightarrow Y = 255287, \ X = 1783470$$

Hence company should purchase £1,783,470 nominal of security A and £255,287 nominal of security B for Redington's first two conditions to be satisfied.

(iv) Redington's third condition is that the convexity of the asset cash flow series is greater than the convexity of the liability cash flow series. Therefore the convexities of the asset cash flows and the liability cash flows will need to be calculated and compared.

Answer 9:

(i) No, because the spread (convexity) of the liabilities would always be greater than the spread (convexity) of the assets then the 3rd Redington condition would never be satisfied.

(ii) Work in £millions

Let proceeds from four-year bond = X

Let proceeds from 20-year bond = Y

Require PV Assets = PV Liabilities

$$Xv^4 + Yv^{20} = 10v^3 + 20v^6 \qquad (1)$$

Require DMT Assets = DMT Liabilities

$$\Rightarrow 4Xv^4 + 20Yv^{20} = 30v^3 + 120v^6 \qquad (2)$$

$(2) - 4 \times (1)$

$$\Rightarrow 16Yv^{20} = 40v^6 - 10v^3$$

$$\Rightarrow Y = \frac{40v^6 - 10v^3}{16v^{20}} = \frac{31.61258 - 8.88996}{7.30219} = £3.11175\text{m}$$

From (1):

$$X = \frac{10v^3 + 20v^6 - Yv^{20}}{v^4} = \frac{8.88996 + 15.80629 - 1.42016}{0.8548042} = £27.22973\text{m}$$

So amount to be invested in 4-year bond is

$Xv4$=£23.27609m

And amount to be invested in 20-year bond is

Yv^{20} = £1.42016m

Require Convexity of Assets > Convexity of Liabilities

$$\Rightarrow 20Xv^6 + 420Yv^{22} > 120v^5 + 840v^8$$

LHS = 981.869 > 712.411 = RHS

Therefore condition is satisfied and so above strategy will immunise company against small changes in interest rates.

Or state that spread of assets ($t = 4$ to $t = 20$) is greater than spread of liabilities ($t = 3$ to $t = 6$).

Answer 10:

(i) The present value of the assets is equal to the present value of the liabilities at the starting rate of interest.

The duration /discounted mean term/volatility of the assets is equal to that of the liabilities.

The convexity of the assets (or the spread of the timings of the asset cashflows) around the discounted mean term is greater than that of the liabilities.

(ii) (a) PV of liabilities is: $£100m\ a_{\overline{40|}}$ at 4%

$$= £100\text{m} \times 19.7928$$

$$= £1{,}979.28\text{m}$$

(b) The duration of the liabilities is:

$$\sum_{t=1}^{t=40} 100t\, v^t \Big/ \sum_{t=1}^{t=40} 100v^t \quad \text{(working in £m)}$$

$$= \frac{100 \sum_{t=1}^{t=40} t\, v^t}{1{,}979.28} = \frac{100(Ia)_{\overline{40|}}}{1{,}979.28} \text{ at 4\%}$$

$$= \frac{100 \times 306.3231}{1{,}979.28} = 15.4765 \text{ years}$$

(iii) Let x = nominal amount of five-year bond

y = nominal amount of 40-year bond.

working in £m

$$1{,}979.28 = xv^5 + yv^{40} \qquad \text{—(1)}$$

$$30{,}632.31 = 5xv^5 + 40yv^{40} \qquad \text{—(2)}$$

multiply equation (1) by 5.

$$9{,}896.4 = 5xv^5 + 5yv^{40} \qquad \text{—(1a)}$$

subtract (1a) from (2) to give

$$20735.91 = 35yv^{40}$$

$$\frac{20{,}735.91}{35 \times v^{40}} = y$$

with $v^{40} = 0.20829$

$y = 2{,}844.38$

Substitute into (1) to give:

$1{,}979.28 = Xv^5 + 2{,}844.38 \times 0.20829$

$v^5 = 0.82193$

$$\frac{1{,}979.28 - 2{,}844.38 \times 0.20829}{0.82193} = x = 1{,}687.28$$

Therefore £1,687.28m nominal of the five-year bond and £2,844.38*m* nominal of the 40-year bond should be purchased.

(iv) (a) The duration of the liabilities is 15.4765

Therefore the volatility of the liabilities is $\frac{15.4765}{1.04}$ =14.88125%

The value of the liabilities would therefore change by:

$1.5 \times 0.1488125 \times 1{,}979.28m = £441.81m$

and the revised present value of the liabilities will be £2,421.09*m*.

(b) PV of liabilities is: $£100m\ a_{\overline{40|}}$ at 2.5%

$= £100\text{m} \times \frac{1 - 1.025^{-40}}{0.025}$

= £2,510.28*m*.

(c) The PV of liabilities has increased by £531*m*. This is significantly greater than that estimated in (iv) (a). This estimation will be less valid for large changes in interest rates as in this case.