Lecture

INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

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Chapter: Unit 4 Chapter 2

Chapter Name: Immunisation theory

Today’s Agenda

1. Notations

1. Interest rate risk
   1. Effective duration
   2. Duration

3. Convexity

3. Immunization
   1. Conditions for Redington’s immunization
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1 Notations

- $\mathbf{y_t}$ = t-year spot rate of interest
- $\mathbf{f_{t,r}}$ = discrete time forward rate (the annual interest rate agreed at time 0 for an investment made at time t > 0 for a period of r years).
- $\mathbf{Y_t}$ = t-year spot force of interest
- $\mathbf{F_{t,r}}$ = continuous time forward rate (the force of interest equivalent to the annual forward rate of interest)
- $\mathbf{F_t}$ = instantaneous forward rate

2 Interest rate risk

Suppose an institution holds assets of value V_A, to meet liabilities of value V_L. Since both V_A and V_L represent the discounted value of future cashflows, both are sensitive to the rate of interest. We assume that the institution is healthy at time 0 so that currently $V_A > V_L$.

If rates of interest fall, both V_A and V_L will increase. If rates of interest rise then both will decrease. We are concerned with the risk that following a downward movement in interest rates the value of assets increases by less than the value of liabilities, or that, following an upward movement in interest rates the value of assets decreases by more than the value of the liabilities.

2.1 Effective duration

One measure of the sensitivity of a series of cashflows , to movements in the interest rates, is the effective duration (or volatility). Consider a series of cashflows $\{C_{tk}\}$ for k=1,2,........ n. Let A be the present value of the payments at rate (yield to maturity) i, so that:

$$A = \sum_{k=1}^{n} C_{t_k} v_i^{t_k}$$

Then the effective duration is defined to be:

$$\nu(i) = -\frac{1}{A}\frac{d}{di}A = -\frac{A'}{A}$$

$$= \left(\frac{1}{\sum_{k=1}^{n} C_{t_k} v_i^{t_k}} \right) \left(\sum_{k=1}^{n} C_{t_k} t_k v_i^{t_k+1} \right)$$

2.1 Effective duration

This is a measure of the rate of change of value of A with i, which is independent of the size of the present value. assumes that the cashflows do not depend on the rate of interest.

For a small movement ε in interest rates, from i to i + ε , the relative change in value of the present value is approximately $-\varepsilon\, v(i)$ so the new present value is approximately $A\,(1 - \varepsilon v(i))$

2.2 Duration

Another measure of interest rate sensitivity is the duration, also called Macaulay Duration or discounted mean term (DMT). This is the mean term of the cashflows $\{C_{tk}\}$, weighted by present value. That is, at rate i, the duration of the cashflow sequence $\{C_{tk}\}$ is:

$$\tau = \frac{\sum_{k=1}^{n} t_k C_{t_k} v_i^{t_k}}{\sum_{k=1}^{n} C_{t_k} v_i^{t_k}}$$

Comparing this expression with the equation for the effective duration it is clear that:

$$\tau = (1+i)v(i)$$

2.2 Duration

Another way of deriving the Macaulay duration is in terms of the force of interest, δ

$$\tau = -\frac{1}{A}\frac{d}{d\delta}A = \frac{di}{d\delta}v(i)$$

$$i = e^{\delta} - 1 \quad \Rightarrow \quad \frac{di}{d\delta} = e^{\delta}$$

$$\Rightarrow \quad \tau = e^{\delta}v(i) = (1+i)v(i)$$

The equation for τ in terms of the cashflows C_{tk} may be found by differentiating A with respect to δ, recalling that $v_i^{tk} = e^{-\delta tk}$

2.2 Duration

The duration of an n year coupon paying bond, with coupons of D payable annually, redeemed at R, is:

$$\tau = \frac{D(Ia)_{\overline{n}|} + Rnv^n}{Da_{\overline{n}|} + Rv^n}$$

The modified duration but can be expressed in terms of the Macaulay Duration as:

$$\frac{\tau}{1 + \frac{i^{(p)}}{p}}$$

3 Convexity

The convexity of the cashflow series $\{C_{tk}\}$ is defined as:

$$c(i) = \frac{1}{A}\frac{d^2}{di^2}A = \frac{A''}{A}$$

$$= \left(\frac{1}{\sum_{k=1}^{n} C_{t_k} v_i^{t_k}}\right)\left(\sum_{k=1}^{n} C_{t_k} t_k (t_k + 1) v_i^{t_k+2}\right)$$

Combining convexity and duration gives a more accurate approximation to the change in A following a small change in interest rates. For small ε:

$$\frac{A(i+\varepsilon) - A(i)}{A} = \frac{\partial A}{\partial i} \times \frac{1}{A} \times \varepsilon + ½ \times \frac{\partial^2 A}{\partial i^2} \times \frac{1}{A} \times \varepsilon^2 + \cdots$$

$$\approx -\varepsilon v(i) + \varepsilon^2 \times ½ \times c(i)$$

3 Convexity

Convexity gives a measure of the change in duration of a bond when the interest rate changes. Positive convexity implies that $\tau(i)$ is a decreasing function of i. This means, for example, that A increases more when there is a decrease in interest rates than it falls when there is an increase of the same magnitude in interest rates.

4 Immunisation

Let:

A_{tk} = Asset cash flows

L_{tk} = Liability cash flows

$V_A(i)$ = The present value of the assets at effective rate of interest i

$V_L(i)$ = The present value of the liabilities at rate i

$v_A(i)$ = The volatility of the asset cash flows

$v_L(i)$ = The volatility of the liability cash flows

$c_A(i)$ = The convexity of the asset cash flows

$c_L(i)$ = The convexity of the liability cash flows

4.1 Conditions for Redington's immunisation

$V_A\ (i_0) = V_L\ (i_0)$ – The value of the assets at the starting rate of interest is equal to the value of the liabilities.

$v_A\ (i_0) = v_L\ (i_0)$ – The volatilities of the asset and liability cash flow series are equal.

$c_A(i_0) > c_L(i_0)$ The convexity of the asset cashflow series is greater than the convexity of the liability cashflow series.

4.2 Limitations of immunisation

- There may be options or other uncertainties in the assets or in the liabilities, making the assessment of the cashflows approximate rather than known.

- Assets may not exist to provide the necessary overall asset volatility to match the liability volatility.

Question 1

A pension fund has to pay out benefits at the end of each of the next 40 years. The benefits payable at the end of the first year total £1 million. Thereafter, the benefits are expected to increase at a fixed rate of 3.8835% per annum compound.
(i) Calculate the discounted mean term of the liabilities using a rate of interest of 7% per annum effective. [5]

The pension fund can invest in both coupon-paying and zero-coupon bonds with a range of terms to redemption. The longest-dated bond currently available in the market is a zero-coupon bond redeemed in exactly 15 years.

(ii) Explain why it will not be possible to immunize this pension fund against small changes in the rate of interest. [2]

(iii) Describe the other practical problems for an institutional investor who is attempting to implement an immunization strategy. [3]

Solution

(i) DMT of liabilities is given by:

$$\frac{1\times 1\times v_{7\%}+2\times(1.038835)\times v_{7\%}^{2}+3\times(1.038835)^{2}\times v_{7\%}^{3}+\ldots+40\times(1.038835)^{39}\times v_{7\%}^{40}}{1\times v_{7\%}+(1.038835)\times v_{7\%}^{2}+(1.038835)^{2}\times v_{7\%}^{3}+\ldots+(1.038835)^{39}\times v_{7\%}^{40}}$$

$$=\frac{(1.038835)^{-1}\times\left[\left(\frac{1.038835}{1.07}\right)+2\times\left(\frac{1.038835}{1.07}\right)^{2}+3\times\left(\frac{1.038835}{1.07}\right)^{3}+\ldots+40\times\left(\frac{1.038835}{1.07}\right)^{40}\right]}{(1.038835)^{-1}\times\left[\left(\frac{1.038835}{1.07}\right)+\left(\frac{1.038835}{1.07}\right)^{2}+\left(\frac{1.038835}{1.07}\right)^{3}+\ldots+\left(\frac{1.038835}{1.07}\right)^{40}\right]}$$

$$=\frac{v_{i^*}+2\times v_{i^*}^{2}+3\times v_{i^*}^{3}+\ldots+40\times v_{i^*}^{40}}{v_{i^*}+v_{i^*}^{2}+v_{i^*}^{3}+\ldots+v_{i^*}^{40}}$$

$$=\frac{(Ia)_{\overline{40}|}^{i^*}}{a_{\overline{40}|}^{i^*}}$$

Solution

where $v_{i^*} \equiv \frac{1}{1+i^*} = \frac{1.038835}{1.07} \Rightarrow i^* = \frac{1.07}{1.038835} - 1 = \frac{0.07 - 0.038835}{1.038835} = 0.03$.

Hence, DMT of liabilities is:

$$\frac{(Ia)_{\overline{40}|}^{3\%}}{a_{\overline{40}|}^{3\%}} = \frac{384.8647}{23.1148} = 16.65 \text{ years}$$

(ii) Even if the fund manager invested entirely in the 15-year zero-coupon bond, the DMT of the assets will be only 15 years (and, indeed, any other portfolio of securities will result in a lower DMT).

Thus, it is not possible to satisfy the second condition required for immunisation (i.e. DMT of assets = DMT of liabilities).

Hence, the fund cannot be immunised against small changes in the rate of interest.

Solution

(iii) The other problems with implementing an immunization strategy in practice include:

- the approach requires a continuous re-structuring of the asset portfolio to ensure that the volatility of the assets remains equal to that of the liabilities over time
- for most institutional investors, the amounts and timings of the cash flows in respect of the liabilities are unlikely to be known with certainty
- institutional investor is only immunized for small changes in the rate of interest
- the yield curve is unlikely to be flat at all durations
- changes in the term structure of interest rates will not necessarily be in the form of a parallel shift in the curve (e.g. the shape of the curve can also change from time to time)

Question 2

A company has liabilities of £10 million due in three years' time and £20 million due in six years' time. The investment manager for the company is able to buy zero coupon bonds for whatever term he requires and has adequate monies at his disposal.

(i) Explain whether it is possible for the investment manager to immunize the fund against small changes in the rate of interest by purchasing a single zero coupon bond. [2]

The investment manager decides to purchase two zero-coupon bonds, one for a term of four years and the other for a term of 20 years. The current interest rate is 4% per annum effective.

(ii) Calculate the amount that must be invested in each bond in order that the company is immunized against small changes in the rate of interest. You should demonstrate that all three Redington conditions are met. [10]
[Total 12]

Solution

(i) No, because the spread (convexity) of the liabilities would always be greater than the spread (convexity) of the assets then the 3rd Redington condition would never be satisfied.

(i) Work in £millions
Let proceeds from four-year bond = X
Let proceeds from 20-year bond = Y

Require PV Assets = PV Liabilities

$$Xv^4 + Yv^{20} = 10v^3 + 20v^6 \qquad (1)$$

Require DMT Assets = DMT Liabilities

$$\Rightarrow 4Xv^4 + 20Yv^{20} = 30v^3 + 120v^6 \qquad (2)$$

Solution

(2) − 4 × (1)

$\Rightarrow 16Yv^{20} = 40v^6 - 10v^3$

$$\Rightarrow Y = \frac{40v^6 - 10v^3}{16v^{20}} = \frac{31.61258 - 8.88996}{7.30219} = £3.11175\text{m}$$

From (1):

$$X = \frac{10v^3 + 20v^6 - Yv^{20}}{v^4} = \frac{8.88996 + 15.80629 - 1.42016}{0.8548042} = £27.22973\text{m}$$

So amount to be invested in 4-year bond is

$Xv4$=£23.27609m

Solution

And amount to be invested in 20-year bond is

$Yv^{20} = £1.42016\text{m}$

Require Convexity of Assets > Convexity of Liabilities

$$\Rightarrow 20Xv^6 + 420Yv^{22} > 120v^5 + 840v^8$$

LHS = 981.869 > 712.411 = RHS

Therefore condition is satisfied and so above strategy will immunise company against small changes in interest rates.

Or state that spread of assets ($t = 4$ to $t = 20$) is greater than spread of liabilities ($t = 3$ to $t = 6$).