INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Class: FY MSc

Subject : Financial Mathematics

Chapter: Unit 4 Chapter 2

Chapter Name: Capital Budgeting

Today's Agenda

1. Introduction
    1. Reasons for Evaluation
    2. Methods for Project Evaluation
    3. How to begin?

2. Methods of Project Evaluation
    1. Accumulated Value
    2. Net Present Value
    3. Internal Rate of return
    4. Payback periods
    5. Other considerations

3. Measures of Investment Performance
    1. Money weighted rate of return
    2. Time weighted rate of return
    3. Linked internal rate of return

1 Introduction

In this chapter we are concerned with a number of applications of compound interest theory to the assessment of investments and business ventures.

Project appraisal:

The process of critically evaluating various investment opportunities to measure its profitability.

Project appraisal or Capital appraisal is the process of assessing, in a structured way, the case for proceeding with a project or proposal, or the project's viability. It often involves comparing various options, using certain methods generally used for evaluation

1.1 Why to Evaluate?

Investors should evaluate projects before considering them as ideal investments.

Why???

1.1 Reasons for Evaluation

There is a great deal of practical application associated with project evaluations.

- **General Feasibility** – A capital investment appraisal will reveal the project's general feasibility. This will include the projected cash flows and projected quarterly or annual profits. It will normally include a net present value analysis. These will give you a good idea of the probable profitability of the project in both the near and long terms.

- **Alternative Possibilities** – A proposed investment almost never exists without alternatives. When considering an investment, you need to understand how it ranks in relation to others like it.

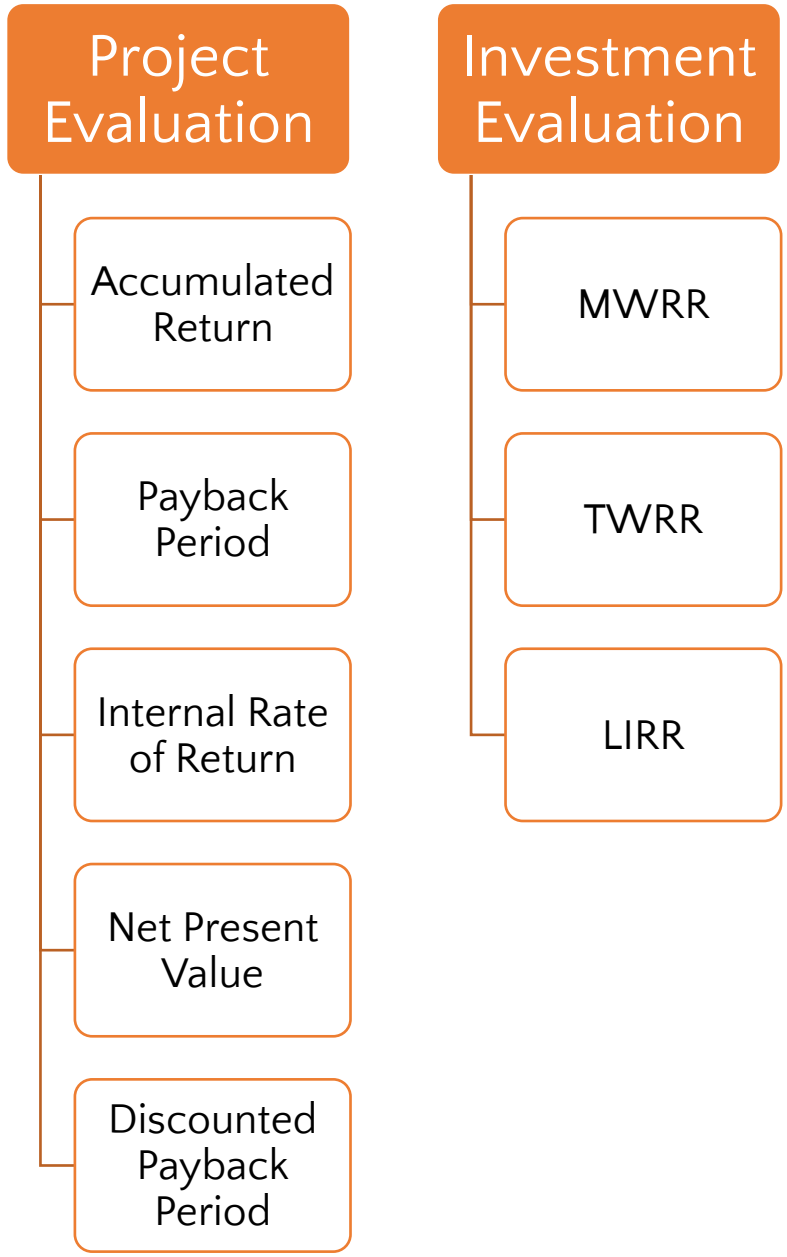
- **Uncertainty Concerns** – Another important function of the investment appraisal concerns uncertainty. Projecting discounted cash flows over a period of time requires assigning specific values to inflation rates, future regulatory costs and other factors that, in reality, remain uncertain.

- **Strategic Fit** – An investment appraisal not only will demonstrate a project's relative financial feasibility, it also will assess how well the particular project fits into a company's strategic plans. Ultimately, how well a proposed project moves forward strategic objectives or how well it fits into the company's social fabric will have wider financial implications.

1.2 Methods for Project Evaluation

What do you think can be the methods we could consider to evaluate different investment options???

1.2 Methods


Project Evaluation
Accumulated Return
Payback Period
Internal Rate of Return
Net Present Value
Discounted Payback Period
Investment Evaluation
MWRR
TWRR
LIRR

1.3 How to begin?

Estimating Cash flows from the Project.

This involves the following aspects

- Understanding of the Business
- Projecting the Cash-Flows
- Assessing the risk associated with the Project

Estimating the Risk Discount Rate

Evaluating the Project using the techniques as mentioned above

1.3 Net Cash Flow

- **Estimation:** Any investment or project will normally require an initial outlay and possibly other outlays in future, which will be followed by receipts (in some cases the pattern of income and outgo is more complicated). The cash flows associated with the investment may be completely fixed, or they may have to be estimated. The estimation of the cash inflows and outflows associated with a business project usually requires considerable experience and judgment, and all relevant factors should be considered.

- **Net cash flow** ct at time t (measured in suitable time units) is:

$$ct = \text{cash inflow at time t} - \text{cash outflow at time t}$$

- If any payments may be regarded as continuous then $\rho(t)$, the net rate of cash flow per unit time at time t, is defined as:

$$\rho(t) = \rho_1(t) - \rho_2(t)$$

where $\rho_1(t)$ and $\rho_2(t)$ denote the rates of inflow and outflow at time t respectively.

Question

Measuring time in years, describe the net cash flows associated with Business venture. The cash flows are given as below:

Year	Contributions	Returns
0	10000	0
1	5000	0
2	1000	0
3	1000	0
4	1000	0
5	1000	0
6	1000	8000
7	1000	9000
8	1000	10000
9	1000	11000
10	0	12000
Total	23000	50000

Solution

Year	*Contributions*	*Returns*	*Net Cash Flow*
0	10,000	0	–10,000
1	5,000	0	– 5,000
2	1,000	0	– 1,000
3	1,000	0	– 1,000
4	1,000	0	– 1,000
5	1,000	0	– 1,000
6	1,000	8,000	7,000
7	1,000	9,000	8,000
8	1,000	10,000	9,000
9	1,000	11,000	10,000
10	0	12,000	12,000
Total	23,000	50,000	27,000

Project PM

Further in this chapter we consider the two hypothetical projects as examples for evaluation.

A mining company is considering an opencast project in the Hatti Gold Mines of Karnataka.

The cash flows for the project are as follows:

Outflow: Beginning of Year 1 (£100,000)
Beginning of Year 2 (£200,000)
Beginning of Year 3 (£200,000)

Inflow: End of Year 3 £950,000 (sales)

Project AM

A businessman is considering a particular project where in the cash flows during the life if the project will be as follows:

Outflow: Beginning of Year 1 (£300,000)
Throughout Year 1 (£50,000)
Throughout Year 2 (£75,000)
Throughout Year 3 (£105,000)

Inflow: End of Year 3 £950,000 (sales)

The staff costs occurred during the year 1, 2 and 3 can be assumed to be paid uniformly throughout the year.

2.1 Accumulated Value (AV)

- Calculating the accumulated profit at the end of the project . This is the accumulated value of the net cash flow (at the time of last payment).
- Can be used for assessment only when there is a fixed time horizon for the project.
- This will not be the case if the time horizon (i.e. the time until the last cashflow payment) is unlimited or the timing of the payments is uncertain.
- Accumulated profits for two different projects cannot be compared directly if they have different time horizons .
- The Accumulated value, at time T, of a cash flow can be expressed as:

$$\mathbf{A(T)} = \sum c_t \, (1+i)^{T-t} + \int_0^T \rho(t)(1+i)^{T-t} \, dt$$

Question

For the projects AM and PM, calculate the accumulated value at the end of the project if the rate of interest is 19%.

Solution

The accumulated value is approx.

PM = 260.2641

AM = 154.9467377

2.1 Drawback

We cannot find the accumulated value for the investment since the returns are paid till perpetuity. Thus, this is a drawback of the accumulated value method for evaluating investments.

- Accumulated profit calculations suffer from the disadvantage that they can only be used in situations where there is a definite fixed time horizon for the project. This will not be the case if the time horizon is unlimited or the timing of the payments is uncertain.

This problem can be avoided by calculating the “net present value” instead.

2.2 Net Present Value (NPV)

- The present value at rate of interest i of the net cash flows is called net present value at i .
- The rate of interest used in NPV is called the risk discount rate.
- A higher net present value indicates a more profitable project.
- Net present value depends solely on the forecasted cash flows from the project and the opportunity cost of capital.
- $\boldsymbol{NPV\ (i) = \sum c_t\ (1+i)^{-t} + \int_0^T \rho(t)(1+i)^{-t}\ dt}$
- N

2.2 Net Present Value (NPV)

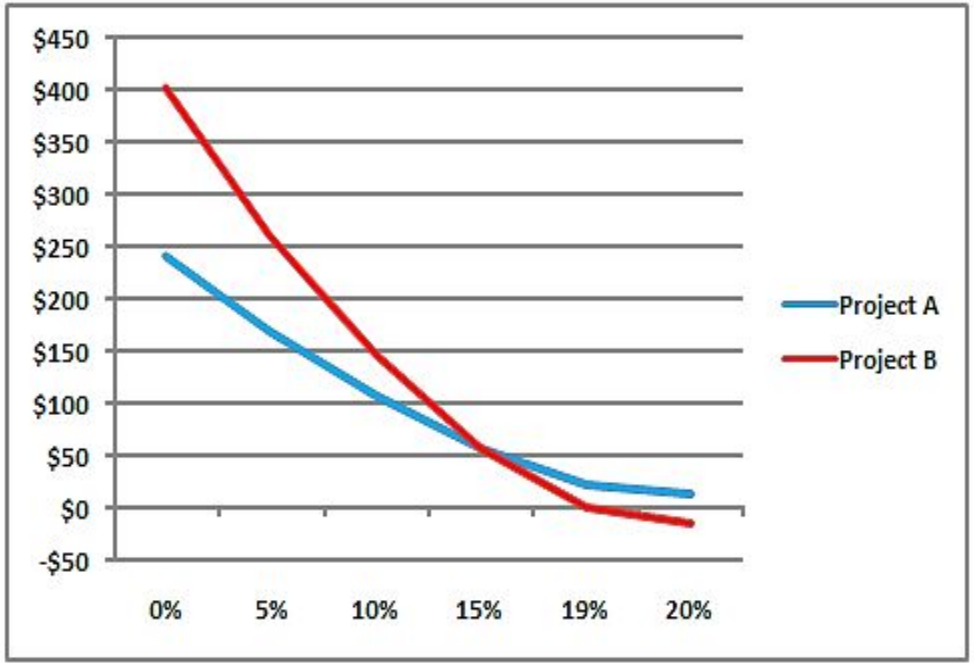
Cross over Points

- **The rate of interest at which the net present value of two projects are same is the crossover rate.**

Question :
Consider a project A which requires an initial investment of $400 million. This project is expected to generate cash flows of $160 million for the next four years.

Consider another project B which requires an initial investment of $400 million and no cash flows in the next three years and $800 million in the last year.

Which project is preferable using net present value as criterion ? Use risk discount rate of
a) 10% b) 15% c) 18% ?



Question

Calculate the net present value for Project AM and Project PM using a risk discount rate of 19% per annum.
Using net present values as a criterion, which project is preferable?

Solution

The NPV is calculated as:

NPV_{PM} = -100 – 200v - $200v^2$ + 950 v^3 @ 19%

= 154.444832

NPV_{AM} = -300 - $50\bar{a}_{\overline{1}|}$ - 75v$\bar{a}_{\overline{1}|}$ - 105 $v^2\bar{a}_{\overline{1}|}$ + 950 v^3 @ 19%

= 91.94784448

The net present values are £15444.48 for project PM and £9194.78 for project AM.

So, using a risk discount rate of 19%, Project R appears more favourable.

Note that the net present value will depend on the risk discount rate used.

2.3 Internal Rate of Return (IRR)

- The yield rate is that rate of interest at which the present value of returns from the investment is equal to the present value of the contributions into the investment. In the business and finance literature is often called the *internal rate of return.*

- From lender's perspective , the higher the yield rate , the more favourable the transaction. From borrower's perspective, the opposite is the case.

- The *internal rate of return rule is to accept an investment project if the opportunity* cost of capital is less than the internal rate of return.

- It is valid to use yield rates to compare alternative investments only if the period of investment is same for all the alternatives

2.3 Internal Rate of Return (IRR)

- IRR isn't necessarily unique. A transaction can have multiple yield rates.
- One situation in which the yield rate will be unique is when all cash flows in one direction ARE made before the cash flows in other direction . Also if the outstanding investment balance is positive at all points throughout the period of investment then the yield rate will be positive.
- A situation where multiple yield rates would arise would be an investment in a physical plant which requires major renovation expenses midway through the investment period.

Question

Find the internal rate of return for Project PM.

Solution

IRR (PM) = $-100 - 200v - 200v^2 + 950\,v^3 = 0$

= 40.3%

Question

Find the internal rate of return for project AM and then compare the two projects based on the IRR criterion.

Solution

IRR (AM) = 28%

Thus project PM is more favorable due to higher IRR.

2.3 Interpretation of NPV & IRR

Practical interpretation of the net present value function NPV(i) and the yield is as follows.

- Suppose that the investor may lend or borrow money at a fixed rate of interest i_1. $NPVi_1$ is the present value at rate of interest i_1 of the net cash flows associated with the project, we conclude that the project will be profitable if and only if:

$$\boldsymbol{NPVi_1 > 0}$$

- Let us now assume that, as is usually the case in practice, the yield i_0 exists and NPV(i) changes from positive to negative when i = i_0. Under these conditions it is clear that the project is profitable if and only if:

$$\boldsymbol{i_1 < i_0}$$

ie the yield exceeds that rate of interest at which the investor may lend or borrow money.

2.3 Comparing two Investments

- Suppose that an investor is comparing the merits of two investments, which we call projects A and B, respectively. We assume that the borrowing powers of the investor are not limited.
- Let $NPV_A(i)$ and $NPV_B(i)$ denote the respective net present value functions, and let i_A and i_B denote the yields (which we shall assume to exist).
- It might be thought that the investor should always select the project with the higher yield, but this is not always the best policy.
- A better criterion to use is the profit at time T (the date when the later of the two projects ends) or, equivalently, the net present value, calculated at the rate of interest $i1$ at which the investor may lend or borrow money. The reason is that A is the more profitable venture if $NPV_A(i1) > NPV_B(i1)$.

Question

An investor is considering whether to invest in either or both of the following instruments:

Investment A – For a purchase price of £10,000 the investor will receive £1,000 per annum payable quarterly in arrears for 15 years.

Investment B – For a purchase price of £11,000, the investor will receive an income of £605 per annum, payable annually in arrears for 18 years, and a return of his outlay at the end of this period.

The investor may lend or borrow money at 4% per annum.

Would you advise him to invest in either loan, and, if so, which would be the more profitable?

Solution

Consider Investment A

$$NPV_A(i) = -10{,}000 + 1{,}000a_{\overline{15}|}^{(4)}$$

and the yield is found by solving the equation $NPV_A(i) = 0$, o $a_{\overline{15}|}^{(4)} = 10$, which gives $i_A \approx 5.88\%$.

For Investment B we have

$$NPV_B(i) = -11{,}000 + 605a_{\overline{18}|} + 11{,}000v^{18}$$

and the yield (i.e., the solution of $NPV_B(i) = 0$) is $i_B = 5.5\%$. The rate of interest at which the investor may lend or borrow money is 4% per annum, which is less than both i_A and i_B, so we compare NPV_A (0.04) and NPV_B(0.04).

Now $NPV_A(0.04) = £1{,}284$ and $NPV_B(0.04) = £2{,}089$, so it follows that, although the yield on Investment B is less than on Investment A, the investor will make a larger profit from loan B. We should therefore advise him that an investment in either loan would be profitable, but that, if only one of them is to be chosen, then loan B will give the higher profit.

2.4 Payback Periods

In many practical problems, the net cash flow changes sign only once, this change being from negative to positive. In these circumstances the balance in the investor's account will change from negative to positive at a unique time t_1 or it will always be negative, in which case the project is not viable.

If this time t_1 exists, it is referred to as the discounted payback period (DPP).

It is the smallest value of t such that A(t) ≥ 0, where

$$\mathbf{A(t) = \sum_{s \le t} c_s\,(1+j1)^{(t-s)} + \int_0^t \rho(s)(1+j1)^{(t-s)}\,ds}$$

where j_1 = rate of interest applicable to investors borrowings.

Other things being equal, a project with a shorter discounted payback period is preferable to a project with a longer discounted payback period because it will start producing profits earlier.

2.4 Payback Period

- Some companies require that the initial outlay on any project should be recoverable within a specified period.

- The **payback period of a project is found by counting the number of years it takes before the cumulative forecasted cash flow equals the initial investment.**

Advantages :
- Simple calculation

Disadvantages :
- The payback rule ignores all cash flows after the cut-off date.
- The payback rule gives equal weight to all cash flows before the cut-off date(i.e. Ignores time value of money).

2.4 Payback Period

	Cash Flows ($)			
Project	C_0	C_1	C_2	C_3
A	–2,000	500	500	5,000
B	–2,000	500	1,800	0
C	–2,000	1,800	500	0

Assess the projects using the criteria of payback period.

Question

An investment of £100,000 will produce an annuity of £10,500 annually in arrears for 25 years. Find the discounted payback period when the interest rate on borrowed money is 9% per annum. Find also the accumulated profit after 25 years if money may be invested at 7% per annum.

Solution

The discounted payback period is the smallest integer t such that

$$10{,}500a_{\overline{t}|} \geq 100{,}000 \qquad \text{at } 9\%$$

From compound interest tables we see that the discounted payback period is 23 years, i.e.,

$$a_{\overline{23}|} = 9.5802 \text{ at } i = 9\%.$$

The accumulated profit after 25 years is (from Eqs 6.5.3 and 6.5.5)

$$P = \left[-100{,}000(1.09)^{23} + 10{,}500s_{\overline{23}|0.09}\right](1.07)^2 + 10{,}500s_{\overline{2}|0.07} = £26{,}656$$

2.5 Other Considerations

The investors will also need to take into account a number of other factors, such as:

- **Cash flows** – Are the cash flow requirements for the project consistent with the business's other needs?
- **Borrowing requirements** – Can the business raise the necessary cash at the times required?
- **Resources** – Are the other resources required for the project available?
- **Risk** – What are the financial risks involved in going ahead with the project (and in doing nothing)?
- **Investment conditions** – What is the economic climate? Are interest rates likely to rise or fall?
- **Cost vs benefit** – Is the project worth doing at all? Do the costs outweigh the benefits?
- **Indirect benefits** – Will the project bring any additional benefits?

Question

A company is undertaking a new project. The project requires an investment of £5m at the outset, followed by £3m three months later.
It is expected that the investment will provide income over a 15 year period starting from the beginning of the third year. Net income from the project will be received continuously at a rate of £1.7m per annum. At the end of this 15 year period there will be no further income from the investment.

Calculate at an effective rate of interest of 10% per annum:
(i) the net present value of the project [3]
(ii) the discounted payback period [4]

A bank has offered to loan the funds required to the company at an effective rate of interest of 10% per annum. Funds will be drawn from the bank when required and the loan can be repaid at any time. Once the loan is paid off, the company can earn interest on funds from the venture at an effective rate of interest of 7% per annum.

(iii) Calculate the accumulated profit at the end of the 17 years. [4]
[Total 11]

Solution

(i) $\text{NPV} = -5 - 3v^{1/4} + 1.7\overline{a}_{\overline{15}|}v^2 \ @10\%$

$$\text{NPV} = -5 - 3 \times 0.976454 + 1.7 \times 0.82645 \times \frac{i}{\delta} a_{\overline{15}|} \ @10\%$$

$$= -5 - 2.929362 + 1.404965 \times 1.049206 \times 7.6061$$

$$= -7.929362 + 11.21213458$$

$$= 3.282772575$$

NPV = £3.283m

Solution

(ii) DPP is $t+2$ such that

$$1.7\bar{a}_{\overline{t}|}v^2 = 5+3v^{\frac{1}{4}} \Rightarrow 1.474097708a_{\overline{t}|} = 7.929362 \ @10\%$$

$$\frac{1-1.1^{-t}}{0.1} = 5.379129 \Rightarrow 1-1.1^{-t} = 0.5379129$$

$$\Rightarrow 0.4620871 = 1.1^{-t} \Rightarrow \ln 0.4620871 = -t \ln 1.1$$

$$\Rightarrow t = 8.100$$

$$\therefore DPP = 10.1 \text{ years}$$

Solution

(iii) Accumulated profit 17 years from start of project:

$$= 1.7\overline{s}_{\overline{6.9}|7\%} = 1.7 \times \frac{\left(1.07^{6.9} - 1\right)}{\delta} \ @\ 7\%$$

$$= 1.7 \times \frac{\left(1.07^{6.9} - 1\right)}{0.067659}$$

$$= 1.7 \times 8.79346$$

$$= £14.95m$$

Question

A particular charity invests its assets in a fund on which it has a target rate of return of 8% per annum effective. From time-to-time, the charity also invests in projects that help achieve its charitable objectives whilst providing a rate of return. Projects that are accepted by the charity must fulfil each of the following criteria:
1. a minimum annual effective internal rate of return of 2% less than the target return on the investment fund.
2. a payback period of no more than ten years.
3. a positive cash flow during the fifth year or earlier.

The charity is considering investing in a social enterprise project that involves providing loans to farmers in low-income countries to help them develop better resilience against poor weather conditions. The details are as follows:

- The project involves making loans of £1m at the start of each year for three years, the first loan being made at the beginning of 2017.
- The loans will be paid back from the extra income obtained by the farmers from the beginning of 2020.
- The repayments in each year will be through level monthly instalments paid in advance with the rate of payment of the instalments increasing by 1% per year for 10 years after which the payments stop.
- The annual rate of repayment in 2020 will be £495,000.
- The charity will also incur costs at the end of each of the years in which income is received of £50,000 per annum.

(i) Explain why, in general, the payback period is not an appropriate decision criterion for an investment project. [2]
(ii) Determine which of the three criteria used by the charity are met in this case. [12]
[Total 14]

Solution

(i) The payback period simply tells an investor when the total cash inflows from the investment have exceeded the total cash outflows. This tells the investor nothing about the overall profitability of the project. [2]

Solution

NPV of cash flows at 6% = 3.2877 – 3.1424 = £0.1453m = £145,300 [1]

The project has a positive NPV at 6% and therefore an IRR higher than 6% and the first criteria is met. [½]

By the end of the 10^{th} year, the total outgoing cash flows will have been: £3,000,000 plus 7 × £50,000 or £3,350,000. [1]

Total incoming cash flows are:

$495{,}000 \times (1 + 1.01 + 1.01^2 + \ldots + 1.01^6)$ (i.e. rate of payment of £495,000 rising by 1% per year for seven years). [1]

Solution

Geometric progression with common ratio 1.01 and seven terms

$= 495{,}000(1 - 1.01^7)/(1 - 1.01) = £3{,}570{,}700$ [1]

This is greater than total outgoing cash flows and therefore second criterion is met. [½]

There is clearly a positive cash flow in the fifth year as the incoming cash flows will be greater than £495,000 and the outgoing cash flows will be £50,000. [1]

Therefore final criterion is met. [½]

[Total 14]

3 Measures of Investment Performance

It is often necessary to be able to measure the investment performance of a fund (for example a pension fund, or the funds of an insurance company) over a period. In this section we will look at three measures of investment performance.

1) Money-weighted rate of return
2) Time-weighted rate of return
3) Linked Internal rate of return

Solution

(ii) The present value of outgoing cash flows at a rate of return of 6% per annum effective is as follows (in £m):

$$\ddot{a}_{\overline{3}|} + 0.05\left(a_{\overline{13}|} - a_{\overline{3}|}\right)$$

$$= 1.06 \times 2.6730 + 0.05(8.8527 - 2.6730) = 3.14238 \qquad [2]$$

The present value of the incoming cash flows is as follows (in £m):

$$= 0.495v^3 \ddot{a}_{\overline{1}|}^{(12)} \left(1 + 1.01v + 1.01^2 v^2 + \ldots + 1.01^9 v^9\right)$$

$$= 0.495v^3 \frac{d}{d^{(12)}} \left(1 + 1.01v + 1.01^2 v^2 + \ldots + 1.01^9 v^9\right)$$

$$= 0.495 \times 0.83962 \times 0.973784 \times (1 - 1.01^{10}/1.06^{10}) \,/\, (1 - 1.01/1.06)$$

$$= 0.404716 \times 8.12352 = 3.2877 \qquad [3½]$$

3.1 Money Weighted Rate of Return

One measure of the performance is the yield earned on the fund over the period. The yield earned on the fund is also called the "money-weighted rate of return" (MWRR).

The money-weighted rate of return is the interest rate satisfying the equation of value incorporating the initial and final fund values and the intermediate net cash flows.

Note that the equation of value used in calculating the MWRR only takes account of new money. New money includes all "extra" money paid into the fund that was not generated by the fund itself. Withdrawals from the fund correspond to negative new money. Any cash flows generated by the fund itself must be ignored.

3.1 Money Weighted Rate of Return

For example, consider a fund with value F_0 at time 0, with net cash flows C_{tk} at times $t1$, $t2$, ….., tn and fund value F_T at time T ≥ tn, then the equation of value, equating values at time T, is:

$$\boldsymbol{F_0\,(1+i)^T + C_{t1}\,(1+i)^{T-t1} + C_{t2}\,(1+i)^{T-t2} + \ldots\ldots + C_{tn}\,(1+i)^{T-tn} = F_T}$$

where i is the effective annual rate of interest earned by the fund in the interval [0,T].

Question

A fund had a value of £150,000 on 1 July 2009. A net cash flow of £30,000 was received on 1 July 2010, and a further net cash flow of £40,000 was received on 1 July 2011. The fund had a value of £175,000 on 30 June 2010 and a value of £225,000 on 30 June 2011. The value of the fund on 1 January 2012 was £280,000.

Solution

The money-weighted rate of return is i such that

$$150(1+i)^{2.5} + 30(1+i)^{1.5} + 40(1+i)^{0.5} = 280$$

This can be solved through trial and interpolation to find that $i = 12.58\%$ per annum.

3.1 Drawback of MWRR

As a measure of investment performance the money-weighted rate of return is not entirely satisfactory, as it is sensitive to the amounts and timing of the net cash flows. If, say, we are assessing the skill of the fund manager, this is not ideal, as the fund manager does not control the timing or amount of the cash flows – he or she is merely responsible for investing the positive net cash flows and realising cash to meet the negative net cash flows.

A measure which tries to eliminate this effect is the time-weighted rate of return.

3.2 Time Weighted Rate of Return

The time-weighted rate of return is found from the product of the growth factors between consecutive cash flows.

Define F_0, F_T, C_{tk} as above, and let C_0 be the cash flow (if any) at time t = 0; in addition let $F_{tk}-$ be the amount of the fund just before the cash flow due at time tk, so that the amount of the fund just after the receipt of the net cash flow due at time tk is $(F_{tk}-) + C_{tk}$. The "Time-Weighted Rate of Return" (TWRR) is i per annum, where:

$$(1+i)^T = \frac{F_{t1}-}{F_0+C_0}\,\frac{F_{t2}-}{F_{t1}-+C_{t1}} \cdots\cdots \frac{F_T}{F_{tn}-+C_{tn}}$$

Again, note that the cash flows in the formula for calculating the TWRR only include those relating to new money. Any cash flows generated by the fund itself must be taken into account in the figures for the fund value.

Using the TWRR eliminates the effect of the cash flow amounts and timing, and therefore gives a fairer basis for assessing the investment performance for the fund.

Question

For the investment fund described in Previous Question, calculate the time-weighted rate of return per annum earned on the fund between 1 July 2009 and 1 January 2012.

Solution

The time-weighted rate of return per annum is i such that

$$(1+i)^{2.5} = \frac{175}{150} \times \frac{225}{175+30} \times \frac{280}{225+40} = 1.352968$$

hence,

$$i = 12.85\%$$

3.2 Drawbacks

The disadvantages of both the time-weighted and money-weighted rates of return are that the calculation requires information about all the cash flows of the fund during the period of interest. In addition, the TWRR requires the fund values at all the cash flow dates. A disadvantage of the MWRR is that the equation may not have a unique solution – or indeed any solution. If the fund performance is reasonably stable in the period of assessment then the TWRR and the MWRR will give similar results.

3.3 Linked Internal Rate of Return

Linked rates of return provide a way of combining rates of return for successive sub periods to obtain an approximate rate of return over a longer period. The linked (internal) rate of return is found from the product of the yields for the sub periods.

If the rate of return on a fund is measured over a series of intervals (0, $t1$), ($t1$,$t2$), ….., ($tn-1$, tn) such that the annual effective rate of interest earned by a fund in the interval ($tr-1$, tr) is ir then the "Linked Internal Rate of Return" is i per annum, where:

$$(1+i)^{tn} = (1+i1)^{t1}\,(1+i2)^{t2-t1} \ldots\ldots (1+in)^{tn-tn-1}$$

The linked internal rate of return will be equal to the TWRR if the subintervals ($tr-1$, tr) are the same in each calculation. In practice, the yields ir may be calculated by approximate methods, and then, if the subintervals used are sufficiently short, the linked internal rate of return will be close to the TWRR.

Question

A fund had assets totalling £600m on 1 January 2011. It received net income of £40m on 1 January 2012 and £100m on 1 July 2012. The value of the fund was £450m on 31 December 2011, £500m on 30 June 2012, and £800m on 31 December 2012.

(a) Calculate the time-weighted rate of return per annum for the period 1 January 2011 to 31 December 2012.

(b) Calculate the linked-internal rate of return over the same period, using subintervals of a calendar year

Solution

(a) The time-weighted rate of return per annum is i such that

$$(1+i)^2 = \frac{450}{600} \times \frac{500}{450 \times 40} \times \frac{800}{500+100} = 1.020408$$

hence, $i = 1.015\%$ per annum.

(b) The first subinterval is the first year, and the money-weighted rate of return in this period is i_1. We have

$$1+i_1 = \frac{450}{600}, \text{and so } i_1 = -25\%$$

The second subinterval is the second year, and the money-weighted rate of return in this period is i_2. We have

$$490(1+i_2) + 100(1+i_2)^{0.5} = 800$$

This can be solved as a quadratic in $(1+i_2)^{0.5}$, leading to $i_2 = 39.188\%$ per annum.

The linked internal rate of return is then i, such that

$$(1+i)^2 = 0.75 \times 1.39188, \text{leading to } i$$
$$= 2.1719\% \text{ per annum.}$$

Question

Westcott-Smith is a privately held investment management company. Two other investment counselling companies, which want to be acquired, have contacted Westcott-Smith about purchasing their business. Company A's price is £2 million. Company B's price is £3 million. After analysis, Westcott-Smith estimates that Company A's profitability is consistent with a perpetuity of £300,000 a year. Company B's prospects are consistent with a perpetuity of £435,000 a year. Westcott-Smith has a budget that limits acquisitions to a maximum purchase cost of £4 million. Its opportunity cost of capital relative to undertaking either project is 12% per annum.

(a) Determine which company or companies (if any) Westcott-Smith should purchase according to the NPV measure.

(b) Determine which company or companies (if any) Westcott-Smith should purchase according the IRR measure.

(c) State which company or companies (if any) Westcott-Smith should purchase. Justify your answer.

Solution

We have $i = 12\%$ per annum.

(a) *Company A:* The annual payment under the perpetuity is £300,000 (assume in arrears), and the price is £2,000,000. The NPV is then obtained from

$$\text{NPV}_A = -2{,}000{,}000 + \frac{300{,}000}{0.12} = £500{,}000$$

Company B: The annual payment under the perpetuity is £435,000 (assume in arrears), and the price is £3,000,000. The NPV is then obtained from

$$\text{NPV}_B = -3{,}000{,}000 + \frac{435{,}000}{0.12} = £625{,}000$$

Both Company A and Company B would be positive NPV acquisitions, but Westcott-Smith cannot purchase both because the total purchase price of £5 million exceeds the budgeted amount of £4 million. Because Company B's NPV of £625,000 is higher than Company A's NPV of £500,000, Westcott-Smith should purchase Company B according to the NPV rule.

(b) *Company A:* The equation of value is

$$-2{,}000{,}000 + \frac{300{,}000}{\text{IRR}} = 0$$

which can be solved to give the IRR as 15% per annum.

Company B: The equation of value is

$$-3{,}000{,}000 + \frac{435{,}000}{\text{IRR}} = 0$$

which can be solved to give the IRR as 14.5% per annum.

Both Company A and Company B have IRRs that exceed Westcott-Smith's opportunity cost of 12%, but Westcott-Smith cannot purchase both because of its budget constraint. According to the IRR rule, Westcott-Smith should purchase Company A because its IRR of 15% is higher than Company B's IRR of 14.5%.

Solution

(c) Westcott-Smith should purchase Company B. When the NPV and IRR rules conflict in ranking mutually exclusive investments, we should follow the NPV rule because it directly relates to shareholder wealth maximization.