U1 — Measurement
— Real & Discounting
U2 —

Lecture 5

INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Class: FY MSc

Subject : Financial Mathematics

Subject Code: PPSAS102

Chapter: Unit 3 – Chapter 1

Chapter Name: Annuities

Today's Agenda

Annuities

1. Annuities
    1. What is an Annuity?
    2. Types of Annuities

2. Present values of annuities

1. Perpetuities

1. P-thly payable annuities

1. Continuous Annuities

1. Accumulated values of annuities
    1. Annuity value on any date
    2. P-thly payable annuities
    3. Continuously payable annuities
    4. Relationships

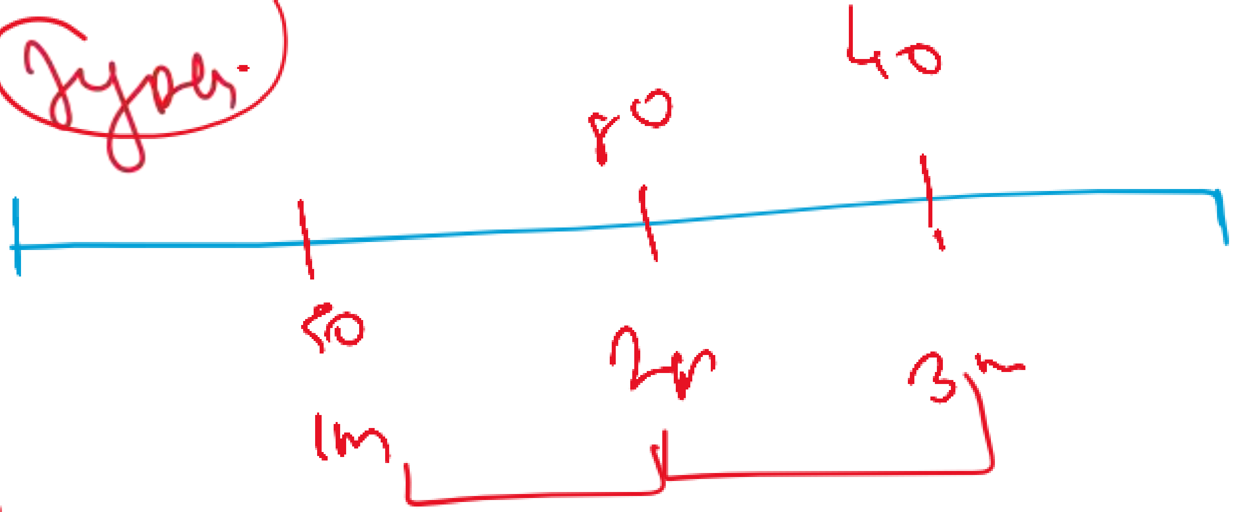
7. Varying rates of interest

7. Uncommon facts about annuities

7. Varying Annuities
    1. Increasing Annuities
    2. Decreasing Annuities
    3. Compound Increasing Annuities

annuity

Types.



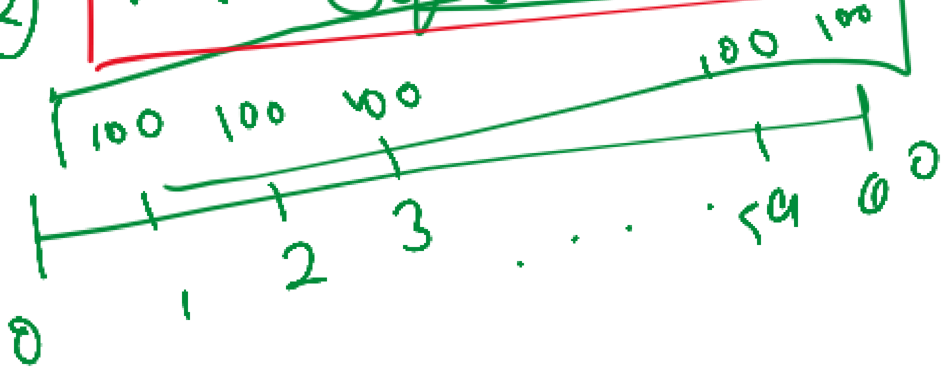
1000

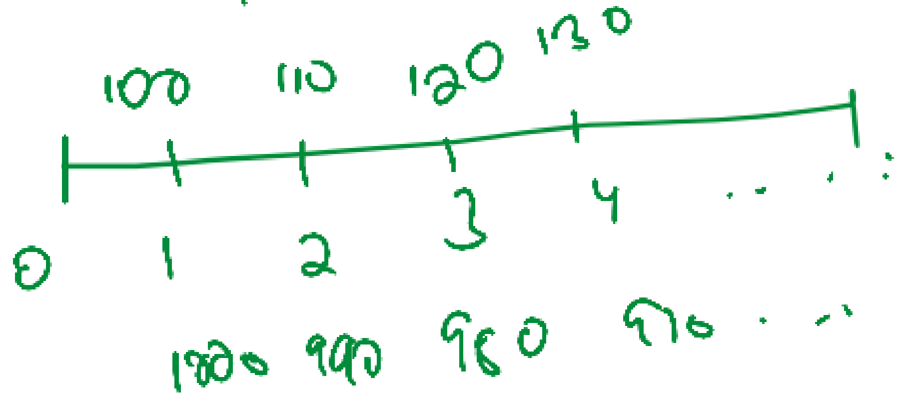
Practical

① Series of payments

100
10

② At equal intervals





5%

bulk payment

80,000

25 26 27 28 29 ... 35 70

1st Sep 25

$80,000(1.05)$

$80,000(1.05)^2$

HDFC Life

10,00,000

7

Gift

Term Life

25 — 70

~~2~~ 2.5 L

nominee

1.1 What is an Annuity?

An annuity may be defined as a series of payments made at equal intervals of time. Annuities are common in our economic life.

House rents, mortgage payments, instalment payments on automobiles, and interest payments on money invested are all examples of annuities.

Types:

① Certainty

- annuity certain
- contingent annuity

② Time of payments

- cont annuity
- Immediate annuity / annuity in arrears.
- Annuity due / annuity in advance

$$\left\{ \begin{array}{l} \sum_{t=0}^{n} C_t v^t \\ \int_0^n \rho(t) v(t)\, dt \end{array} \right.$$

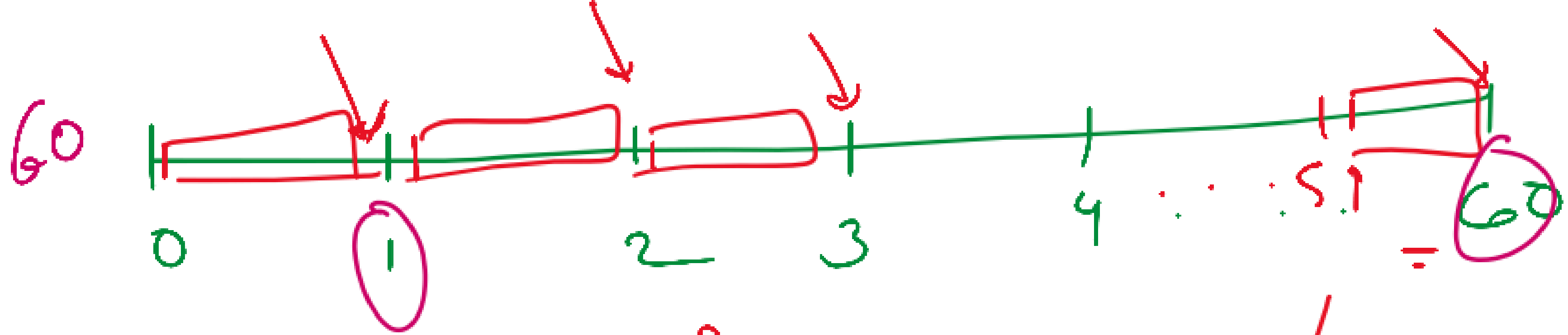
PRLIP.

SOY SOY SOY

0 1 2 3

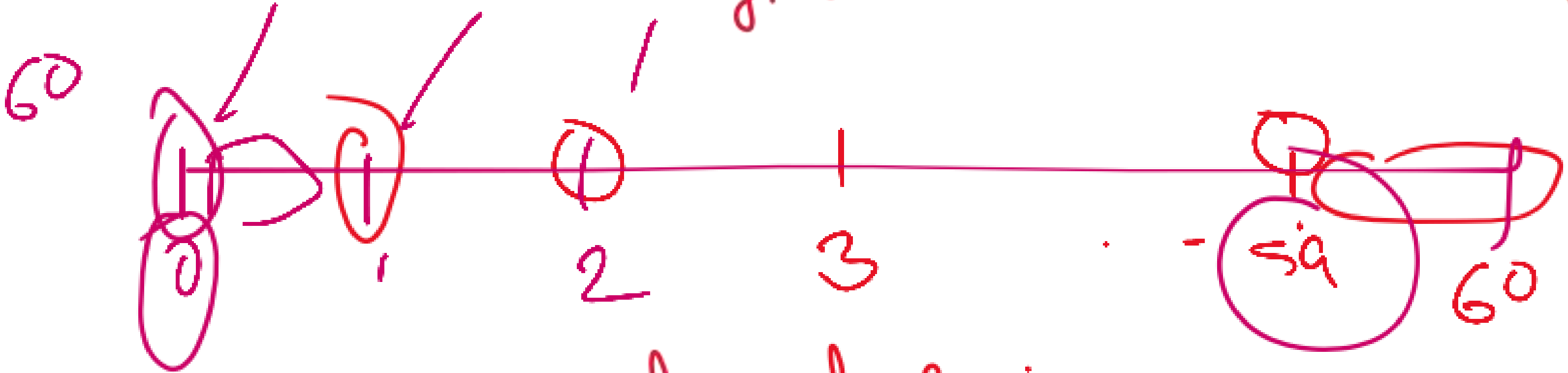
EOY EOY EOY

$\boxed{365}$



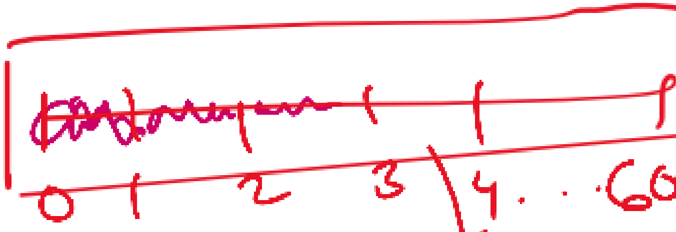
annuity in arrears

Immediate annuity



annuity due

annuity in advance



Cont annuity

daily SIPs

5 years

5 × 365

③ Payments :

U14 .

100

Level annuity

Increasing / decreasing

Same amt

Same rate .

For any type of annuity :

- PV

- AV .

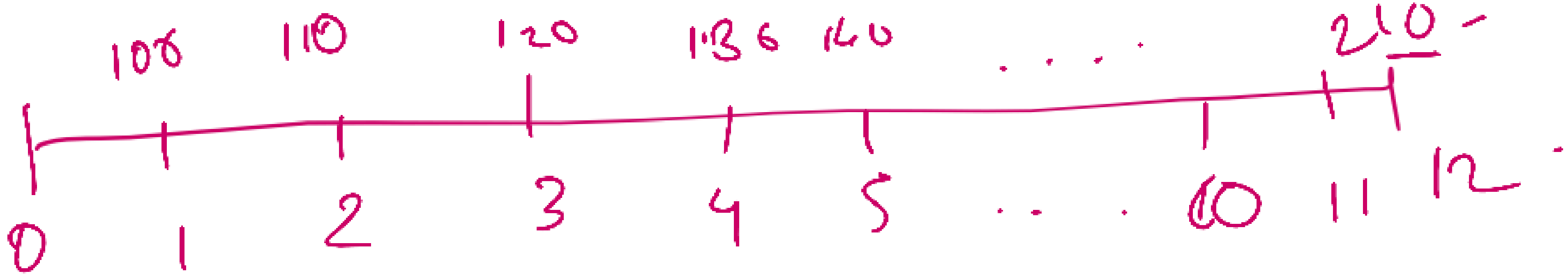
100 , 110 , 120 , 130 . . .

1000 , 980 , 960 , 940 . . .

↑ 10% .

↑ 15% .

100 , 100 × 1.15 , 100×1.15^2 . .



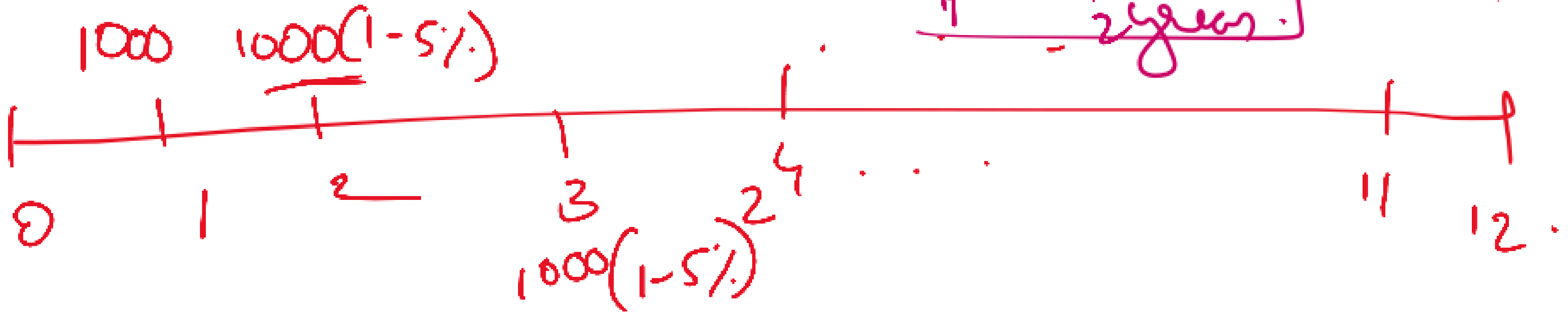
arrears

1st payment = 100

+ 10 $

Big decreasing

2 years



1000 -5%

1.2 Types of Annuities

- Annuity
  - Time of payment
    - Arrear
    - Due
  - Commencing Time
    - Immediate
    - Deferred
  - Payment
    - Fixed
    - Variable
    - Increasing
    - Decreasing

1.2 Types of Annuities

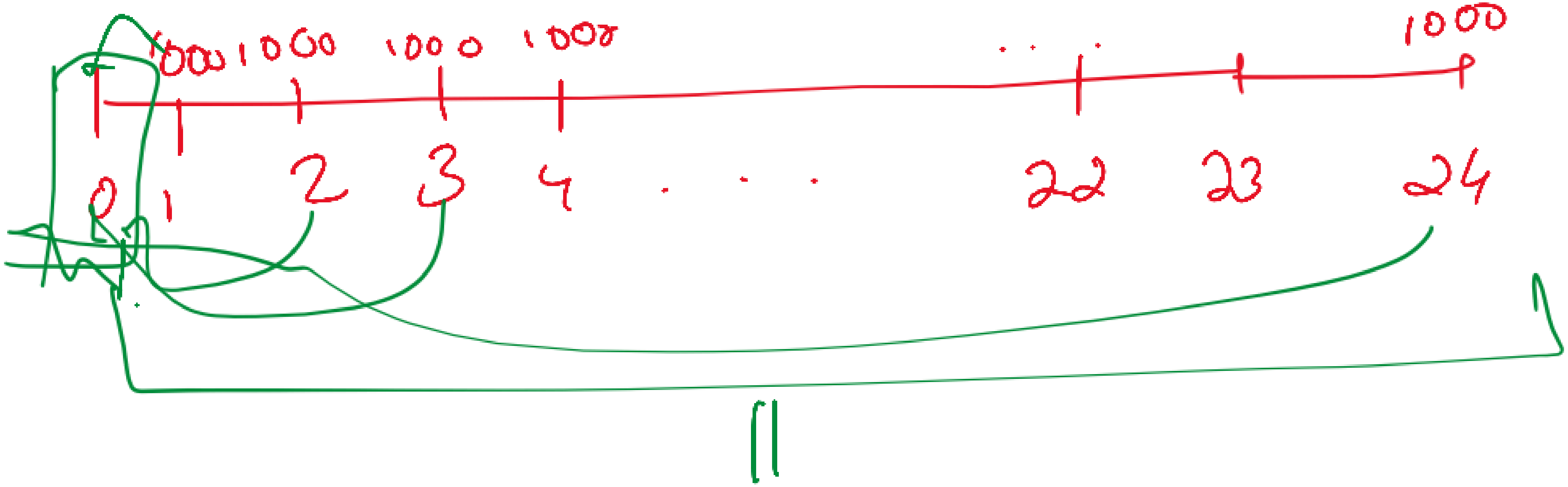
1. Consider an annuity such that payments are certain to be made for a fixed period of time. An annuity with these properties is called an **annuity-certain**. The fixed period of time for which payments are made is called the term of the annuity..

2. An annuity under which the payments are not certain to be made is called a **contingent annuity**. A common type of contingent annuity is one in which payments are made only if a person is alive. Such annuity is called life annuity.

3. If payments are made at the end of each time period, they are paid in arrear. Such an annuity is called an **annuity-immediate**.

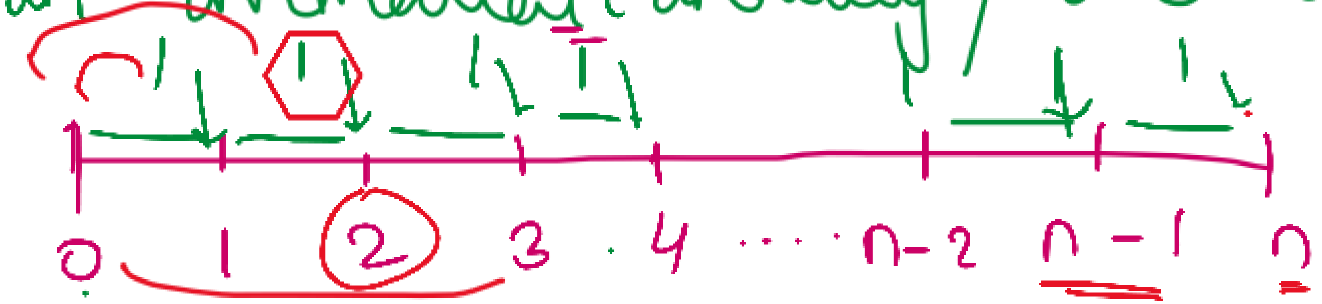
4. If they are made at the beginning of each time period, they are paid in advance. An annuity paid in advance is also known as an **annuity-due**.

5. If each payment is for the same amount, this is a **level annuity.** If payments increase (decrease) each time by the same amount, this is a **simple increasing (decreasing) annuity.**



closed form -

① PV of an immediate annuity / annuity in arrears.



$$\boxed{v = \frac{1}{1+i}}$$

payments are made @ end

n payments.

1st payment → time 1

$$PV = 1v + 1v^2 + 1v^3 + 1v^4 + \dots + 1v^{n-1} + 1v^n$$

$$= 1\left(v + v^2 + v^3 + v^4 + \dots + v^{n-1} + v^n\right)$$

$r = v$ $\quad n$

$a = v$

$$PV = v\left(\frac{1 - v^n}{1 - v}\right)$$

$$PV = v\left(\frac{1-v^n}{1-v}\right)$$

$$v = \frac{1}{1+i}$$

$$= v\left(\frac{1-v^n}{d}\right)$$

$$= v\left(\frac{1-v^n}{iv}\right)$$

$$a_{\overline{n}|}^{@i\%} = \boxed{\frac{1-v^n}{i}}$$

PV of an immediate annuity

$$1-v = 1-\frac{1}{1+i}$$

$$= \frac{1+i-1}{1+i}$$

$$= \boxed{\frac{i}{1+i}} = d$$

$$= i\left(\frac{1}{1+i}\right)$$

$$= iv$$

$1 + x + x^2 + x^3 + x^4 + \dots + x^n$

Geometric progression

Sum of GP - n terms -

$a\left(\frac{r^n - 1}{r - 1}\right)$

if $r > 1$
or $r < -1$

V

$a\left(\frac{1 - r^n}{1 - r}\right)$

if $-1 < r < 1$

V

$\frac{1}{1+i}$

$0 <$ <1

2 Present values of Annuities

1] Annuity Immediate

Consider an annuity under which payments of 1 are made at the end of each period for n periods, where n is a positive integer.

The valuation is illustrated on the following timeline:

Time	0	1	2	……	n
Payment		1	1	…	1
Value					

2

Present values of Annuities

How do we derive the Present Value?

We can derive an expression for as an equation of value at the beginning of the first period.

The present value of a payment of 1 made at the end of the first period is v. The present value of a payment of 1 made at the end of the second period is v^2. This process is continued until the present value of a payment of 1 made at the end of the nth period is v^n.

The total present value $a_{\overline{n|}}$ must equal the sum of the present values of each payment, i.e.

$$a_{\overline{n|}} = v + v^2 + ... + v^n.$$

This is a geometric progression.

2 Present values of Annuities

Derivation

A key algebraic relationship used in valuing a series of payments is the geometric series summation formula:

$$1 + x + x^2 + \ldots + x^n = \frac{1 - x^{n+1}}{1 - x} = \frac{x^{n+1} - 1}{x - 1}$$

Using the above relationship:

$$a_{\overline{n|}} = v + v^2 + \ldots.. + v^n.$$

$$= v\,\frac{1 - v^n}{1 - v}$$

$$= v\,\frac{1 - v^n}{i\,v}$$

$$= \frac{1 - v^n}{i}$$

2 Present values of Annuities

Conditions to be met

The symbol $a_{\overline{n}|}$ can be used to express the present value of an annuity provided the following conditions are met;

1. There are n payments of equal amount.

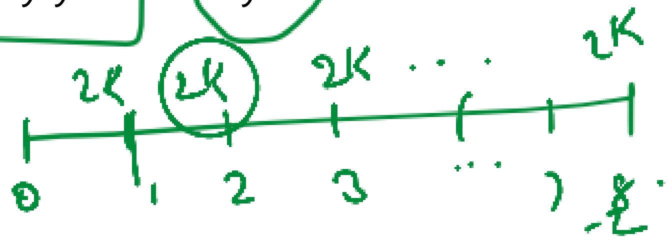
2. The payments are at equal intervals of time, with the same frequency as the frequency of interest compounding.

3. The valuation point is one payment period before the first payment is made.

$a_{\overline{n}|} @ i\%$

Question

Find the present value of annuity which pays Rs. 2000 at the end of every year for 8 years if the rate of interest is 5% p.a. effective.



$$PV = 2000\, a_{\overline{8|}} \ @\, 5\%\, pa$$

$$= 2000 \left[\frac{1 - v^{8}}{5\%} \right]$$

$$= 2000 \left[\frac{1 - \left(\frac{1}{1+5\%}\right)^{8}}{5\%} \right] = 12926.42552.$$

Solution

$$P.V. = 2000\, a_{\overline{8|}} @ 5\%$$

$$= 2000 \times \left(\frac{1 - v^8}{5\%} \right) \qquad v = \frac{1}{1.05}$$

$$= 12926.43$$

Question

Find the present value of annuity which pays $500 at the end of every half year for 20 years if the rate of interest is 9% convertible semiannually.

$\boxed{9200.79221}$

$x\, a_{\overline{n}|} @ i\%$

100

60

$i^{(2)}$ $\boxed{\dfrac{i^{(2)}}{2}}$

$\dfrac{i^{(2)}}{2} =$

1000

$i^{(2)} =$ 6% pa

Solution

$PV = \boxed{500}\, a_{\overline{40}|}$ @ 4.5%

$= 500 \left(\frac{1 - v^{40}}{4.5\%} \right)$ where $v = \frac{1}{1.045}$

$= 9200.79$

$i^{(2)} = 9\%$

$\frac{i^{(2)}}{2} = 4.5\%$

$500 \left[1 - (\right.$

$\left| A \right($

$i^{(2)}$

Question 5.3

Bhavesh has bought a new car and requires a loan of $\boxed{12000 \text{ to}}$ pay for it. The car dealer offers two alternatives to repay:

1. Monthly payments for 3 years, starting one month after purchase, with annual interest rate of 12% compounded monthly. $\boxed{398.57172}$ $\frac{i^{(12)}}{12} = 1\%$ $i^{(m)} = 12\%$
2. Monthly payments for 4 years, starting one month after purchase, with annual interest rate of 15% compounded monthly. 333.969 $\times 48$ $\boxed{\frac{i^{(12)}}{12} = 1\%}$

Calculate:

1. Bhavesh's monthly payment.

2. Total amount paid over the course of repayment period under each of the two options.

$12000 = \frac{398.57172}{\times 36}$ $12000 = x \, a_{\overline{36}|}^{@1\%}$

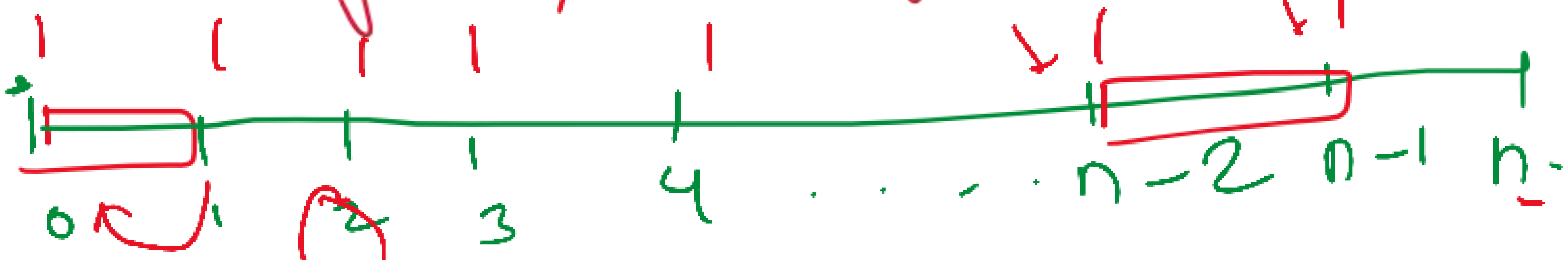
$$12000 = x \cdot \left[\frac{1 - \left(\frac{1}{1+1\%} \right)^{36}}{1\%} \right]$$

$$12000$$

$$12000 = x \cdot 30.10750504$$

$$x = \frac{12000}{30.10750504} =$$

(2) PV of an annuity due / annuity in advance.



n → no. of payments

payments @ start

1st payment → 0 last payment → $n-1$.

$$PV = 1 + 1v + 1v^2 + \dots + 1v^{n-2} + 1v^{n-1}$$

$$= 1\left(\underline{1 + v + v^2 + \dots + v^{n-2} + v^{n-1}}\right)$$

$$= 1\left(\frac{1-v^n}{1-v}\right) = \boxed{\frac{1-v^n}{d}}$$

$\ddot{a}_{\overline{n|}}$ @ i%

$$a\left(\frac{1-r^n}{1-r}\right)$$

2 Present values of Annuities

2] Annuity Due

Consider an annuity under which payments of 1 are made at the start of each period for n periods, where n is a positive integer.

The valuation is illustrated on the following timeline:

Time	0	1	2	......	n-1	n
Payment	1	1	1	...	1	
Value						

2 Present values of Annuities

Derivation

We can derive an expression for $\ddot{a}_{\overline{n|}}$ analogous to annuity immediate as:

$$\ddot{a}_{\overline{n|}} = 1 + \mathrm{v} + v^2 + \ldots\ldots + v^{n-1}$$

Again summing the geometric progression

$$\ddot{a}_{\overline{n|}} = 1 + \mathrm{v} + \mathrm{v}^2 + \ldots\ldots + v^{n-1}.$$

$$= \frac{1 - v^n}{1 - v}$$

$$= \frac{1 - v^n}{i\,v}$$

$$= \frac{1 - v^n}{d}$$

Question

Find the present value of annuity which pays Rs.500 at the start of every year for 5 years if the rate of interest is 10% pa effective.

500

$i = 10\%$ pa.

$@\ i = 10\%$

$$500\,\ddot{a}_{\overline{5}|} = 500\left[\frac{1-\left(\frac{1}{1+10\%}\right)^5}{\frac{10\%}{1+10\%}}\right] = 2084.93$$

Question 5.5

Find the present value of annuity which pays Rs.100 at the ~~end~~ of every quarter for 10 years if the rate of interest is 8% convertible quarterly.

start

$i^{(4)} = 8\%$

$\frac{i^{(4)}}{4} = 2\%$

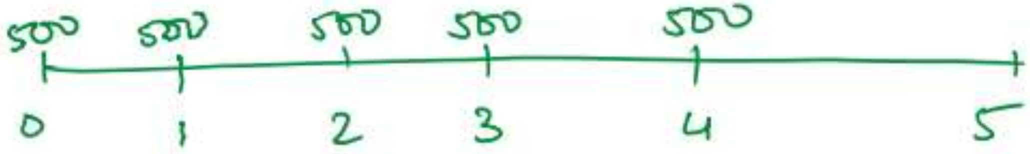
27 90.25 803

$100 \cdot \ddot{a}_{\overline{40}|} \text{ @ } i = 2\%$

$$100 \cdot \left[\frac{1 - \left(\frac{1}{1+2\%}\right)^{40}}{\left(\frac{2\%}{1+2\%}\right)} \right] =$$

$\boxed{2735.55}$

Solution



$i = 10\%$ p.a.

$$500\,\ddot{a}_{\overline{5}|} = 500\left(\frac{1-v^5}{d}\right)$$

$$= 2084.93$$

$$d = \frac{0.1}{1.1}$$

$$500 \times \frac{i}{d} \times a_{\overline{5}|}$$

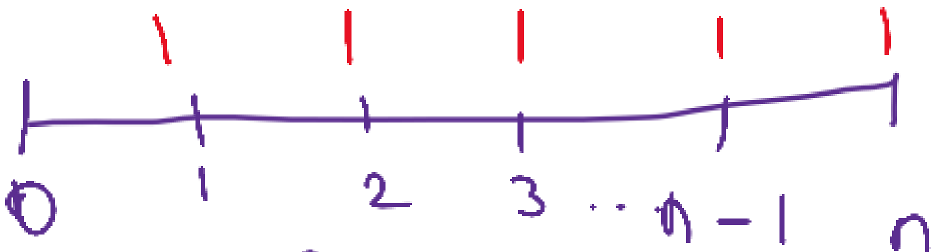
$$500 \times (1+i)\, a_{\overline{5}|}$$

$a_{\overline{n}|}$ @ i%

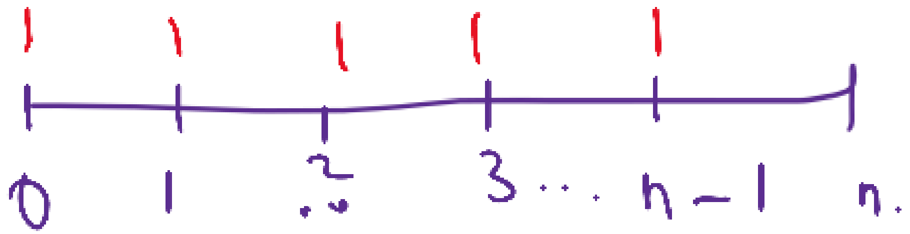
$\ddot{a}_{\overline{n}|}$ @ i%

annuity due.

Relationship



$a_{\overline{n}|}$



$\ddot{a}_{\overline{n}|}$

1

$$a_{\overline{n}|} = v + v^2 + v^3 + \dots v^{n-1} + v^n$$

$$\ddot{a}_{\overline{n}|} = 1 + v + v^2 + v^3 + \dots + v^{n-1}$$

$$a_{\overline{n}|} = v\left(1 + v + v^2 + \dots + v^{n-2} + v^{n-1}\right)$$

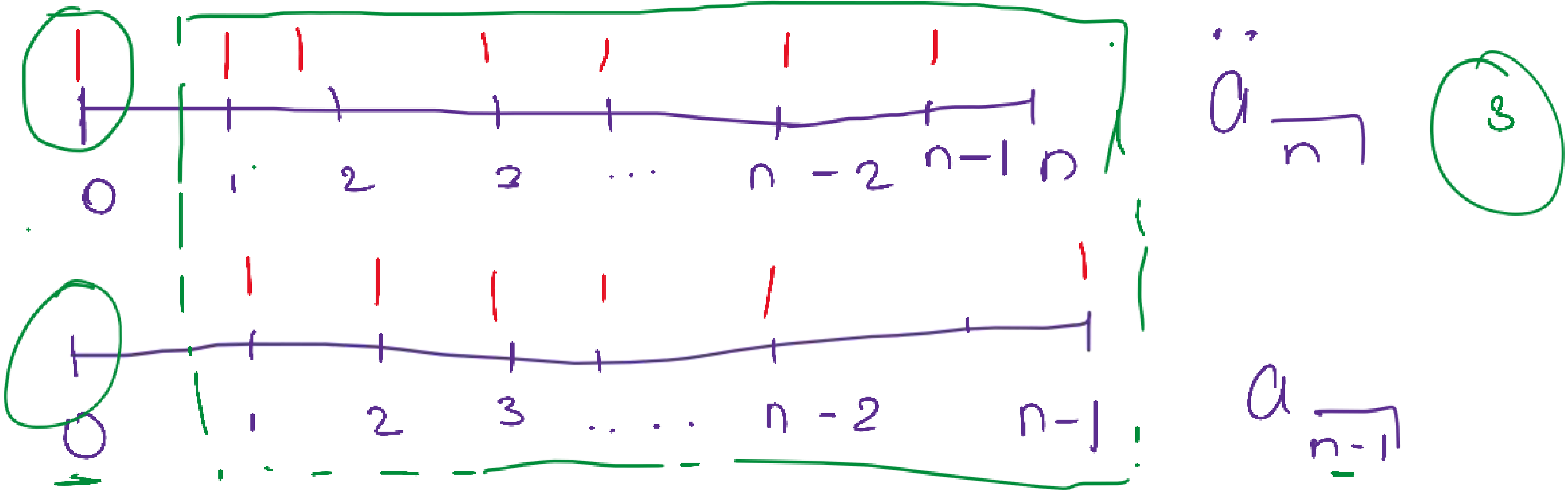
$$a_{\overline{n}|} = v \cdot \ddot{a}_{\overline{n}|}$$

2

$$a_{\overline{n}|} = v\ddot{a}_{\overline{n}|}$$

$$a_{\overline{n}|} = \frac{1}{1+i}\ddot{a}_{\overline{n}|}$$

$$\ddot{a}_{\overline{n}|} = (1+i)a_{\overline{n}|}$$



$$\ddot{a}_{\overline{n}|} = a_{\overline{n-1}|} + 1$$

$$100\,\ddot{a}_{\overline{n}|} = 100\,a_{\overline{n-1}|} + 100$$

$$a_{\overline{n}|} = \frac{1-v^n}{i} \qquad \ddot{a}_{\overline{n}|} = \frac{1-v^n}{d}$$

4

$$\frac{i}{d} a_{\overline{n}|} = \frac{i}{d}\left(\frac{1-v^n}{i}\right) = \frac{1-v^n}{d} = \ddot{a}_{\overline{n}|}$$

$$\ddot{a}_{\overline{n}|} = \frac{i}{d} a_{\overline{n}|}$$

Question 5.4

$500\ddot{a}_{\overline{5}|}$ @ 10%

Find the present value of annuity which pays Rs.500 at the ~~end~~ start of every year for 5 years if the rate of interest is 10% pa effective.

$500 + a_{\overline{4}|}$

$500 +$

$500(3.1699)$

2084.94

$500\ddot{a}_{\overline{5}|} = 50$

$500\ddot{a}_{\overline{5}|}$

10%

i	0.100 000
$i^{(2)}$	0.097 618
$i^{(4)}$	0.096 455
$i^{(12)}$	0.095 690
δ	0.095 310
$(1+i)^{1/2}$	1.048 809
$(1+i)^{1/4}$	1.024 114
$(1+i)^{1/12}$	1.007 974
v	0.909 091
$v^{1/2}$	0.953 463
$v^{1/4}$	0.976 454
$v^{1/12}$	0.992 089

n	$(1+i)^n$	v^n	$s_{\overline{n}|}$	$a_{\overline{n}|}$	$(Ia)_{\overline{n}|}$	$(Da)_{\overline{n}|}$	n
1	1.100 00	0.909 09	1.000 0	0.909 1	0.909 1	0.909 1	1
2	1.210 00	0.826 45	2.100 0	1.735 5	2.562 0	2.644 6	2
3	1.331 00	0.751 31	3.310 0	2.486 9	4.815 9	5.131 5	3
4	1.464 10	0.683 01	4.641 0	$\boxed{3.169\ 9}$	7.548 0	8.301 3	4
5	1.610 51	0.620 92	6.105 1	3.790 8	10.652 6	12.092 1	5
6	1.771 56	0.564 47	7.715 6	4.355 3	14.039 4	16.447 4	6
7	1.948 72	0.513 16	9.487 2	4.868 4	17.631 5	21.315 8	7
8	2.143 59	0.466 51	11.435 9	5.334 9	21.363 6	26.650 7	8
9	2.357 95	0.424 10	13.579 5	5.759 0	25.180 5	32.409 8	9
10	2.593 74	0.385 54	15.937 4	6.144 6	29.035 9	38.554 3	10
11	2.853 12	0.350 49	18.531 2	6.495 1	32.891 3	45.049 4	11
12	3.138 43	0.318 63	21.384 3	6.813 7	36.714 9	51.863 1	12
13	3.452 27	0.289 66	24.522 7	7.103 4	40.480 5	58.966 4	13
14	3.797 50	0.263 33	27.975 0	7.366 7	44.167 2	66.333 1	14
15	4.177 25	0.239 39	31.772 5	7.606 1	47.758 1	73.939 2	15
16	4.594 97	0.217 63	35.949 7	7.823 7	51.240 1	81.762 9	16
17	5.054 47	0.197 84	40.544 7	8.021 6	54.603 5	89.784 5	17
18	5.559 92	0.179 86	45.599 2	8.201 4	57.841 0	97.985 9	18

$$\ddot{a}_{\overline{n}|} = a_{\overline{n-1}|} + 1$$

$$\boxed{\underline{500}}\,\ddot{a}_{\overline{5}|}$$

$$\ddot{a}_{\overline{5}|} = a_{\overline{4}|} + 1$$

$$500\,\ddot{a}_{\overline{5}|} = 500\left[a_{\overline{4}|} + 1\right]$$

$$= 500\left[3.1699 + 1\right]$$

$$500\,\ddot{a}_{\overline{5}|} = 2084.95$$

2 Relationship between $\mathbf{a}_{\overline{n|}}$ and $\ddot{\mathbf{a}}_{\overline{n|}}$

It is simply possible to relate annuity-immediate and annuity-due. One type of relationship is:

$$\ddot{\boldsymbol{a}}_{\overline{\boldsymbol{n}|}} = \boldsymbol{a}_{\overline{\boldsymbol{n}|}}\,(\mathbf{1} + \mathbf{i})$$

Since the payment under $\ddot{\boldsymbol{a}}_{\overline{\boldsymbol{n}|}}$ is made one period earlier than each payment under $\boldsymbol{a}_{\overline{\boldsymbol{n}|}}$, the total present value must be larger by one period's interest.

There is another relationship which states that:

$$\ddot{\boldsymbol{a}}_{\overline{\boldsymbol{n}|}} = \mathbf{1} + \boldsymbol{a}_{\overline{\boldsymbol{n-1}|}}$$

The n payments under $\ddot{a}_{\overline{n|}}$ can be split into the first payment and the remaining n-1 payments. The present value of first payment is 1 and present value of remaing n-1 payments is $a_{\overline{n-1|}}$. The sum must give $\ddot{a}_{\overline{n|}}$.

2 Annuity values on any date

So far we considered evaluating present values of annuities at the beginning of the term (either one period before, or at the date of the first payment). However, it is often necessary to evaluate annuities at other dates.

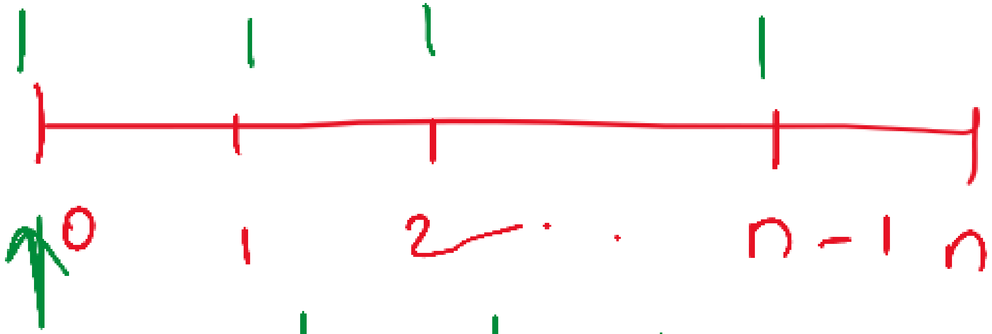
Example

Consider an annuity under which seven payments of 1 are made at the end of the 3rd through the 9th periods inclusive. Below is a is a time diagram for this annuity:

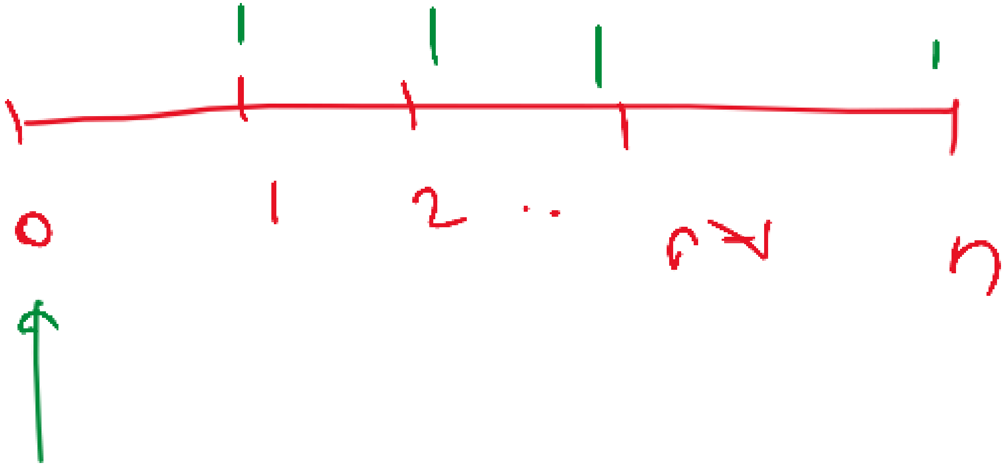
			1	1	1	1	1	1	1		
0	1	2	3	4	5	6	7	8	9	10	11
		↑	↑								

The values at the end of the 2nd and 3rd periods are given directly, either by annuities-immediate or by annuities-due, as labeled on the time diagram.

$\ddot{a}_{\overline{n}|}$

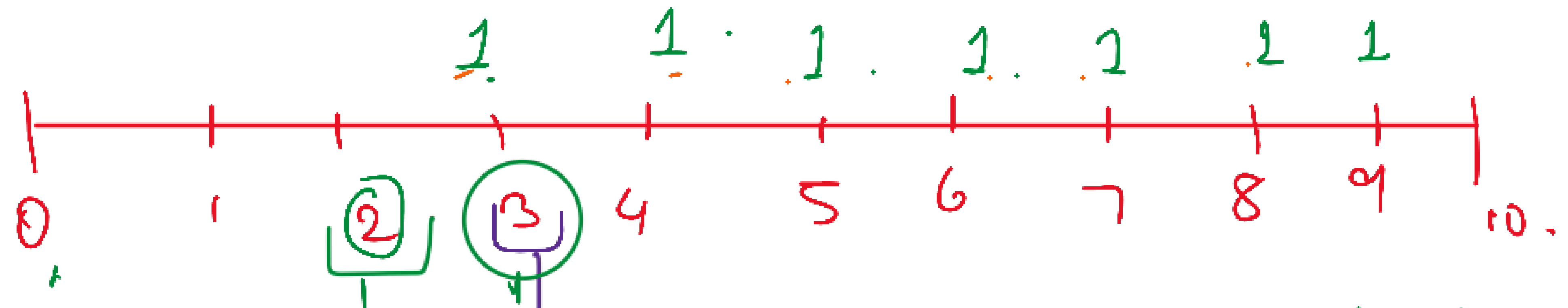


$a_{\overline{n}|}$



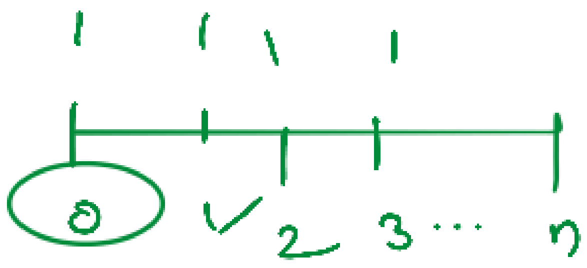
Starting at

$t = 0$



start

start

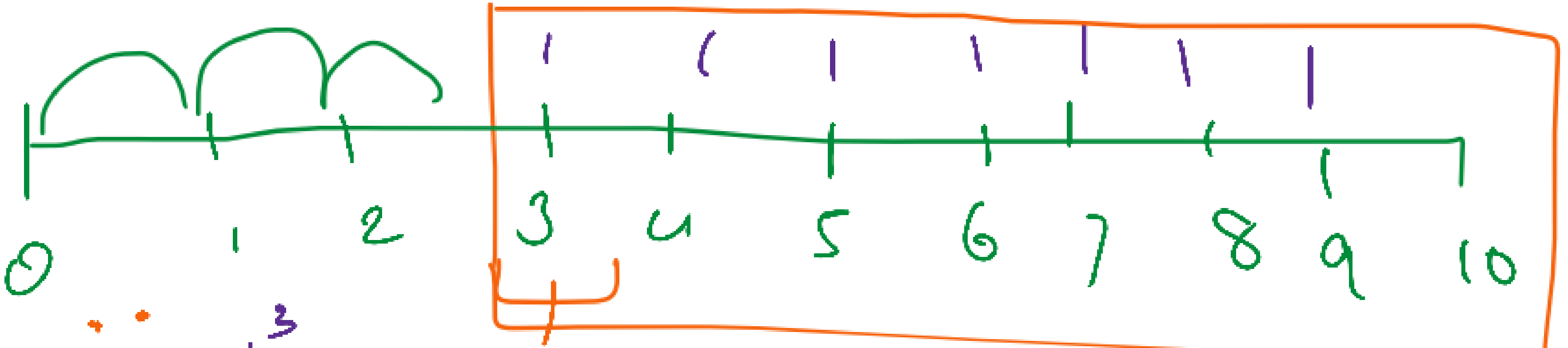


$\ddot{a}_{\overline{7}|}$

$\ddot{a}_{\overline{n}|}$

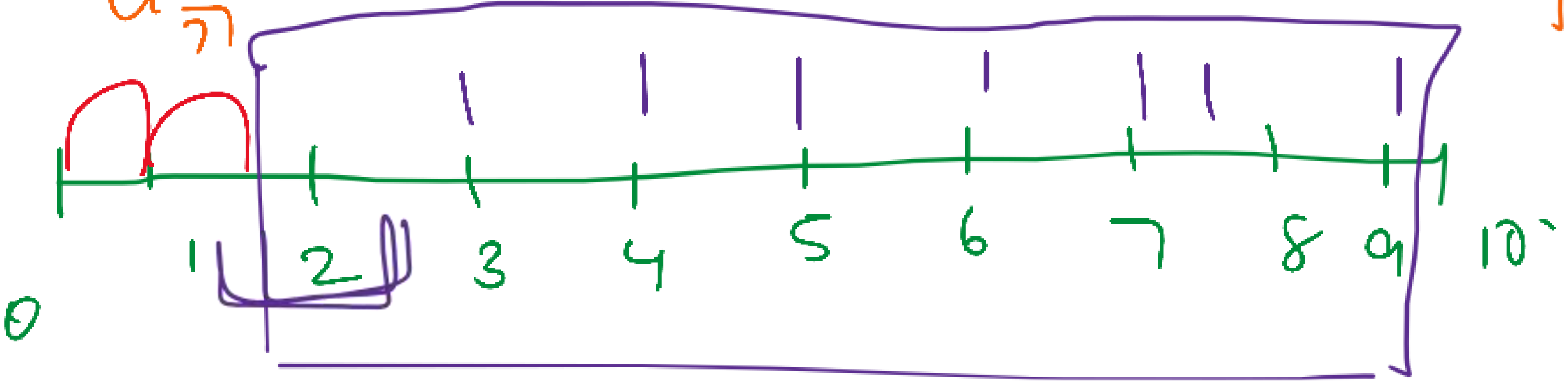
$$PV_8 = \ddot{a}_{\overline{7}|} \implies PV_0 = \ddot{a}_{\overline{7}|}\, v^3$$

$$PV_2 = a_{\overline{7}|} \implies PV_0 = a_{\overline{7}|}\, v^2$$



$3|\ddot{a}_{\overline{7}|}$

$\ddot{a}_{\overline{7}|}\, v^3$



$a_{\overline{7}|}\, v^2$

$2|a_{\overline{7}|}$

$m|a_{\overline{n}|} = a_{\overline{n}|}\, v^m$ | $m|\ddot{a}_{\overline{n}|} = \ddot{a}_{\overline{n}|}\, v^m$

2 Annuity values on any date

Present Value Calculation

In this case, the present value of the annuity at the beginning of the 1st period is seen to be the present value at the end of the 2nd period discounted for two periods, i.e

$$v^2 \, \boldsymbol{a}_{\overline{7}|}$$

It is possible to develop an alternate expression for this present value strictly in terms of annuity values. Temporarily assume that imaginary payments of 1 are made at the end of the 1st and 2nd periods. Then the present value of all nine payments at time t = 0 is $\boldsymbol{a}_{\overline{9}|}$. However, we must remove the present value of the imaginary payments, which is $\boldsymbol{a}_{\overline{2}|}$.

Thus, an alternate expression for the present value is

$$\boldsymbol{a}_{\overline{9}|} - \boldsymbol{a}_{\overline{2}|}$$

2 Annuity values on any date

Deferred Annuity

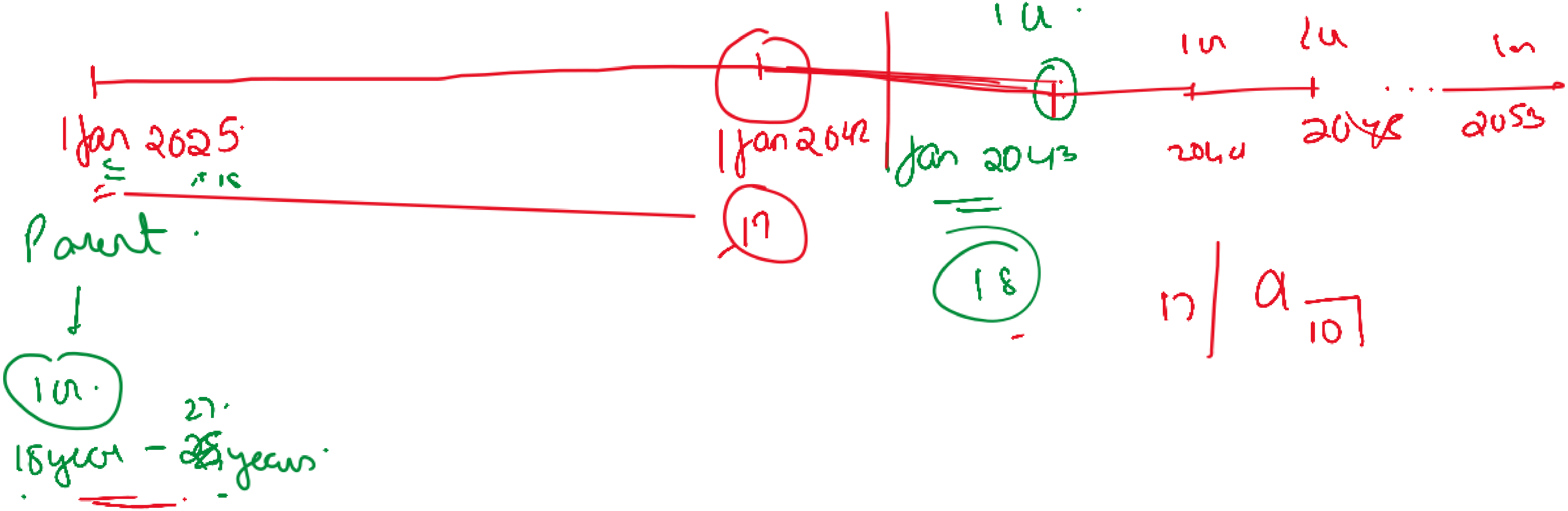
This type of annuity is often called ***a deferred annuity,*** since payments commence only after a deferred period. A symbol given to an annuity-immediate deferred for m periods with a term of n periods after the deferral period is . $_{m|}a_{\overline{n|}}$

Thus, the annuity being illustrated could be labeled $_{2|}a_{\overline{7|}}$

It is also possible to work with a deferred annuity-due. The reader should verify that the answer to this case, expressed as annuity-due, is:

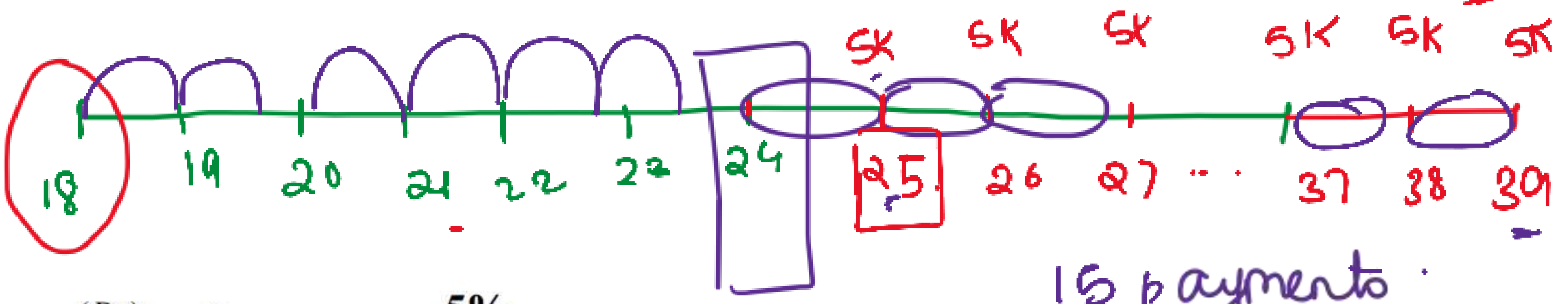
$$v^3\ddot{a}_{\overline{7|}} = \ddot{a}_{\overline{10|}} - \ddot{a}_{\overline{3|}}.$$

A symbol for this annuity would be $_{3|}\ddot{a}_{\overline{7|}}$


1 Jan 2025
1 Jan 2042
Jan 2043
2044
2045
2053
1L
1L
1L
1L
1
17
18
Parent
1L
18 years – 27 years
17
9
10

Question

On her 18th b'day, Latika receives an annuity that is to pay Rs. 5,000 on her 25th through 39th b'days. Calculate the value of the annuity on her 18th b'day using an annual effective interest rate of 5% p.a.



$i = 5\%$ pa

15 payments

$_{6|}a_{\overline{15}|}$

$_{7|}\ddot{a}_{\overline{15}|}$

$5000 \cdot a_{\overline{15}|} \cdot v^6$

$= 5000 \times 10.3797 \times 0.74622 = 38727.6987$

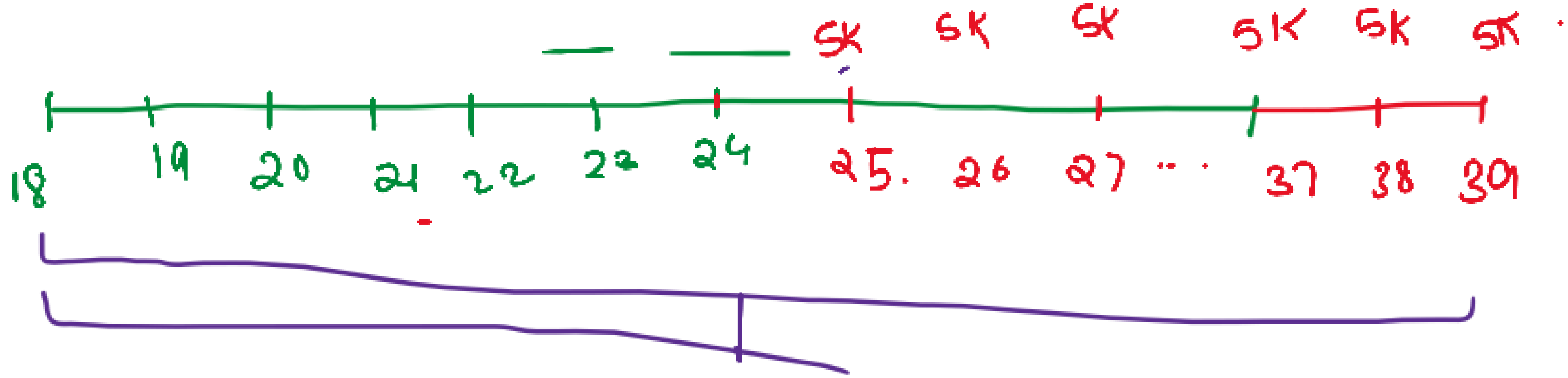
5%

$s_{\overline{n}\vert}$	$a_{\overline{n}\vert}$	$(Ia)_{\overline{n}\vert}$	$(Da)_{\overline{n}\vert}$	n
1.000 0	0.952 4	0.952 4	0.952 4	1
2.050 0	1.859 4	2.766 4	2.811 8	2
3.152 5	2.723 2	5.358 0	5.535 0	3
4.310 1	3.546 0	8.648 8	9.081 0	4
5.525 6	4.329 5	12.566 4	13.410 5	5
6.801 9	5.075 7	17.043 7	18.486 2	6
8.142 0	5.786 4	22.018 5	24.272 5	7
9.549 1	6.463 2	27.433 2	30.735 7	8
11.026 6	7.107 8	33.234 7	37.843 6	9
12.577 9	7.721 7	39.373 8	45.565 3	10
14.206 8	8.306 4	45.805 3	53.871 7	11
15.917 1	8.863 3	52.487 3	62.735 0	12
17.713 0	9.393 6	59.381 5	72.128 5	13
19.598 6	9.898 6	66.452 4	82.027 2	14
21.578 6	10.379 7	73.667 7	92.406 8	15

i	0.050 000
$i^{(2)}$	0.049 390
$i^{(4)}$	0.049 089
$i^{(12)}$	0.048 889
δ	0.048 790
$(1+i)^{1/2}$	1.024 695
$(1+i)^{1/4}$	1.012 272
$(1+i)^{1/12}$	1.004 074
v	0.952 381
$v^{1/2}$	0.975 900

$${}_{6.5|}a_{\overline{15}|}$$

$$5000\left(a_{\overline{21}|} - a_{\overline{6}|}\right) = 37827.50$$

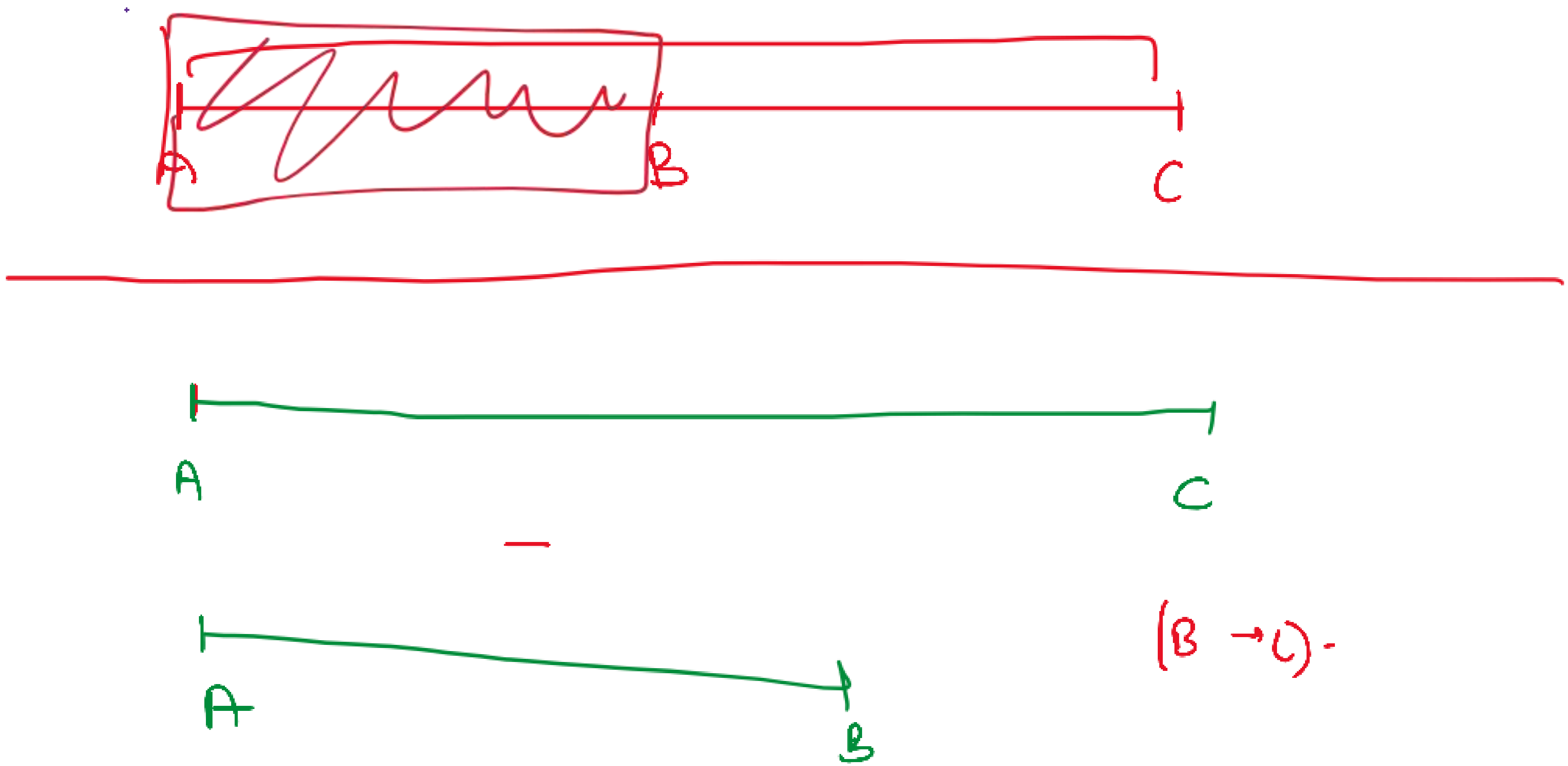


$i = 5\%$ pa

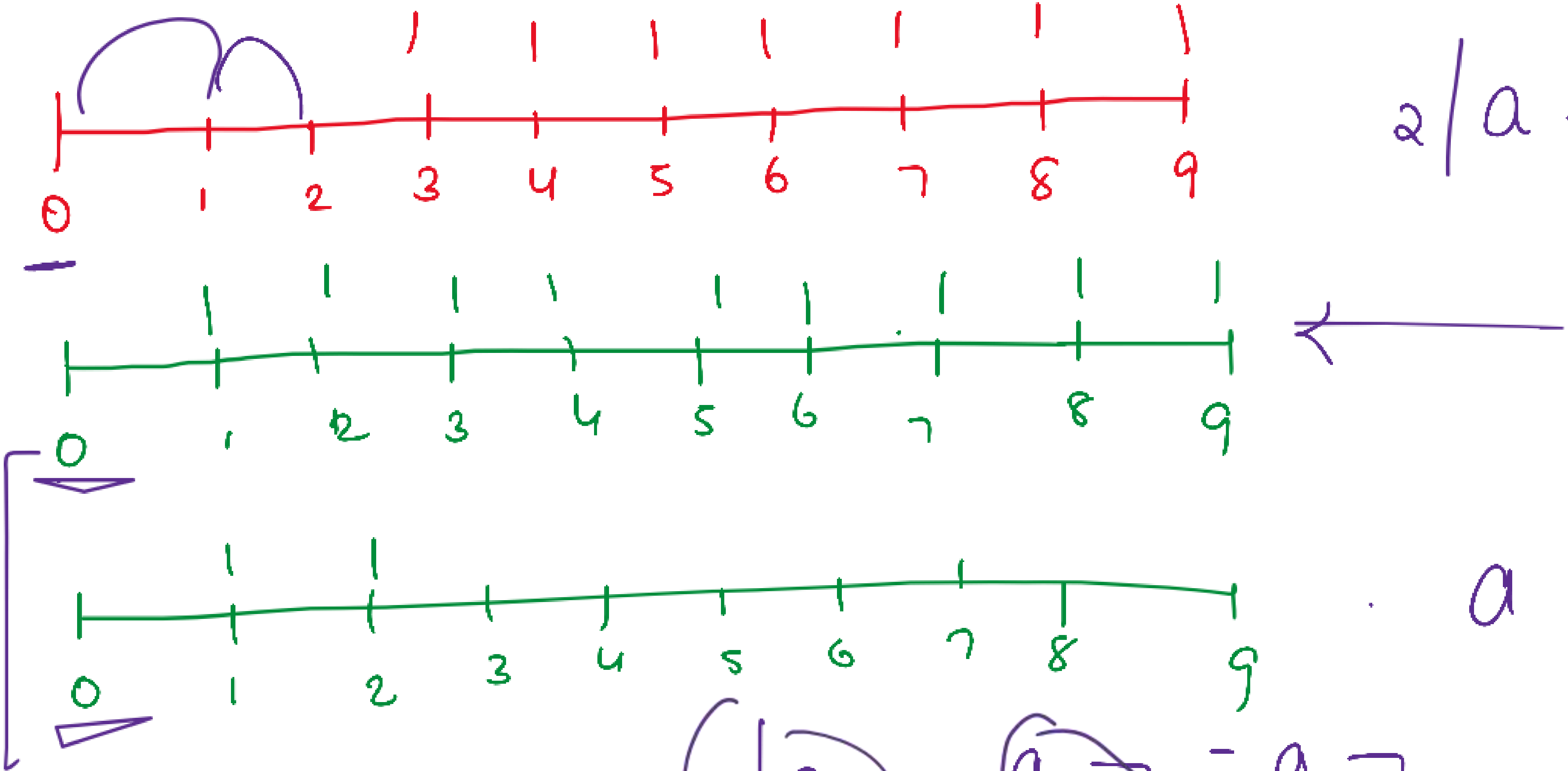
$$a_{\overline{15+6}|} - a_{\overline{6}|}$$

$${}_{7|}\ddot{a}_{\overline{13}|}$$

$$\boxed{\ddot{a}_{\overline{7+13}|} - \ddot{a}_{\overline{7}|}}$$


A
B
C
A
C
A
B
(B → C)

$${}_{m|}a_{\overline{n}|} = a_{\overline{m+n}|} - a_{\overline{m}|}$$



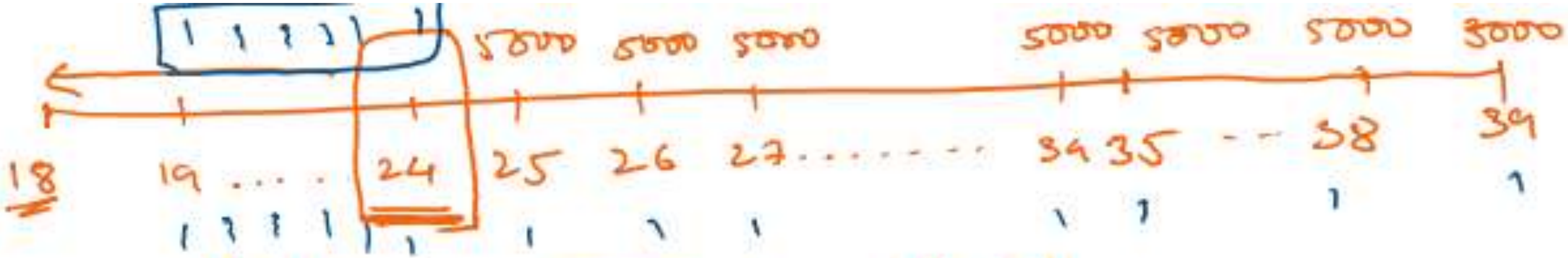
$${}_{2|}a_{\overline{7}|}$$

$$a_{\overline{9}|}$$

$$a_{\overline{2}|}$$

$${}_{2|}a_{\overline{7}|} = a_{\overline{9}|} - a_{\overline{2}|}$$

Solution



PV on 24^{th} b'day $= 5000\, a_{\overline{15}|}$ @5%

PV on 18^{th} b'day $= v^6\, 5000\, a_{\overline{15}|} = 38{,}727.30$

$PV_{18} = 5000\left(a_{\overline{21}|} - a_{\overline{6}|}\right) = 38{,}727.30$

Perpetuities - forever. $n = \infty$

Perpetual immediate annuity — end

Perpetual advance annuity — start

$a_{\overline{n}|}$

$\ddot{a}_{\overline{n}|}$

$m|a_{\overline{n}|}$

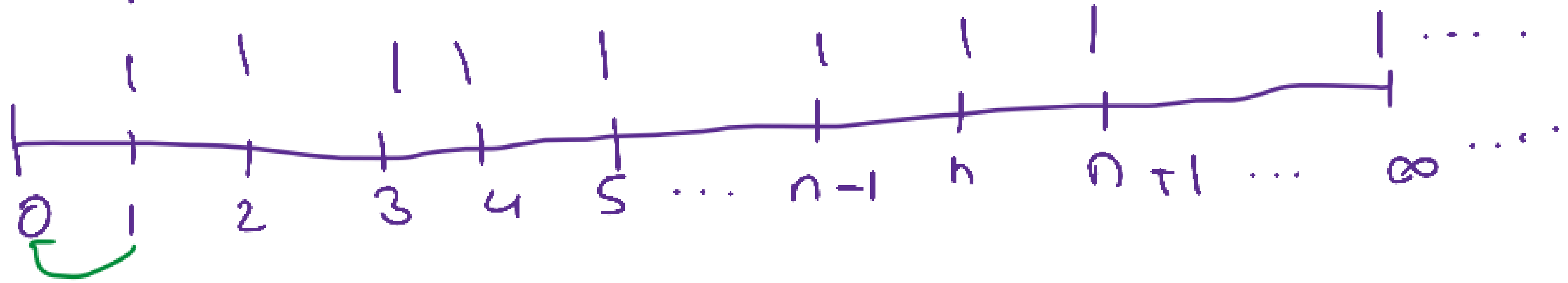
$m|\ddot{a}_{\overline{n}|}$

$a_{\overline{\infty}|}$

$\ddot{a}_{\overline{\infty}|}$

PV of a perpetuity (payments @ end)

$1 - v = d$



$$PV = 1v + 1v^2 + 1v^3 + 1v^4 + \dots$$

$$= 1\underbrace{\left(v + v^2 + v^3 + v^4 + \dots\right)}$$ ← Infinite G.P. series

$$S_\infty = \frac{a}{1-r}$$

$$a_{\overline{\infty|}} = \frac{v}{1-v} = \frac{v}{d} = \frac{v}{iv} = \frac{1}{i}$$

$$a_{\overline{\infty|}} = \frac{1}{i}$$

$$a_{\overline{\infty}|} = \lim_{n\to\infty} a_{\overline{n}|}$$

$$= \lim_{n\to\infty} \frac{1-v^n}{i}$$

$$= \frac{1 - \lim_{n\to\infty} v^n}{i} \quad 0$$

$$a_{\overline{\infty}|} = \frac{1}{i}$$

$$a_{\overline{n}|} = \frac{1-v^n}{i}$$

$$\cancel{\frac{1}{(1+i)^n}} \to 0 \qquad \cancel{\frac{1}{\infty}} \quad 0$$

$$\ddot{a}_{\overline{\infty}|} = \frac{1}{d}$$

$$\ddot{a}_{\overline{\infty}|} = 1 + v + v^2 + v^3 + \dots$$

$$\ddot{a}_{\overline{\infty}|} = \frac{1}{1 - v} = \frac{1}{d}$$

3 Perpetuities

A *perpetuity* is an annuity whose payments continue forever, i.e. the term of the annuity is not finite. Although it seems unrealistic to have an annuity with payments continuing forever, examples do exist in practice. The dividends on preferred stock with no redemption provision and the British consols, which are nonredeemable obligations of the British government, are examples of perpetuities.

Annuity-Immediate

The present value of a perpetuity-immediate is denoted by $\boldsymbol{a}_{\overline{\infty|}}$, which can be evaluated as the sum of an infinite geometric progression, giving

$$\boldsymbol{a}_{\overline{\infty|}} = \mathrm{v} + v^2 + v^3 + \ldots..$$

$$= \frac{v}{1 - v}$$

$$= \frac{v}{iv}$$

$$= \frac{1}{i}$$

3

Perpetuities

Annuity-Due

The present value of a perpetuity-immediate is denoted by $\ddot{a}_{\overline{\infty|}}$, which can be evaluated as the sum of an infinite geometric progression, giving

$$\ddot{a}_{\overline{\infty|}} = 1 + v + v^2 + v^3 + \ldots..$$

$$= \frac{1}{1-v}$$

$$= \frac{1}{d}$$

Question

end

next 10

A leaves an estate of $100000. Interest is paid to beneficiary B for first 10 years, to C for next 10 years and **to D thereafter.**

Find the present value of interest received by B, C and D if it is assumed that the estate will earn a 7% annual effective interest rate.

$7\% \times 100{,}000$

7000

$= A = \$100{,}000$

$7\% \times 100{,}000$

7K

B – 10 year

C – 10 year

D

$7000 \times v^{10}$

$\boxed{35}$ $a_{\overline{10}|}$

v^{10}

B

49165.07

C

24993.129

D

25842

$B = 10$

$C = 10$

$D =$ perpeuity

$$7000\, a_{\overline{\infty}|}\, v^{20}$$

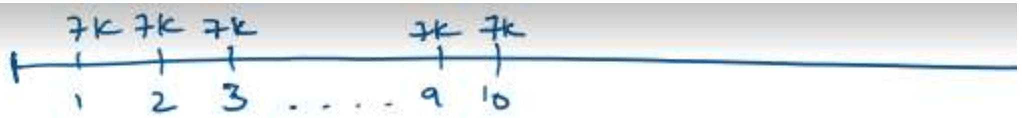
$$\frac{7000}{7\%} \times 0.25842 = 25842$$

$$10 \mid a_{\overline{10}|}$$

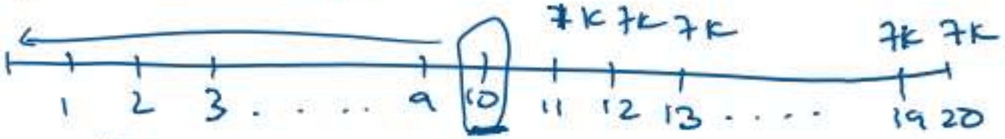
$$7000\, a_{\overline{10}|}\, v^{10}$$

3 5 . . .

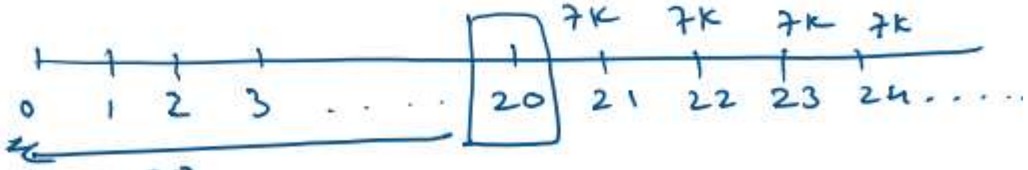
Solution



$$PV_B = 7000\, a_{\overline{10|}} = 49{,}165$$



$$PV_c = \underline{v^{10}\, 7000\, a_{\overline{10|}}} \quad \text{or} \quad a_{\overline{20|}} - a_{\overline{10|}} = 24{,}993$$



$$PV_D = v^{20}\, \underline{7000\, a_{\overline{\infty|}}} = 25{,}842$$

$$7000 \times \frac{1}{7\%} \times v^{20}$$

3.1 Differing Interest and Payment periods

We saw annuities where the quoted compounding interest period is the same as the annuity payment period. It may often be the case that the quoted compounding interest period may not coincide with the annuity payment period.

For the purpose of numerical evaluation of annuity, we focus on the annuity payment period and determine and use the interest rate per payment period that is equivalent to the quoted rate of interest

p-thly payable annuities

500

$\boxed{\$\,1000}$ → semi-annually with an eff. int rate of 7% pa

500   500   500   500

0   0.3   1   1.5   2

@ $\frac{i^{(2)}}{2}$

$500\, a_{\overline{4}|}$

$x\, a^{(p)}_{\overline{n}|}$ @ $i^{(p)}$

annual

(4) @ 12% pa

$x\, a_{\overline{4}|}$

1000 every year

quarterly

for 4 years.

$i^{(4)} = 12\%$ p.a.

end

CFs: $\frac{1}{p}$ $\frac{1}{p}$ $\frac{1}{p}$ $\frac{1}{p}$ $\frac{1}{p}$ ......

0 $\frac{1}{p}$ $\frac{2}{p}$ $\frac{3}{p}$ $\frac{4}{p}$ $\frac{5}{p}$ ... $\frac{(n-1)p}{p}$ $\frac{np}{p}$

annuity of 1$ per year payable @ end 'p' times a year for 'n' years

5

1

monthly pay

for 5 years

$\frac{1}{12}$ $\frac{1}{12}$ $\frac{1}{12}$ ... $\times 12$ times

qtly

4

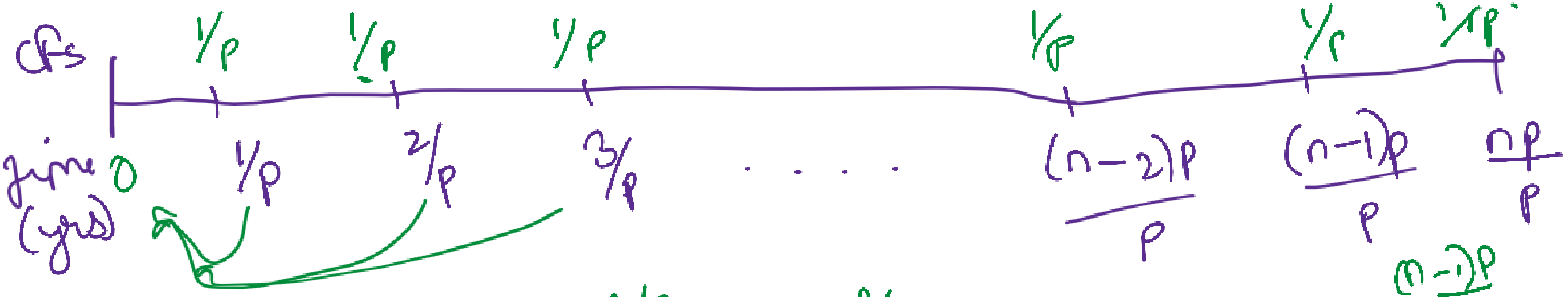
$\frac{1}{12}$ $\frac{2}{12}$ $\frac{3}{12}$ $\frac{4}{12}$. $\frac{1}{12}+\frac{1}{12}+\frac{1}{12}\ldots = \frac{12}{12} = \$1$

$4\times 4 = 16$

$= 4$yth

$\frac{4\times 12}{12}$ $\frac{\times p}{p}$

4



$$PV_0 = \frac{1}{p}v^{\frac{1}{p}} + \frac{1}{p}v^{2/p} + \frac{1}{p}v^{3/p} + \cdots + \frac{1}{p}v^{\frac{(n-1)p}{p}} + \frac{1}{p}v^{\frac{np}{p}}$$

60

$$\boxed{\frac{v^{2/p}}{v^{1/p}} = v^{1/p}}$$

$$= \frac{1}{p}\left(v^{\frac{1}{p}} + v^{2/p} + v^{3/p} + \cdots + v^{\frac{(n-1)p}{p}} + v^{\frac{np}{p}}\right)$$

$$= \frac{1}{p}\left(\frac{v^{\frac{1}{p}}\left(1-\left(v^{\frac{1}{p}}\right)^{np}\right)}{1-v^{\frac{1}{p}}}\right).$$

$$a\left(\frac{1-r^n}{1-r}\right).$$

$$PV_0 = \frac{1}{p}\left[\frac{v^{1/p}\left[1-\left(v^{\frac{1}{p}}\right)^{np}\right]}{1-v^{\frac{1}{p}}}\right]$$

$$\frac{1}{v^{\frac{1}{p}}} = \frac{1}{\left(\frac{1}{1+i}\right)^{\frac{1}{p}}} = (1+i)^{\frac{1}{p}}$$

$$\boxed{x\,a^{(p)}_{\overline{n}|}}$$

u

annual years

$$= \frac{1}{p}\left[\frac{1-v^n}{\left(\frac{1-v^{\frac{1}{p}}}{v^{1/p}}\right)}\right]$$

$$\ddot{a}^{(d)}_{\overline{n}|} = \frac{1-v^n}{d^{(n)}}$$

$$= \frac{1}{p}\left[\frac{1-v^n}{\frac{1}{v^{1/p}}-1}\right] = \frac{1}{p}\left[\frac{1-v^n}{(1+i)^{\frac{1}{p}}-1}\right]$$

$$a^{(p)}_{\overline{n}|} = \frac{(1-v^n)}{p\left[(1+i)^{\frac{1}{p}}-1\right]} = \frac{1-v^n}{i^{(p)}}$$

4 P-thly Payable Annuities

P-thly payable Annuity-Immediate

It is denoted as $a_{\overline{n|}}^{(p)}$.

- The effective annual interest rate is i.
- Payments of amount 1/p each occur at the end of every 1/p – year period. (Total amount paid per year is 1).
- Payments are made or received for n years.
- There would be total m x n payments.
- Valuation point is the time one payment period before the first payment.

Question 5.2

Find the present value of annuity which pays $\boxed{\text{Rs. 500}}$ at the end of each half-year for 20 years if the rate of interest is 9% p.a. convertible semiannually.

500 is being paid every 6 months

20 years

$n =$ years

$i^{(2)} = 9\%$ pa

(2) @ $i^{(2)} = 9\%$ pa

@ 4.5% i

$500\, a_{\overline{40|}}$

$$1000\, a_{\overline{20|}}$$

$$= 1000 \left[\frac{1 - v^{20}}{i^{(2)}} \right]$$

$$= 1000 \left[\frac{1 - \left(\frac{1}{1+i}\right)^{20}}{9\%} \right]$$

$$1000\, a^{(2)}_{\overline{20}|} = 1000\left[\frac{1-v^{20}}{i^{(2)}}\right]$$

$$v = \frac{1}{1+i}$$

off 20%

pa

$$(1+i) = \left(1+\frac{i^{(p)}}{p}\right)^{p}$$

$$i = \left(1+\frac{i^{(2)}}{2}\right)^{2} - 1$$

$$i = \left(1+\frac{9\%}{2}\right)^{2} - 1$$

$$\boxed{9200.79}$$

4.5%

$$500\, a_{\overline{40}|}$$

9200.71

4

P-thly Payable Annuities

Present value of P-thly payable Annuity-Immediate is found by summing a geometric progression as:

$$a_{\overline{n|}}^{(p)} = \frac{1}{p}\left[v^{(\frac{1}{p})} + v^{(\frac{2}{p})} + \text{........} + v^{(\frac{n-1}{p})} + v^{n} \right]$$

$$= \frac{1}{p}\left[\frac{v^{(\frac{1}{p})} - v^{(n+1/p)}}{1 - v^{(\frac{1}{p})}} \right]$$

$$= \frac{1 - v^{n}}{p\left[(1+i)^{\frac{1}{p}} - 1 \right]}$$

$$= \frac{1 - v^{n}}{i^{(p)}}$$

4

P-thly Payable Annuities

P-thly payable Annuity-Due

It is denoted as $\ddot{a}_{\overline{n}|}^{(p)}$.

- The effective annual interest rate is i.
- Payments of amount 1/p each occur at the start of every 1/p – year period. (Total amount paid per year is 1).
- Payments are made or received for n years.
- There would be total m x n payments.
- Valuation point is the time at t=0.

4

P-thly Payable Annuities

Present value of P-thly payable Annuity-Due is found similarly by summing a geometric progression as:

$$\ddot{a}_{\overline{n|}}^{(p)} = \frac{1}{p}\left[1 + v^{\left(\frac{1}{p}\right)} + v^{\left(\frac{2}{p}\right)} + \dots\dots.. + v^{\left(\frac{n-1}{p}\right)} + v^{n} \right]$$

Solving further gives;

$$\ddot{a}_{\overline{n|}}^{(p)} = \frac{1 - v^{n}}{d^{(p)}}$$

4 P-thly Payable Annuities

Perpetuities payable p-thly

- The present value of payments of 1 *pa* payable in instalments of 1/*p* at the end of each *p*-thly time period forever is:

$$a^{(p)}_{\overline{\infty|}} = \frac{\mathbf{1}}{\boldsymbol{i}^{(\boldsymbol{p})}}$$

- The present value of payments of 1 *pa* payable in instalments of 1/*p* at the start of each *p*-thly time period forever is:

$$\ddot{a}^{(p)}_{\overline{\cancel{n}|}} = \frac{\mathbf{1}}{\boldsymbol{d}^{(\boldsymbol{p})}}$$

∞

Question

Describe what the following quantity represents and compute the value for i = 5% per annum.

$$a_{\overline{10|}}^{(2)}$$

7.8107

n	$(1+i)^n$	v^n	$s_{\overline{n}|}$	$a_{\overline{n}|}$	$(Ia)_{\overline{n}|}$	$(Da)_{\overline{n}|}$	n
1	1.050 00	0.952 38	1.000 0	0.952 4	0.952 4	0.952 4	1
2	1.102 50	0.907 03	2.050 0	1.859 4	2.766 4	2.811 8	2
3	1.157 63	0.863 84	3.152 5	2.723 2	5.358 0	5.535 0	3
4	1.215 51	0.822 70	4.310 1	3.546 0	8.648 8	9.081 0	4
5	1.276 28	0.783 53	5.525 6	4.329 5	12.566 4	13.410 5	5
6	1.340 10	0.746 22	6.801 9	5.075 7	17.043 7	18.486 2	6
7	1.407 10	0.710 68	8.142 0	5.786 4	22.018 5	24.272 5	7
8	1.477 46	0.676 84	9.549 1	6.463 2	27.433 2	30.735 7	8
9	1.551 33	0.644 61	11.026 6	7.107 8	33.234 7	37.843 6	9
10	1.628 89	0.613 91	12.577 9	7.721 7	39.373 8	45.565 3	10
11	1.710 34	0.584 68	14.206 8	8.306 4	45.805 3	53.871 7	11
12	1.795 86	0.556 84	15.917 1	8.863 3	52.487 3	62.735 0	12
13	1.885 65	0.530 32	17.713 0	9.393 6	59.381 5	72.128 5	13
14	1.979 93	0.505 07	19.598 6	9.898 6	66.452 4	82.027 2	14
15	2.078 93	0.481 02	21.578 6	10.379 7	73.667 7	92.406 8	15
16	2.182 87	0.458 11	23.657 5	10.837 8	80.997 5	103.244 6	16
17	2.292 02	0.436 30	25.840 4	11.274 1	88.414 5	114.518 7	17
18	2.406 62	0.415 52	28.132 4	11.689 6	95.893 9	126.208 3	18
19	2.526 95	0.395 73	30.539 0	12.085 3	103.412 8	138.293 6	19
20	2.653 30	0.376 89	33.066 0	12.462 2	110.950 6	150.755 8	20
21	2.785 96	0.358 94	35.719 3	12.821 2	118.488 4	163.576 9	21
22	2.925 26	0.341 85	38.505 2	13.163 0	126.009 1	176.739 9	22
23	3.071 52	0.325 57	41.430 5	13.488 6	133.497 3	190.228 5	23
24	3.225 10	0.310 07	44.502 0	13.798 6	140.938 9	204.027 2	24
25	3.386 35	0.295 30	47.727 1	14.093 9	148.321 5	218.121 1	25
26	3.555 67	0.281 24	51.113 5	14.375 2	155.633 7	232.496 3	26
27	3.733 46	0.267 85	54.669 1	14.643 0	162.865 6	247.139 3	27
28	3.920 13	0.255 09	58.402 6	14.898 1	170.008 2	262.037 5	28
29	4.116 14	0.242 95	62.322 7	15.141 1	177.053 7	277.178 5	29
30	4.321 94	0.231 38	66.438 8	15.372 5	183.995 0	292.551 0	30

5%	
i	0.050 000
$i^{(2)}$	0.049 390
$i^{(4)}$	0.049 089
$i^{(12)}$	0.048 889
δ	0.048 790
$(1+i)^{1/2}$	1.024 695
$(1+i)^{1/4}$	1.012 272
$(1+i)^{1/12}$	1.004 074
v	0.952 381
$v^{1/2}$	0.975 900
$v^{1/4}$	0.987 877
$v^{1/12}$	0.995 942
d	0.047 619
$d^{(2)}$	0.048 200
$d^{(4)}$	0.048 494
$d^{(12)}$	0.048 691
$i/i^{(2)}$	1.012 348
$i/i^{(4)}$	1.018 559
$i/i^{(12)}$	1.022 715

1.012348

$\times\ 7.\ a_{\overline{10|}}$

7.7217

Solution

$$= \frac{1 - v^{10}}{i^{(2)}} = \frac{1 - 0.61391}{0.049390} = 7.82$$

$$a_{\overline{n}|} = \frac{1-v^n}{i}$$

$$\ddot{a}_{\overline{n}|} = \frac{1-v^n}{d}$$

$$a^{(p)}_{\overline{n}|} = \frac{1-v^n}{i^{(p)}}$$

$$\ddot{a}^{(p)}_{\overline{n}|} = \frac{1-v^n}{d^{(p)}}$$

Relationships b/a $a_{\overline{n}|}$ and $\ddot{a}_{\overline{n}|}$

$$a_{\overline{\infty}|} = \frac{1}{i} \quad \text{and} \quad \ddot{a}_{\overline{\infty}|} = \frac{1}{d}$$

$$a^{(p)}_{\overline{\infty}|} = \frac{1}{i^{(p)}} \qquad \ddot{a}^{(p)}_{\overline{\infty}|} = \frac{1}{d^{(p)}}$$

annual amt. ~~amt~~ annual amt.

$$m|a_{\overline{n}|} = a_{\overline{n}|} \cdot v^m$$

$$a_{\overline{m+n}|} - a_{\overline{m}|}$$

Question

At what annual effective rate of interest is the present value of a series of payment of $1 every six months forever, with the first payment made immediately, equal to $10?

To find : i → eff roi pa

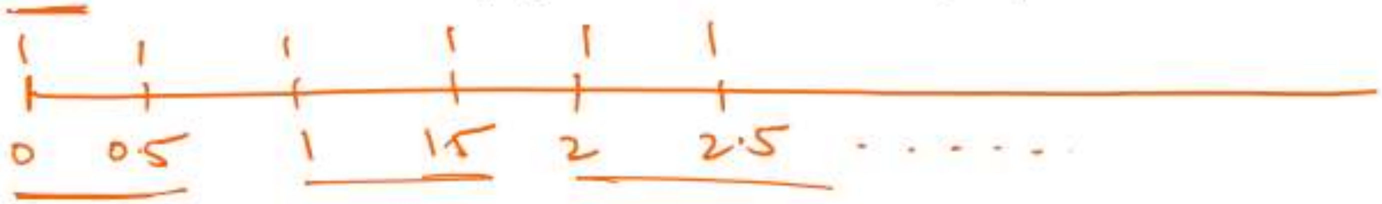
$$\boxed{2\ddot{a}^{(2)}_{\overline{\infty}|} = 10}$$

10.8033%

0

23.4567%

Solution



P.V. = \$10

P.V. $= 2\,\ddot{a}^{(2)}_{\overline{\infty}|}$ $\therefore$ $10 = \dfrac{2}{d^{(2)}}$

$d^{(2)} = 0.2$ $\quad (1+i) = \left(1 - \dfrac{d^{(2)}}{2}\right)^{-2}$

$i = 0.2346$ i.e 23.46%

$$\ddot{a}^{(p)}_{\overline{\infty}|} = \frac{1}{d^{(p)}}$$

$$(1+i) = \left(1 - \frac{d^{(2)}}{2}\right)^{-2}$$

5 Continuous Annuities

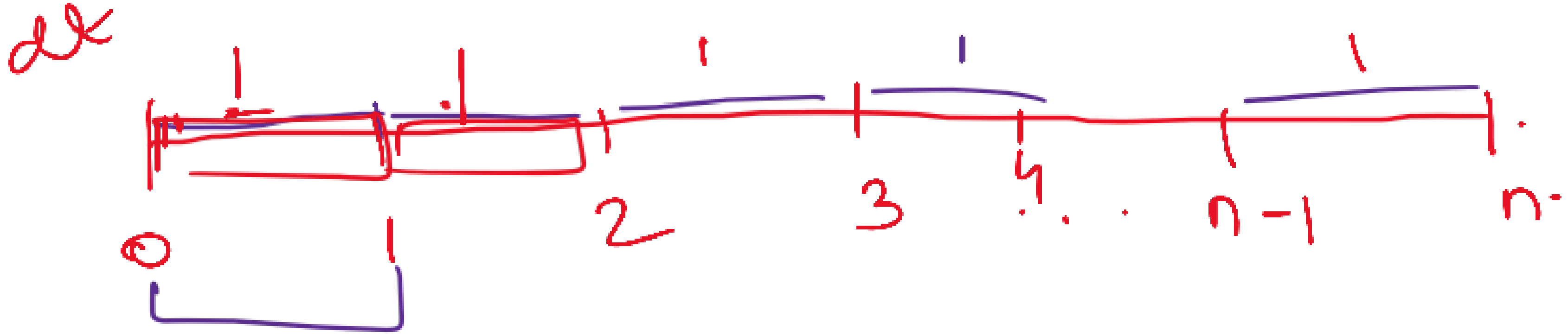
A special case of annuities payable more frequently than interest is convertible is one in which the frequency of payment becomes infinite, i.e. payments are made continuously. Although difficult to visualize in practice, a continuous annuity is of considerable theoretical and analytical significance. Also, it is useful as an approximation to annuities payable with great frequency, such as daily.

We will denote the present value of an annuity payable continuously for *n* interest conversion periods, such that the total amount paid during each interest conversion period is 1, by the symbol $\bar{a}_{\overline{n|}}$.

An expression for $\bar{a}_{\overline{n|}}$ is

$$\bar{a}_{\overline{n|}} = \int_0^n v^t \, dt$$

since the differential expression $v^t dt$ is the present value of the payment *dt* made at exact moment t.



$a^{(p)}_{\overline{n}|}$ — $p=2$, $p=4$, $p=12$

Cont. annuity

$$a^{(\infty)}_{\overline{n}|} = \lim_{p\to\infty} a^{(p)}_{\overline{n}|} = \lim_{p\to\infty}\left[\frac{1-v^n}{i^{(p)}}\right]$$

$$= \frac{1-v^n}{\lim\limits_{p\to\infty} i^{(p)}} = \frac{1-v^n}{\delta} = \bar{a}_{\overline{n}|}$$

$$\bar{a}_{\overline{n}|} = \int_0^n 1\, v^t\, dt$$

$$1\,dt \qquad \frac{1}{} \qquad 1 \cdot dt$$

$$= \int_0^n e^{-\delta t}\, dt$$

$$= \left[\frac{e^{-\delta t}}{-\delta}\right]_0^n = \left[\frac{e^{-\delta n} - 1}{-\delta}\right] = \frac{1 - e^{-\delta n}}{\delta}$$

$$\boxed{\bar{a}_{\overline{n}|} = \frac{1 - v^n}{\delta}}$$

5 Continuous Annuities

A simplifies expression for $\bar{a}_{\overline{n|}}$ is obtained by performing the integration

$$\bar{a}_{\overline{n|}} = \int_0^n v^t \ dt$$

$$= [\frac{v^t}{\ln v}]_0^n$$

$$= \frac{1 - v^n}{\delta}$$

Thus, the continuous annuity is seen to be the limiting case of annuities payable mthly as m -> ∞.

Question

Smit deposits 12 every day in an account during the year 2005 and 2006. The account earns interest from the exact time of deposit, with interest quoted as an effective annual rate. The rate is 12% in 2005 and 2006.

Find the present value of amount in the account

a) Exactly based on daily deposits

b) assuming the approximation that the deposits are made continuously

365

$p = 365$

$(12 \times 365)\, a_{\overline{2}|}^{(365)}$

$i = 12\%$ pa effective

$= \left(1 + \frac{i^{(p)}}{p}\right)^p$

$12\, a_{\overline{730}|}$ @ $\frac{i^{(365)}}{365}$

0, 1d, 2d, 3d, 4d, . . . , 364, 365: 12, 12, 12, 12, . . . , 12, 12

$= 7836.9648$

$i^{(4)} = 12\%$ pa

$\frac{12\%}{4}$

$$(1+i) = \left(1+\frac{i^{(365)}}{365}\right)^{365}$$

$$\left[(1+12\%)^{\frac{1}{365}} - 1\right] = \frac{i^{(365)}}{365}$$

$$\frac{i^{(365)}}{365} = 0.03105\%$$

$$12\, a_{\overline{730}|}^{@0.03105\%} = 12\left[\frac{1-v^{730}}{0.03105\%}\right] = 12\left[\frac{1-\left(\frac{1}{1+0.03105\%}\right)^{730}}{0.03105\%}\right]$$

$$=$$

$$\boxed{v = \left(\frac{1}{1+i}\right) : 1- \quad v = (1-d)}$$

$$\boxed{(12 \times 365)}\, \bar{a}_{\overline{2}|} = (12 \times 365)\left[\frac{1 - v^2}{\delta}\right] \qquad \bar{a}_{\overline{n}|}$$

$$\delta = \ln(1+i) \qquad \left(\frac{1}{1+i}\right) \qquad = (12 \times 365)\left[\frac{1 - \left(\frac{1}{1+12\%}\right)^2}{\ln(1+12\%)}\right]$$

$$(1+i)^2 = e^{\delta}$$

$$2\ln(1+i) = \delta$$

$$= 7838.2291$$

$$18$$

Solution

(a) $12 a_{\overline{730}|}$ @ $\frac{i^{(365)}}{365} = 12\left[\frac{1-v^{730}}{i}\right] = 7837.07$

$i = 12\%$ $\quad \frac{i^{(365)}}{365} = 0.03105\%$ $\quad (1+0.03105\%)^{-730}$

(b) $4380\, \bar{a}_{\overline{2}|} = 4380\left(\frac{1-v^2}{\delta}\right)$

$i = 12\% \Rightarrow \delta =$ $\quad = 4380\left(\frac{1-0.797194}{0.113329}\right)$

$= 7838.31$

PV factors

∞

PV.

$$\left[\begin{array}{ll} \ddot{a}_{\overline{n}|} & a_{\overline{n}|} \\ a^{(p)}_{\overline{n}|} & \ddot{a}^{(p)}_{\overline{n}|} \\ a_{\overline{\infty}|} & \ddot{a}_{\overline{\infty}|} \\ {}_{m|}a_{\overline{n}|} & {}_{m|}\ddot{a}_{\overline{n}|} \end{array}\right.$$

$$\bar{a}_{\overline{n}|}$$

$=$

6 Accumulated values of Annuities

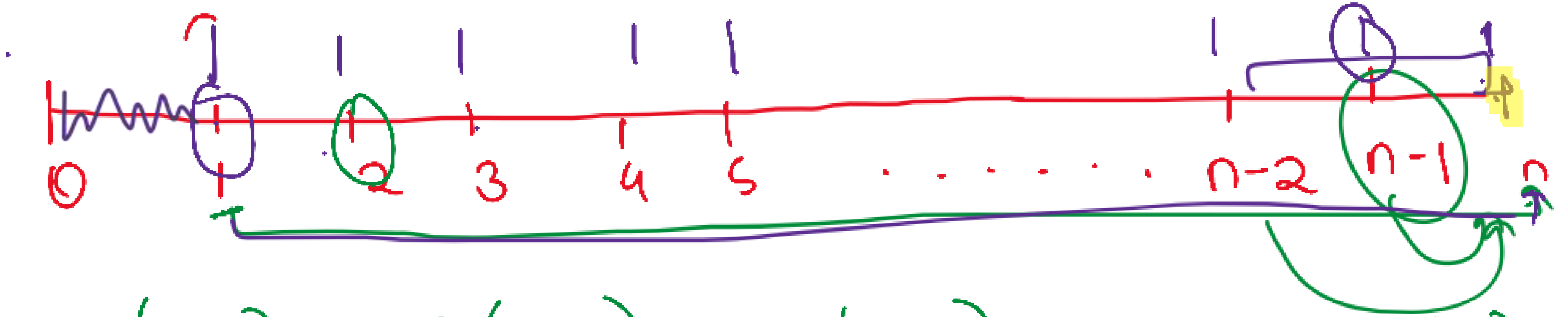
Up till here we considered the present values of different types of annuities. Now we look at the accumulated values of annuities for it's various types.

1] Annuity Immediate

Consider an annuity under which payments of 1 are made at the end of each period for n periods, where n is a positive integer.

The valuation is illustrated on the following timeline:

Time	0	1	2	……	n
Payment		1	1	…	1
Value					



AV fo imm annuity

$$AV = 1\,A(n-1) + 1\,A(n-2) + 1\,A(n-3) + \cdots + 1\,A(1) + 1$$

$$= 1(1+i)^{n-1} + 1.(1+i)^{n-2} + 1(1+i)^{n-3} + \cdots + 1(1+i)^{1} + 1$$

$$= 1 + 1(1+i)^{1} + 1(1+i)^{2} + \cdots + 1(1+i)^{n-3} + 1(1+i)^{n-2} + 1(1+i)^{n-1}$$

$$= 1\underbrace{\left[1 + (1+i)^{1} + (1+i)^{2} + \cdots + (1+i)^{n-3} + (1+i)^{n-2} + (1+i)^{n-1}\right]}_{GP}.$$

$r = (1+i)$

$$\boxed{S_n = a\left(\frac{1-r^n}{1-r}\right) \qquad S_n = a\left(\frac{r^n-1}{r-1}\right)}$$

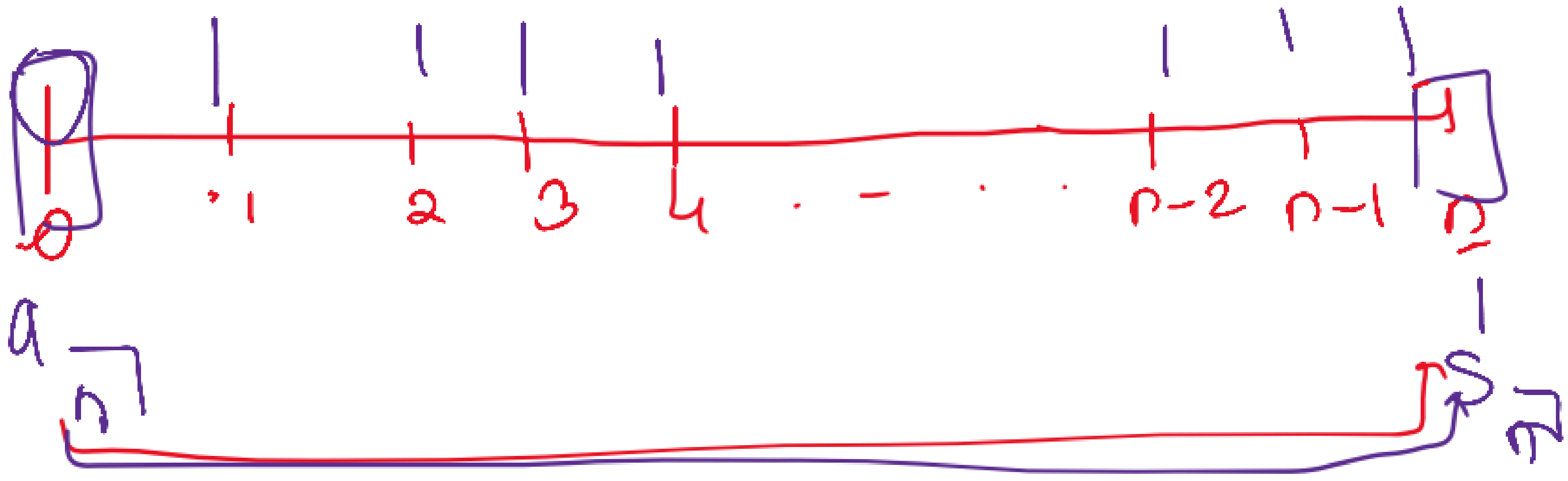
$-1 < r < 1$

$r > 1$

$$AV = 1 + (1+i)^1 + (1+i)^2 + \cdots + (1+i)^{n-2} + (1+i)^{n-1}$$

$$= 1\left[\frac{(1+i)^n - 1}{(1+i) - 1}\right] = \boxed{\frac{(1+i)^n - 1}{i} = s_{\overline{n}|}}$$

AV of imm. annuity



$$a_{\overline{n}|}(1+i)^n = s_{\overline{n}|}$$

6 Accumulated values of Annuities

An expression for $S_{\overline{n|}}$ can be derived in an analogous manner as an equation of value at the end of the nth period. The accumulated value of a payment of 1 made at the end of the first period is $(1+i)^{n-1}$ The accumulated value of a payment of 1 made at the end of the second period is $(1+i)^{n-2}$. This process is continued until the accumulated value of a payment of 1 made at the end of the *n*th period is just 1. The total accumulated value $S_{\overline{n|}}$ must equal the sum of the accumulated values of each payment, i.e.

$$S_{\overline{n|}} = 1 + (1+i) + \ldots\ldots + (1+i)^{n-2} + (1+i)^{n-1}$$

Again, a more compact expression can be derived by summing the geometric progression

$$S_{\overline{n|}} = 1 + (1+i) + \ldots\ldots + (1+i)^{n-2} + (1+i)^{n-1}$$

$$= \frac{(1+i)^n - 1}{(1+i)-1}$$

$$= \frac{(1+i)^n - 1}{i}$$

Question

Find the accumulated value of annuity which pays Rs.500 at the end of every year for 5 years if the rate of interest is 10% pa effective.

500×5

$2500 >$

$@\ i = 10\%\ pa$

$500\, s_{\overline{5|}}$

$= 500 \left[\dfrac{(1+10\%)^5 - 1}{10\%} \right] = 3052.55$

$500\, a_{\overline{5|}} \ @\ i = 10\% \ (1+10\%)^5$

10%

		n	$(1+i)^n$	v^n	$s_{\overline{n\|}}$	$a_{\overline{n\|}}$	$(Ia)_{\overline{n\|}}$	$(Da)_{\overline{n\|}}$
i	0.100 000	1	1.100 00	0.909 09	1.000 0	0.909 1	0.909 1	0.909 1
$i^{(2)}$	0.097 618	2	1.210 00	0.826 45	2.100 0	1.735 5	2.562 0	2.644 6
$i^{(4)}$	0.096 455	3	1.331 00	0.751 31	3.310 0	2.486 9	4.815 9	5.131 5
$i^{(12)}$	0.095 690	4	1.464 10	0.683 01	4.641 0	3.169 9	7.548 0	8.301 3
		5	1.610 51	0.620 92	6.105 1	3.790 8	10.652 6	12.092 1

Solution

$i = 10\%$ p.a.

$$500\, s_{\overline{5}|} = 500\left(\frac{(1.1)^5 - 1}{0.1}\right)$$

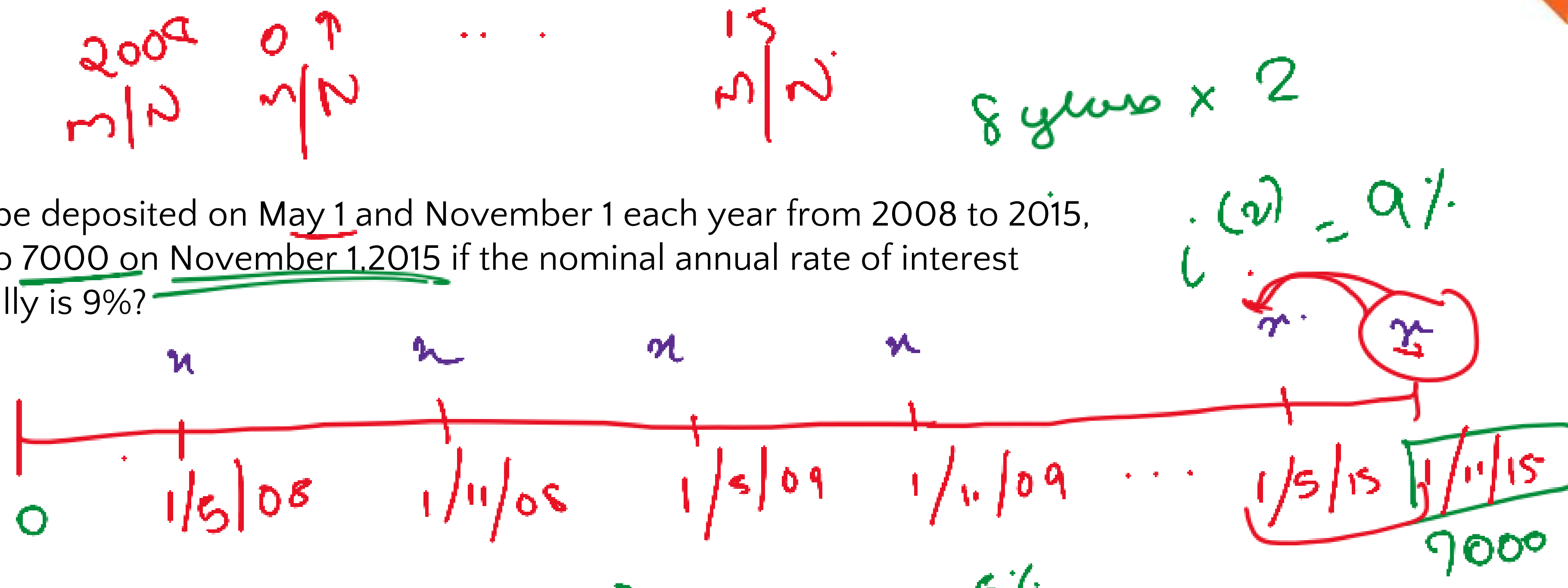
$$= 3052.55$$

Question

What level amount must be deposited on May 1 and November 1 each year from 2008 to 2015, inclusive, to accumulate to 7000 on November 1,2015 if the nominal annual rate of interest compounded semi-annually is 9%?

8 years × 2

$i^{(2)} = 9\%$



$\boxed{308.107}$

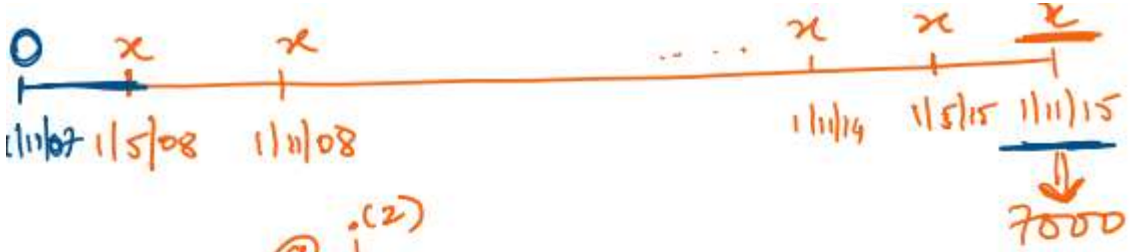
$$x \cdot \left[\frac{(1+4.5\%)^{16}-1}{4.5\%}\right] = 7000$$

$$x = 308.$$

$$x\, s_{\overline{16}|} \;@\; \frac{i^{(2)}}{2} = \frac{9\%}{2} = 4.5\%$$

$$= 7000$$

Solution



$x \, s_{\overline{16}|} \; @ \; \frac{i^{(2)}}{2}$

308.10

6 Accumulated values of Annuities

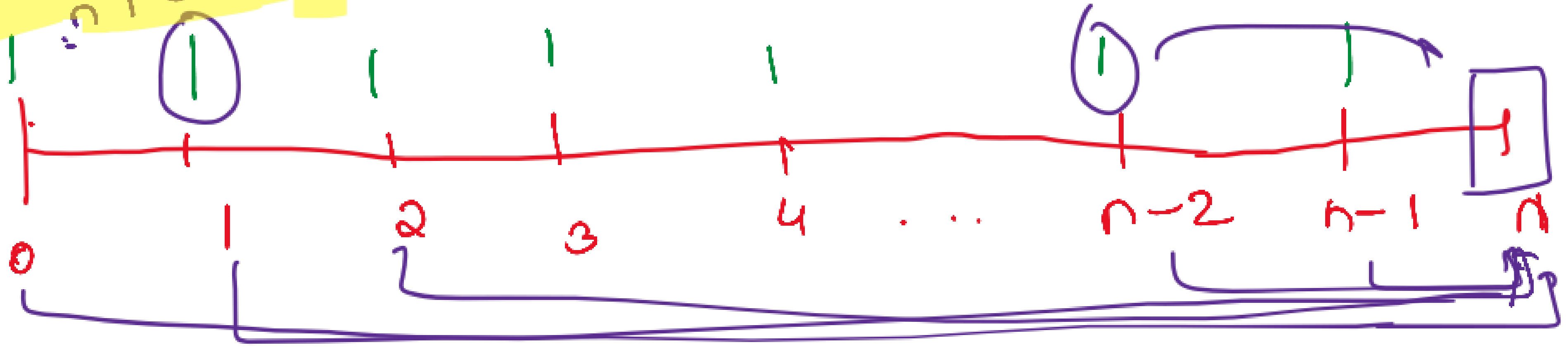
2] Annuity-Due

Consider an annuity under which payments of 1 are made at the start of each period for n periods, where n is a positive integer.

The valuation is illustrated on the following timeline:

Time	0	1	2	……	n-1	n
Payment	1	1	1	…	1	
Value						

$$\ddot{s}_{\overline{n}|} = \ddot{a}_{\overline{n}|}(1+i)^n$$



$$AV = 1\times A(n) + 1\times A(n-1) + 1A(n-2) + \dots + 1A(2) + 1A(1)$$

$$= 1\cdot(1+i)^n + 1\times(1+i)^{n-1} + 1(1+i)^{n-2} + \dots + 1(1+i)^2 + 1(1+i)^1$$

$$\boxed{a\left(\frac{r^n-1}{r-1}\right)}$$

$$= \underbrace{(1+i)^1 + (1+i)^2 + \dots + (1+i)^{n-2} + (1+i)^{n-1} + (1+i)^n}_{GP}$$

$$= (1+i)\left[\frac{(1+i)^n-1}{1+i-1}\right] = (1+i)\left[\frac{(1+i)^n-1}{i}\right] = \frac{(1+i)^n-1}{\left(\frac{i}{1+i}\right)} = \frac{(1+i)^n \dots}{d}$$

$$\ddot{s}_{\overline{n}|}$$

6 Accumulated values of Annuities

The accumulated value of an annuity-due is given by $\ddot{S}_{\overline{n|}}$, it can evaluated as

$$\ddot{S}_{\overline{n|}} = (1+i) + \ldots\ldots + (1+i)^{n-2} + (1+i)^{n-1} + (1+i)^{n}$$

$$= (1+i)\,\frac{(1+i)^{n}-1}{(1+i)-1}$$

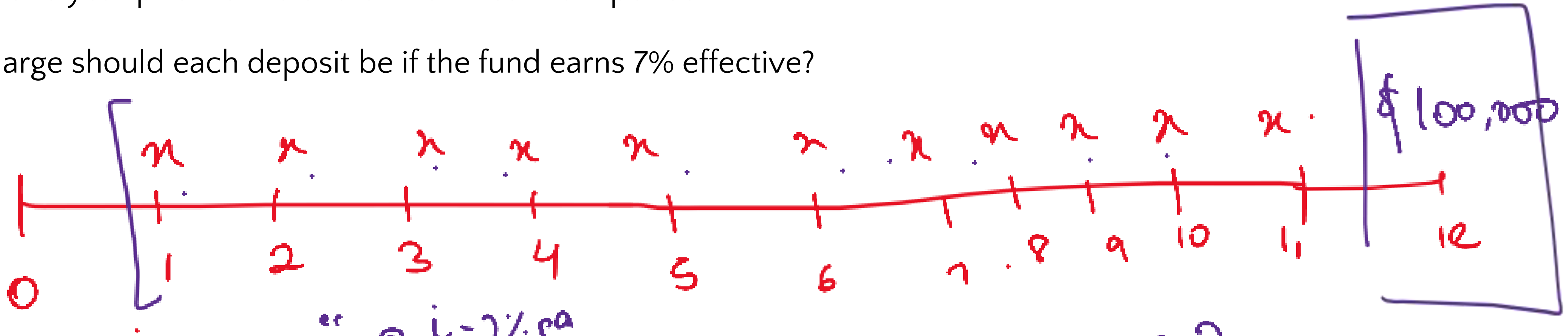
$$= \frac{(1+i)^{n}-1}{i\,v}$$

$$= \frac{(1+i)^{n}-1}{d}$$

Question

An investor wishes to accumulate $100,000 in a fund for retirement at the end of 12 years. To accomplish this the investor plans to make deposits at the end of each year, the final payment to be made one year prior to the end of the investment period.

How large should each deposit be if the fund earns 7% effective?



$x\, \ddot{s}_{\overline{11}|}$ @ $i = 7\%$ pa $= 100,000$

$= \dfrac{(1+i)^{?} - 1}{d}$

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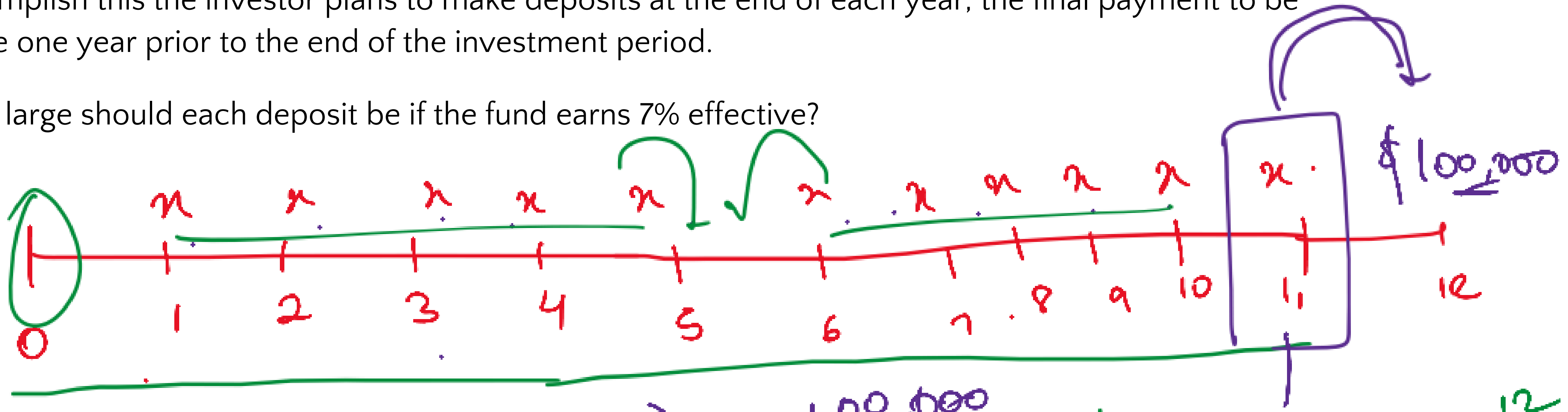
How large should each deposit be if the fund earns 7% effective?

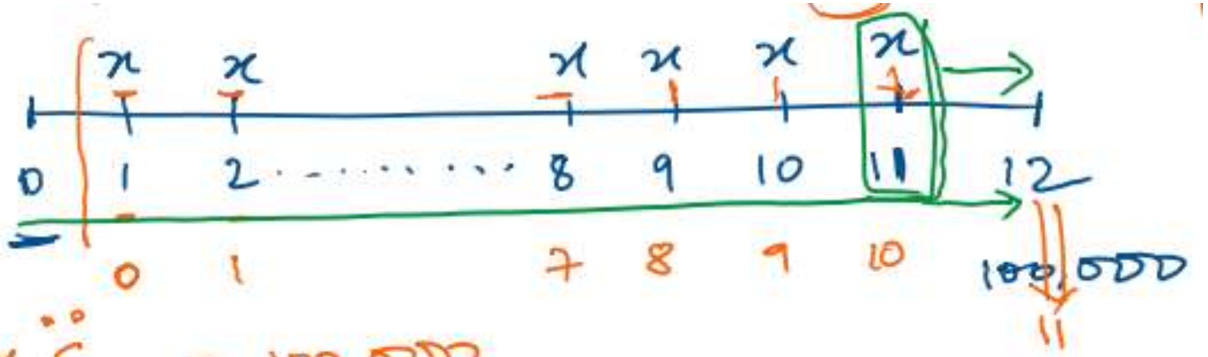


$x\, s_{\overline{11}|}(1+i) = 100,000$

$x\, a_{\overline{11}|} = 100000(1+i)^{-12}$

$x \cdot a_{\overline{11}|}(1+i)^{12} = 100,000$

Solution



$$x\,\ddot{s}_{\overline{11}|} = 100,000$$

$$100,000 = x\left(\frac{(1.07)^{11}-1}{d}\right) = 5921.2061$$

6 Relationships

Note the following relationships between present values and/or accumulated values of annuities. The proofs of these are very intuitive, you can try to derive.

- $S_{\overline{n}|} = a_{\overline{n}|}\,(1+i)^n$
- $\boxed{\dfrac{1}{a_{\overline{n}|}} = \dfrac{1}{S_{\overline{n}|}} + i}$
- $\ddot{S}_{\overline{n}|} = \ddot{a}_{\overline{n}|}\,(1+i)^n$
- $\boxed{\dfrac{1}{\ddot{a}_{\overline{n}|}} = \dfrac{1}{\ddot{S}_{\overline{n}|}} + d}$ 1
- $\boxed{\ddot{a}_{\overline{n}|} = a_{\overline{n}|}(1+i)}$
- $\boxed{\ddot{S}_{\overline{n}|} = S_{\overline{n}|}(1+i)}$
- $\ddot{a}_{\overline{n}|} = 1 + a_{\overline{n-1}|}$

6.1 Annuity values on any date

Accumulated values of annuities can be evaluated on dates other than payment dates.

Consider the earlier example where an annuity under which seven payments of 1 are made at the end of the 3rd through the 9th periods inclusive. Below is a is a time diagram for this annuity

			1	1	1	1	1	1	1			
0	1	2	3	4	5	6	7	8	9	10	11	12

@12 @12

$s_{\overline{10}|} - s_{\overline{3}|}$

$AV_{12} = \boxed{s_{\overline{7}|}}(1+i)^3$ arrears

$AV_{12} = \ddot{s}_{\overline{7}|}(1+i)^2$ due

$\ddot{s}_{\overline{7}|}(1+i)^2$ @12

$s_{\overline{7}|}$

6.1 Annuity values on any date

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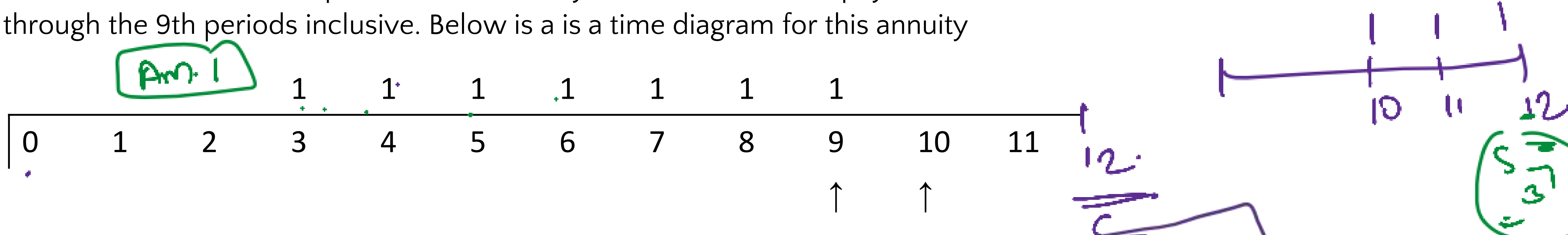
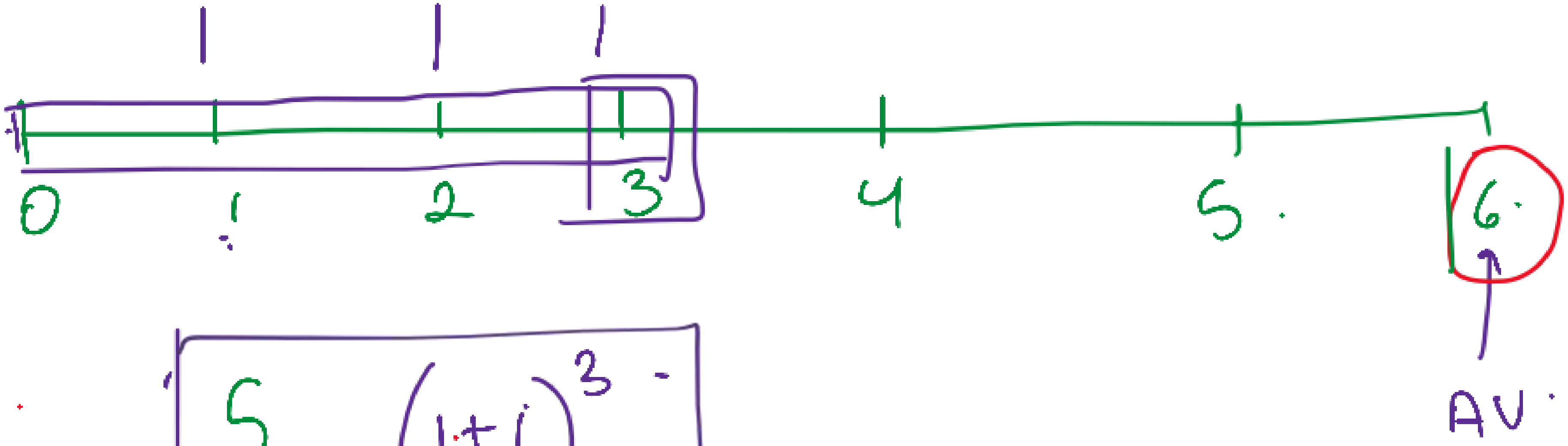


$$= \boxed{\ddot{s}_{\overline{10}|} - \ddot{s}_{\overline{3}|}} = AV_{12} \longrightarrow \boxed{s_{\overline{7}|}(1+i)^3}$$

$s_{\overline{10}|}$

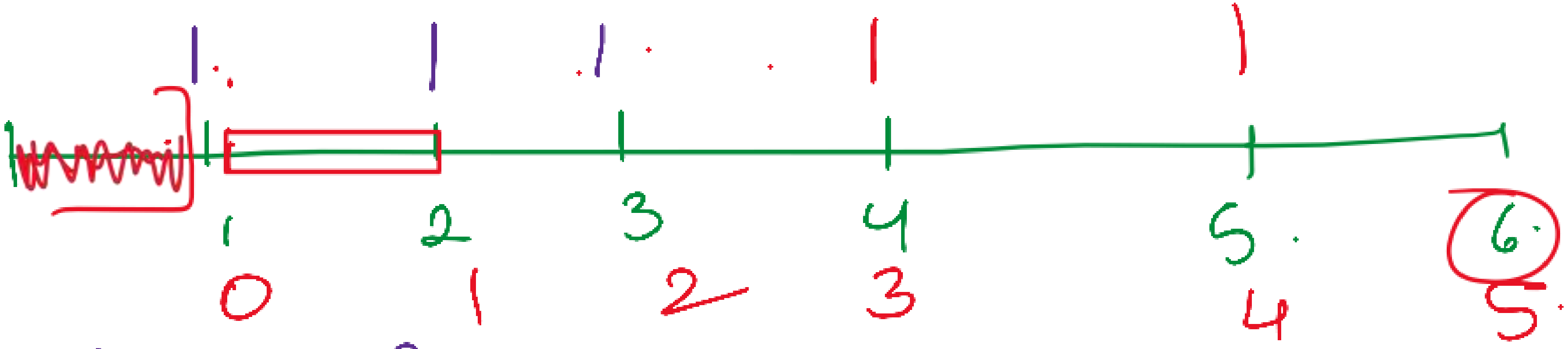
AV of an annuity on any date.



$$\boxed{s_{\overline{3}|}(1+i)^3}$$

$$\boxed{s_{\overline{6}|} - s_{\overline{3}|}}$$

AV of an annuity on any date.



$$\ddot{s}_{\overline{3}|}(1+i)^2$$

$$\ddot{s}_{\overline{5}|} - \ddot{s}_{\overline{2}|}$$

6.1 Annuity values on any date

Accumulated Value Calculation

In this case, the accumulated value of the annuity at the end of the 12th period is seen to be the accumulated value at the end of the 9th period, accumulated for three periods, i.e.

$$\boldsymbol{s_{\overline{7|}}\,(1+i)^3}.$$

Here it is also possible to develop an alternate expression strictly in terms of annuity values. Temporarily, assume that imaginary payments of 1 are made at the end of the 10th, 11th, and 12th periods. Then the accumulated value of all 10 payments is $s_{\overline{10|}}$. However, we must remove the accumulated value of the imaginary payments, which is $s_{\overline{3|}}$. Thus, an alternate expression for the accumulated value is

$$\boldsymbol{s_{\overline{10|}} - s_{\overline{3|}}}$$

Again, it is also possible to work with annuities-due instead of annuities- immediate.. The reader should verify that the answer in this case, expressed as an annuity-due, is

$$\boldsymbol{\ddot{s}_{\overline{9|}} - \ddot{s}_{\overline{2|}}}$$

6.2 P-thly Payable Annuities

We now look at evaluating annuities when the compounding interest periods and the annuity payment periods are differing.

- Accumulated value of p-thly payable Annuity-Immediate, soon after the last payment is made is denoted by $s_n^{(p)}$ and we have:

$$\boxed{s_n^{(p)} = a_{\overline{n}|}^{(p)} (1+i)^n}$$

$$= \frac{(1+i)^n - 1}{i^{(p)}}$$

$$\frac{x}{1} s^{(p)}_{\overline{n}|} = \frac{(1+i)^n - 1}{i^{(p)}}$$

eff roi pa.

annual yearly

- Accumulated value of p-thly payable Annuity-Due, one pth of an interest conversion period after the last payment is made is denoted as $\ddot{s}_n^{(p)}$ and we have:

$$\boxed{\ddot{s}_n^{(p)} = \ddot{a}_{\overline{n}|}^{(p)} (1+i)^n}$$

$$= \frac{(1+i)^n - 1}{d^{(p)}}$$

$$\frac{x}{1} \ddot{s}^{(p)}_{\overline{n}|} = \frac{(1+i)^n - 1}{d^{(p)}}$$

eff roi pa.

annual yearly

Question

An investor effects a contract under which he pays £50 to a savings account on 1 July 2006, and at 3-monthly intervals thereafter, the final payment being made on 1 October 2019. On 1 January 2020 the investor will be paid the accumulated amount of the account.

Calculate how much the investor will receive if the account earns interest at the rate of

(a) 12% per annum effective,

(b) 12% per annum convertible quarterly.

Solution

(a) Work in time units of 1 year. The accumulation is

$$200\ddot{s}^{(4)}_{\overline{13.5}|} \text{ at } 12\% = 200\left[\frac{(1+i)^{13.5}-1}{d^{(4)}}\right] \text{ at } 12\%$$

$$= £6,475.64$$

(b) Work in time units of a half-year. The accumulation is

$$100\ddot{s}^{(2)}_{\overline{27}|} \text{ at } 6\% = £6,655.86$$

(c) Work in time units of a quarter-year. The accumulation is

$$50\ddot{s}_{\overline{54}|} \text{ at } 3\% = £6,753.58$$

AV of:

$$s_{\overline{n}|} = \frac{(1+i)^n - 1}{i}$$

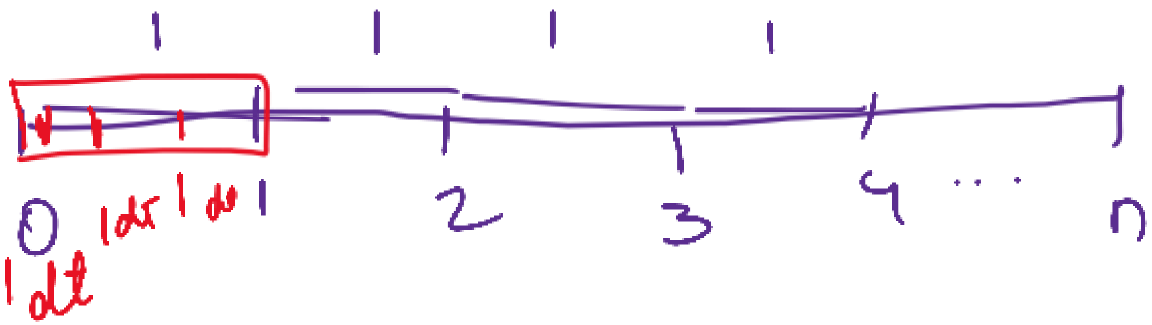
$$a_{\overline{n}|} = \frac{1-v^n}{i}$$

$$\ddot{s}_{\overline{n}|} = \frac{(1+i)^n - 1}{d}$$

$$\ddot{a}_{\overline{n}|} = \frac{1-v^n}{d}$$

$$\bar{s}_{\overline{n}|} = \frac{(1+i)^n - 1}{\delta}$$

$$\bar{a}_{\overline{n}|} = \frac{1-v^n}{\delta}$$



$$\bar{s}_n = \int_0^n 1\, e^{\delta t}\, dt \qquad e^{\delta n} = (1+i)^n$$

$$= \left[\frac{e^{\delta t}}{\delta}\right]_0^n$$

$$= \frac{e^{\delta n} - 1}{\delta} = \frac{(1+i)^n - 1}{\delta} = \bar{s}_{\overline{n}|}$$

6.3 Continuously payable Annuities

The accumulated value of a continuous annuity at the end of the term of the annuity is denoted by $\bar{S}_{\overline{n}|}$. The following relationships hold:

$$\bar{S}_{\overline{n}|} = \int_0^n (1+i)^t \, \mathrm{dt}$$

$$= \left[\frac{(1+i)^t}{\ln(1+i)}\right]_0^n$$

$$= \frac{(1+i)^n - 1}{\delta}$$

6.4 Relationships

- $a^{(p)}_{\overline{n|}} = \frac{i}{i^{(p)}} a_{\overline{n|}}$
- $\ddot{a}^{(p)}_{\overline{n|}} = \frac{i}{d^{(p)}} a_{\overline{n|}}$
- $s^{(p)}_{\overline{n|}} = \frac{i}{i^{(p)}} s_{\overline{n|}}$
- $\ddot{s}^{(p)}_{\overline{n|}} = \frac{i}{d^{(p)}} s_{\overline{n|}}$
- $\bar{a}_{\overline{n|}} = \frac{i}{\delta} a_{\overline{n|}}$
- $\bar{s}_{\overline{n|}} = \frac{i}{\delta} s_{\overline{n|}}$

$$a_{\overline{n|}} = \frac{1-v^n}{i} \times \frac{i}{d}$$

$$\ddot{a}_{\overline{n|}} = a_{\overline{n|}} \times \frac{i}{d}$$

$$\bar{a}_{\overline{n|}} \qquad \times \frac{i}{\delta}$$

7 Varying Rates of Interest

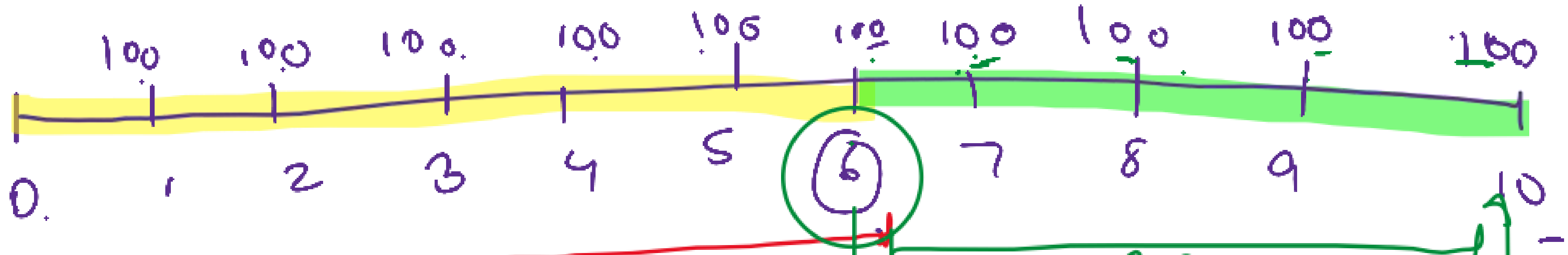
So far we have assumed a level rate of interest throughout the term of the annuity. Now we look at situations where rate of interest is different, but compound interest is still inn effect.

We understand the concept better through an example:

Question

Find the accumulated value of a 10-year annuity-immediate of $100 per year if the effective rate of interest is 5% for the first 6 years and 4% for the last 4 years.

annuity arrears. - 10 year.



$i = 5\%\ \text{pa}$ $\qquad$ $i = 4\%\ \text{pa}$

$$100\, s_{\overline{6}|}^{@5\%}\,(1+i)^{4}\ \boxed{@4\%} + 100\, s_{\overline{4}|}^{@4\%}$$

Solution

The value of first six payments is

$$100\, s_{\overline{6}|.05}\,.$$

This value is accumulated to the end of the 10 years at 4%, giving

$$100\, s_{\overline{6}|.05}\,(1.04)^{4}.$$

The accumulated value of the last four payments is

$$100\, s_{\overline{4}|.04}\,.$$

Thus, the answer is

$$100\left[s_{\overline{6}|.05}(1.04)^{4} + s_{\overline{4}|.04} \right]$$

$$= 100\left[(6.8019)(1.16986)+(4.2465)\right]$$

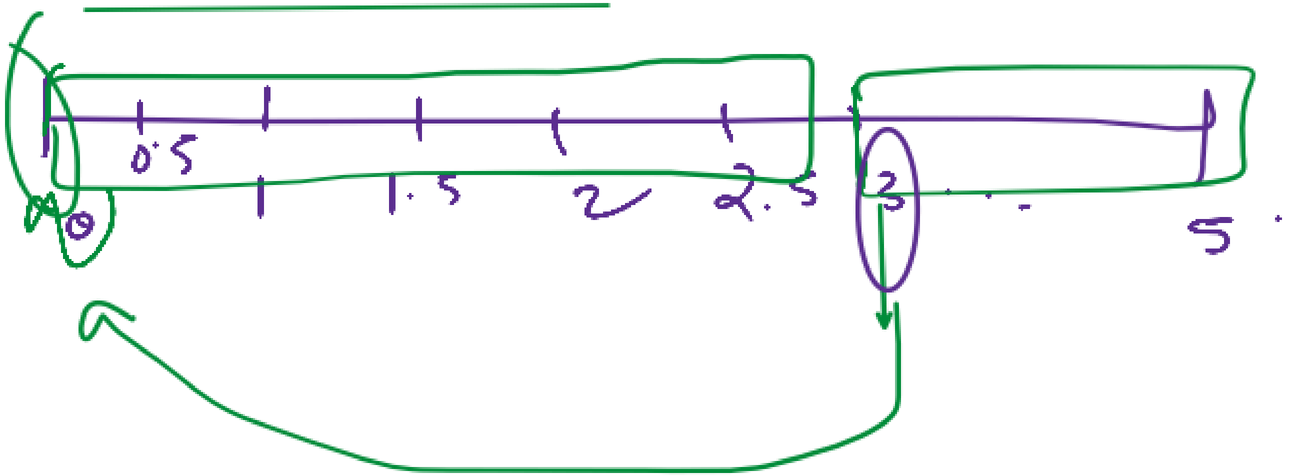
$$= \$1220.38.$$

Question 5.16

Find the present value of an annuity-immediate which pays 1 at the end of each half-year for five years, if the rate of interest is 8% convertible semiannually for first three years and 7% convertible semiannually for the last two years.

$$a_{\overline{6}|} @ \frac{i^{(2)}}{2} + a_{\overline{4}|} @ \frac{i^{(2)}}{2} \left(1+\frac{i^{(2)}}{2}\right)^{-6}$$

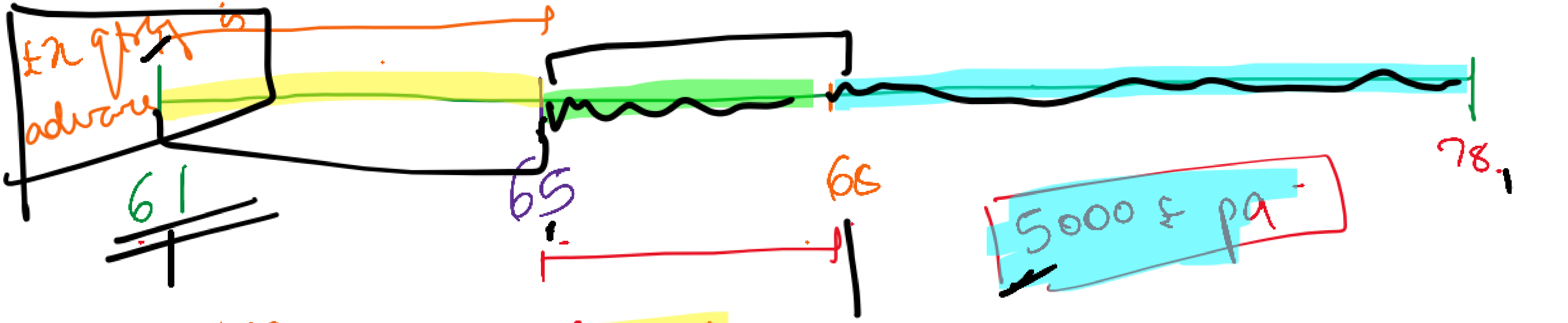
$\boxed{8.144971}$



$\boxed{11.99471}$

An individual intends to retire on his 65th birthday in exactly four years' time. The government will pay a pension to the individual from age 68 of £5,000 per annum monthly in advance. The individual would like to purchase an annuity certain so that his income, including the government pension, is £8,000 per annum paid monthly in advance from age 65 until his 78th birthday. He is to purchase the annuity by a series of payments made over four years quarterly in advance starting immediately.

Calculate the quarterly payments the individual has to make if the present value of these payments is equal to the present value of the annuity he wishes to purchase at a rate of interest of 5% per annum effective. Mortality should be ignored. [6]



$i = 5\%$ pa

£8000pa monthly in adv

£3000pa

$$4x \, \ddot{a}^{(4)}_{\overline{4}|} @5\% = 8000 \, \ddot{a}^{(12)}_{\overline{3}|} @5\% \; v^4 + 3000 \, \ddot{a}^{(12)}_{\overline{10}|} @5\% \; v^7$$

$$4x\, \ddot{a}^{(4)}_{\overline{4|}} = 8000\, \ddot{a}^{(12)}_{\overline{3|}} v^4 + 3000\, \ddot{a}^{(12)}_{\overline{10|}} v^7.$$

$$4x\left[\frac{1-0.8227}{4.8494\%}\right] = 8000\left[\frac{1-0.86384}{4.8691\%}\right]\times 0.8227$$

$$+\ 3000\left[\frac{1-0.61391}{4.8691\%}\right]\times 0\ldots$$

$$x(14.62448963) = 18404.8525 + 16905.7798$$

$$\boxed{x = 2414.6916}.$$

$3000 \times v^7 \times \ldots$

$3000 \times 0.71068 \times 1\ldots$

An investor has a choice of two 15-year savings plans, A and B, issued by a company. In both plans, the investor pays contributions of $100 at the start of each month and the contributions accumulate at an effective rate of interest of 4% per annum before any allowance is made for expenses.

In plan A, the company charges for expenses by deducting 1% from the annual effective rate of return.

In plan B, the company charges for expenses by deducting $15 from each of the first year's monthly contributions before they are invested. In addition it deducts 0.3% from the annual effective rate of return.

Calculate the percentage by which the accumulated amount in Plan B is greater than the accumulated amount in Plan A, at the end of the 15 years.

[6]

Subject CT1 September 2016 Question 2

The nominal rate of interest per annum convertible quarterly is 2%.

Calculate the present value of a payment stream paid at a rate of €100 per annum, monthly in advance for 12 years. [4]

Subject CT1 April 2015 Question 5

An investor pays £120 per annum into a savings account for 12 years. In the first four years, the payments are made annually in advance. In the second four years, the payments are made quarterly in advance. In the final four years, the payments are made monthly in advance.

The investor achieves a yield of 6% per annum convertible half-yearly on the investment.

Calculate the accumulated amount in the savings account at the end of 12 years. [7]

8 Uncommon facts about annuities

- ❖ With annuities, most of the time you don't give up access to your original deposit
- ❖ Not all annuities have surrender charges
- ❖ Deferred annuities provide tax-advantaged saving and lifetime income.
- ❖ A fixed annuity is good for principal protection.
- ❖ A variable annuity has investment risk.
- ❖ Annuities are only as strong as the companies that issue them.
- ❖ Not all annuity payments expire when you do.

Recap - Formulas

$$a_{\overline{n|}} = \frac{1-v^n}{i} \qquad \ddot{a}_{\overline{n|}} = \frac{1-v^n}{d} \qquad \bar{a}_{\overline{n|}} = \frac{1-v^n}{\delta}$$

$$s_{\overline{n|}} = \frac{(1+i)^n - 1}{i} \qquad \ddot{s}_{\overline{n|}} = \frac{(1+i)^n - 1}{d} \qquad \bar{s}_{\overline{n|}} = \frac{(1+i)^n - 1}{\delta}$$

$$a_{\overline{n|}}^{(p)} = \frac{1-v^n}{i^{(p)}} \qquad \ddot{a}_{\overline{n|}}^{(p)} = \frac{1-v^n}{d^{(p)}}$$

$$s_{\overline{n|}}^{(p)} = \frac{(1+i)^n - 1}{i^{(p)}} \qquad \ddot{s}_{\overline{n|}}^{(p)} = \frac{(1+i)^n - 1}{d^{(p)}}$$

Extra Reading

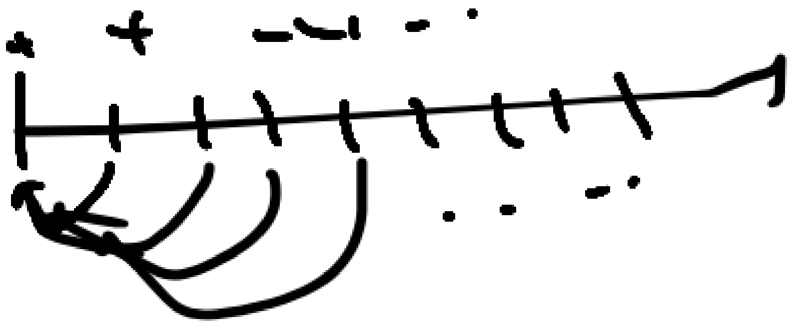
Calculating Present and Future Values of Annuities
https://www.investopedia.com/retirement/calculating-present-and-future-value-of-annuities/

All about annuities
https://www.annuity.org/annuities/

9 Varying Annuities

- So far all the annuities considered have had level payments.
- Now we remove this restriction and consider annuities with varying payments.
- For an annuity in which the payments are not all of an equal amount, it is a simple matter to find the present (or accumulated) value from first principles. For example, the present value of such an annuity may always be evaluated as

$$\sum_{i=1}^{n} X_i \, v^{t_i}$$



where the ith payment, of amount Xi, is made at time ti.

In particular, we consider two types of varying annuities which are commonly encountered in practice.

1] Increasing Annuities

2] Decreasing Annuities

9.1 Increasing Annuities

In the particular case when Xi = ti = i, the annuity is known as an increasing annuity, and its present value is denoted by $(Ia)_{\overline{n|}}$ and accumulated value as $(IS)_{\overline{n|}}$.

It can be shown on the timeline as:

Time	0	1	2	……	n-1	n
Payment		1	2	…	n-1	n
Value						

PV of a level increasing annuity in arrears

$$1 \cdot (Ia)_{\overline{n}|} = 1v^1 + 2v^2 + 3v^3 + \cdots + nv^n \quad \text{—— (1)}$$

multiply throughout by $(1+i)$

$$(1+i)(Ia)_{\overline{n}|} = 1 + 2v + 3v^2 + 4v^3 + nv^{n-1} \quad \text{—— (2)}$$

(2) − (1)

$$(1+i)(Ia)_{\overline{n}|} - (Ia)_{\overline{n}|} = 1 + v + v^2 + v^3 + \cdots + v^{n-1} -$$

$$(1+i)(Ia)_{\overline{n}|} - (Ia)_{\overline{n}|} = \ddot{a}_{\overline{n}|} - nv^n$$

$$(Ia)_{\overline{n}|}(1+i-1) = \ddot{a}_{\overline{n}|} - nv^n$$

$$(Ia)_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{i}$$

$$(I\ddot{a})_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{d} \qquad \Bigg| \qquad (I\bar{a})_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{\delta}$$

9.1 Increasing Annuities

Level Increasing Annuities

$(Ia)_{\overline{n|}}$ = 1v + $2v^2$ + $3v^3$ + ... + nv^n -> 1

Now multiplying on both sides by (1+i)

$(Ia)_{\overline{n|}}$ (1+i) = 1 + 2v + $3v^2$ + ... + nv^{n-1} -> 2

Subtracting 2 from 1

$(Ia)_{\overline{n|}}$ (1+i) - $(Ia)_{\overline{n|}}$ = 1 + v + v^2 + v^3 + ... + v^{n-1} - nv^n

i $(Ia)_{\overline{n|}}$ = $\ddot{a}_{\overline{n|}}$ - nv^n

Thus:

$$\mathbf{(Ia)_{\overline{n|}} = \frac{\ddot{a}_{\overline{n|}} - n v^n}{i}}$$

9.1 Increasing Annuities

Payments varying in Arithmetic Progression

The present value of any annuity payable in arrears for n time units for which the amounts of successive payments form an arithmetic progression can be expressed in terms of $a_{\overline{n|}}$ and $(\mathrm{Ia})_{\overline{n|}}$.

If the first payment of such an annuity is P and the second payment is (P + Q), the t th payment is (P - Q) + Qt, and the present value of the annuity is therefore

$$\mathbf{(P\text{-}Q)}\ \boldsymbol{a_{\overline{n|}}}\ \ \mathbf{+\ Q}\ \boldsymbol{(Ia)_{\overline{n|}}}$$

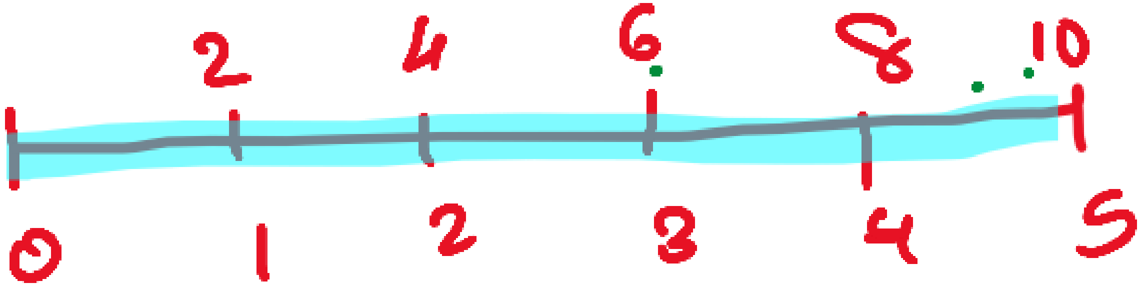
Question

$2 \times 11.7469 + 2 \times 4.1002$

$\boxed{31.6942}$

Measuring time in years, calculate the present value of a cash flow stream Rs. 4, 6, 8, 10, and 12 paid at times 1, 2, 3, 4, and 5 if i = 7% per annum.

$2\,(Ia)_{\overline{5}|7} + 2\,a_{\overline{5}|7}$



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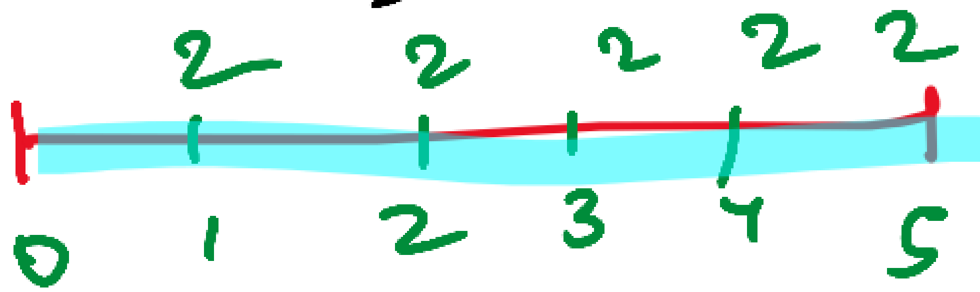


$2\left[\dfrac{\ddot{a}_{\overline{5}|} - 5\cdot v^5}{7\%}\right] + \left[\dfrac{1-v^5}{7\%}\right]\times 2$

· annuity starts at time 1 = 500 \$

Increase = 50 \$

Payments are made till $t = 20$

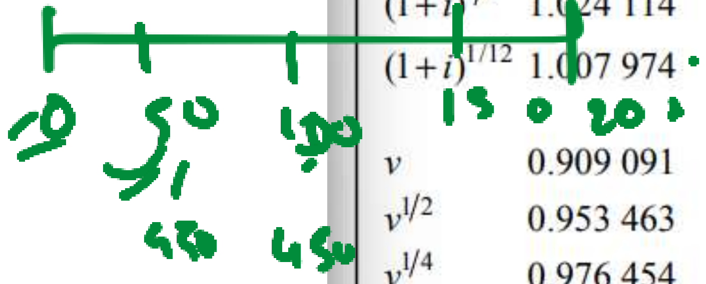
$i = 10\%$ pa eff.

Find the PV.

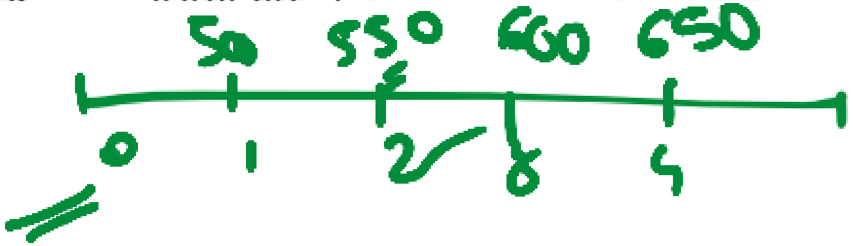
10%

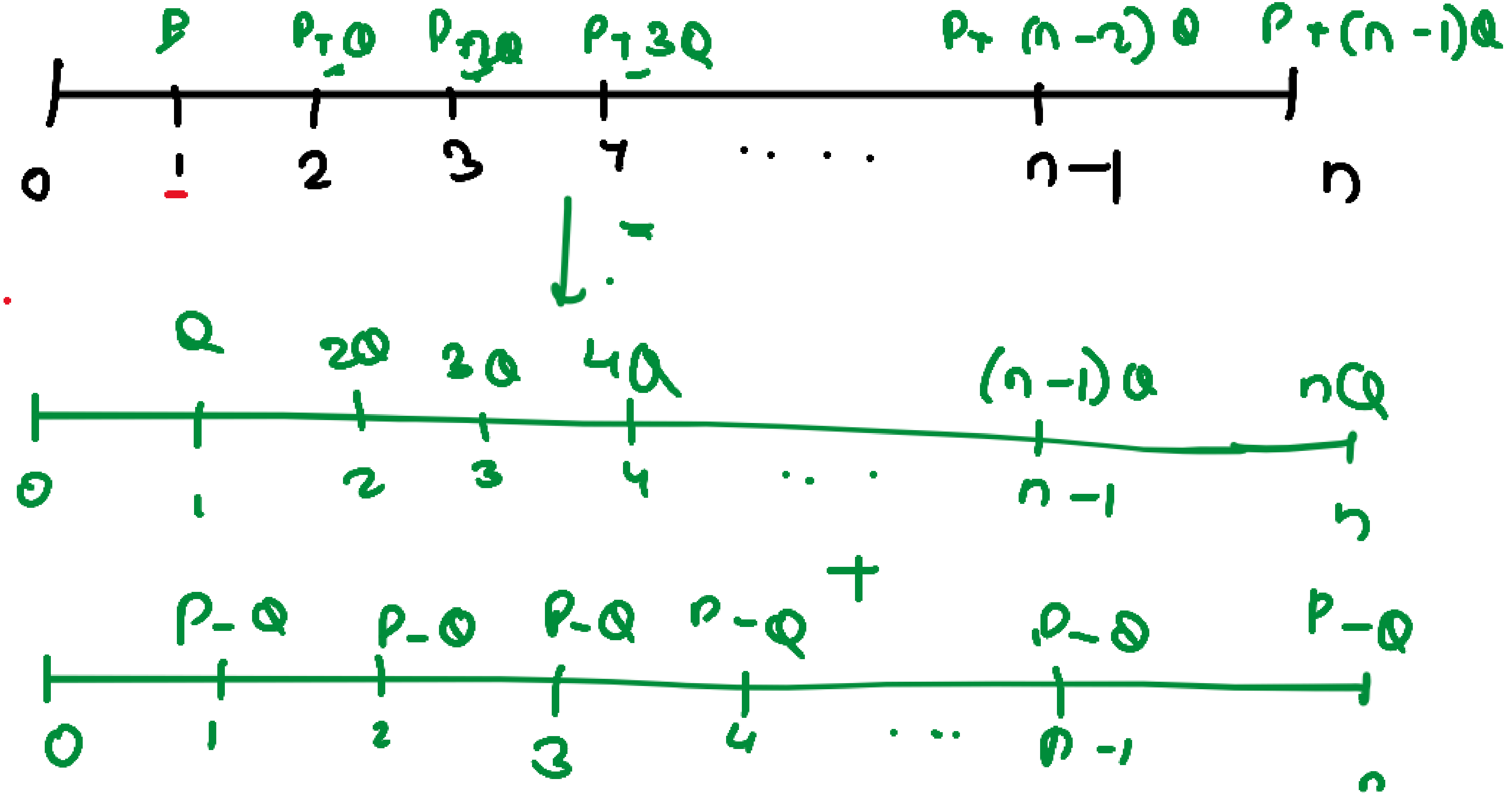
i	0.100 000
$i^{(2)}$	0.097 618
$i^{(4)}$	0.096 455
$i^{(12)}$	0.095 690
δ	0.095 310
$(1+i)^{1/2}$	1.048 809
$(1+i)^{1/4}$	1.024 114
$(1+i)^{1/12}$	1.007 974
v	0.909 091
$v^{1/2}$	0.953 463
$v^{1/4}$	0.976 454
$v^{1/12}$	0.992 089
d	0.090 909

n	$(1+i)^n$	v^n	$s_{\overline{n}\vert}$
1	1.100 00	0.909 09	1.000 0
2	1.210 00	0.826 45	2.100 0
3	1.331 00	0.751 31	3.310 0
4	1.464 10	0.683 01	4.641 0
5	1.610 51	0.620 92	6.105 1
6	1.771 56	0.564 47	7.715 6
7	1.948 72	0.513 16	9.487 2
8	2.143 59	0.466 51	11.435 9
9	2.357 95	0.424 10	13.579 5
10	2.593 74	0.385 54	15.937 4
11	2.853 12	0.350 49	18.531 2
12	3.138 43	0.318 63	21.384 3
13	3.452 27	0.289 66	24.522 7
14	3.797 50	0.263 33	27.975 0
15	4.177 25	0.239 39	31.772 5
16	4.594 97	0.217 63	35.949 7
17	5.054 47	0.197 84	40.544 7
18	5.559 92	0.179 86	45.599 2
19	6.115 91	0.163 51	51.159 1
20	6.727 50	0.148 64	57.275 0



$$\boxed{450\, a_{\overline{20|}}} + \boxed{50\,(Ia)_{\overline{20|}}}$$





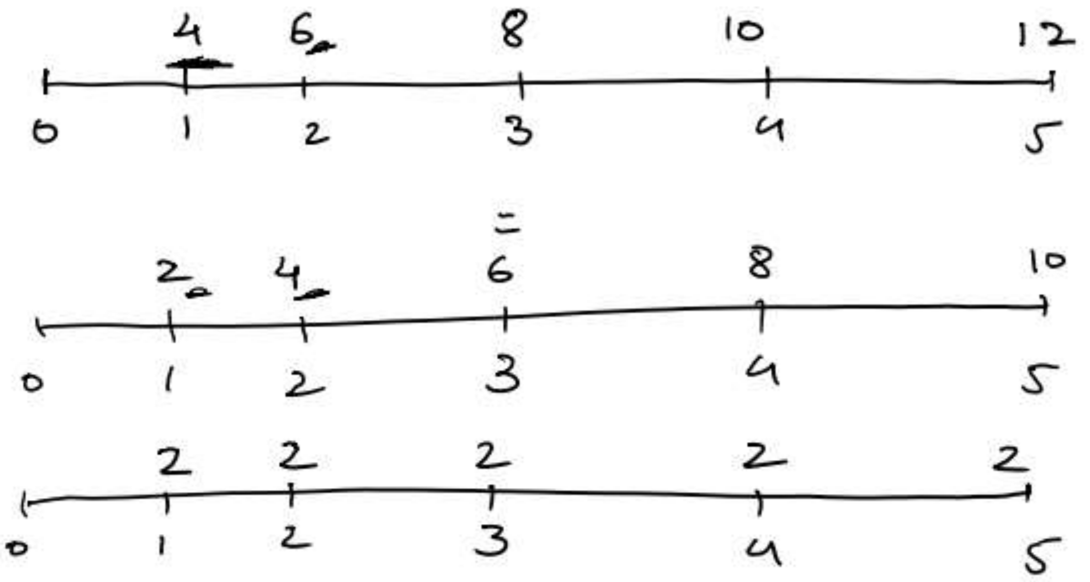
P = 50

Q = 5

$$(P-Q)\,a_{\overline{n}|} + Q\,(Ia)_{\overline{n}|}$$

$450\,a_{\overline{n}|} + 50($

Solution



$$\Rightarrow 2(Ia)_{\overline{5|}} + 2a_{\overline{5|}}$$

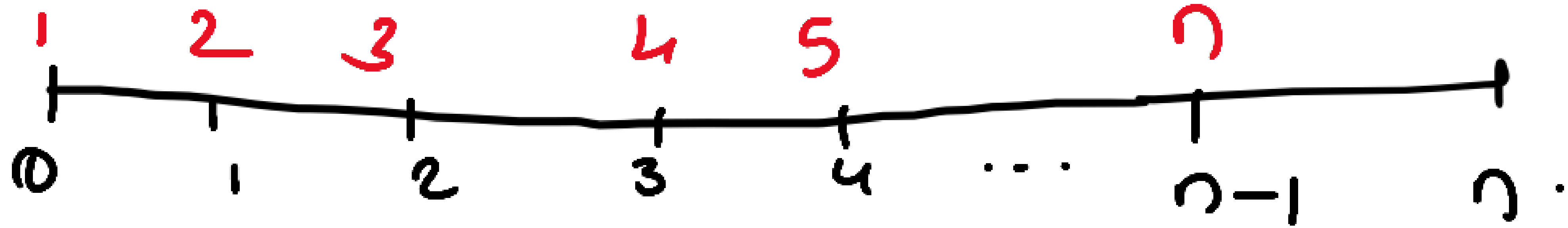
$P = 4$

$Q = 2$

$$(P-Q)a_{\overline{5|}} + Q(Ia)_{\overline{5|}}$$

$$2a_{\overline{5|}} + 2(Ia)_{\overline{5|}}$$

PV of level increasing annuity in due



$$(I\ddot{a})_{\overline{n}|} = 1 + 2v + 3v^2 + 4v^3 + \dots + nv^{n-1} \quad — ($$

multiply by v $[1/(1+i)]$

$$v(I\ddot{a})_{\overline{n}|} = v + 2v^2 + 3v^3 + 4v^4 + \dots + nv^n \quad —$$

1 − 2

$$(I\ddot{a})_{\overline{n}|} - v(I\ddot{a})_{\overline{n}|} = 1 + v + v^2 + v^3 + \dots + v^{n-1} - n$$

$$(I\ddot{a})_{\overline{n|}} - v(I\ddot{a})_{\overline{n|}} = \ddot{a}_{\overline{n|}} - nv^n$$

$$(I\ddot{a})_{\overline{n|}}(1-v) = \ddot{a}_{\overline{n|}} - nv^n$$

$$(I\ddot{a})_{\overline{n|}} = \frac{\ddot{a}_{\overline{n|}} - nv^n}{d}$$

9.1 Increasing Annuities

Increasing Annuity Due

The notation $(I\ddot{a})_{\overline{n|}}$ is used to denote the present value of an increasing annuity due payable for n time units, the nth payment (of amount n) being made at time n - 1.

Present Value

$(I\ddot{a})_{\overline{n|}}$ = 1 + 2v + $3v^2$ + ... + nv^{n-1} -> 1

Multiplying both sides by v

V $(I\ddot{a})_{\overline{n|}}$ = v + $2v^2$ + $3v^3$ + ... + $(n-1)v^{n-1}$ + nv^n -> 2

Subtracting 1 from 2

$(I\ddot{a})_{\overline{n|}}$ (1-v) = 1 + v + v^2 + v^3 + ... + v^{n-1} - nv^n = $\ddot{a}_{\overline{n|}}$ - nv^n

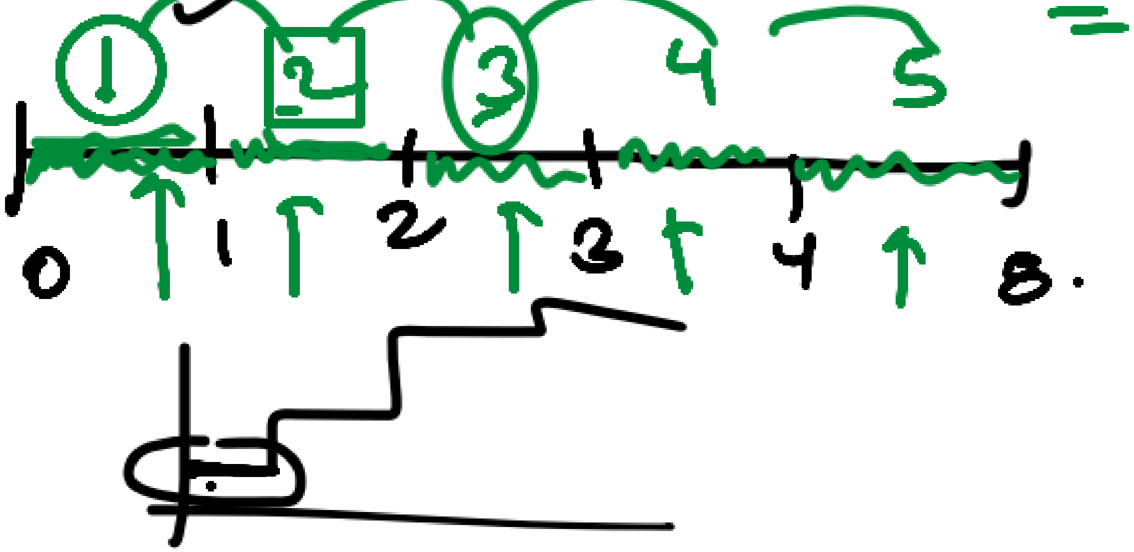
$$\boldsymbol{(I\ddot{a})_{\overline{n|}} = \frac{\ddot{a}_{\overline{n|}} - \mathbf{n}v^n}{d}}$$

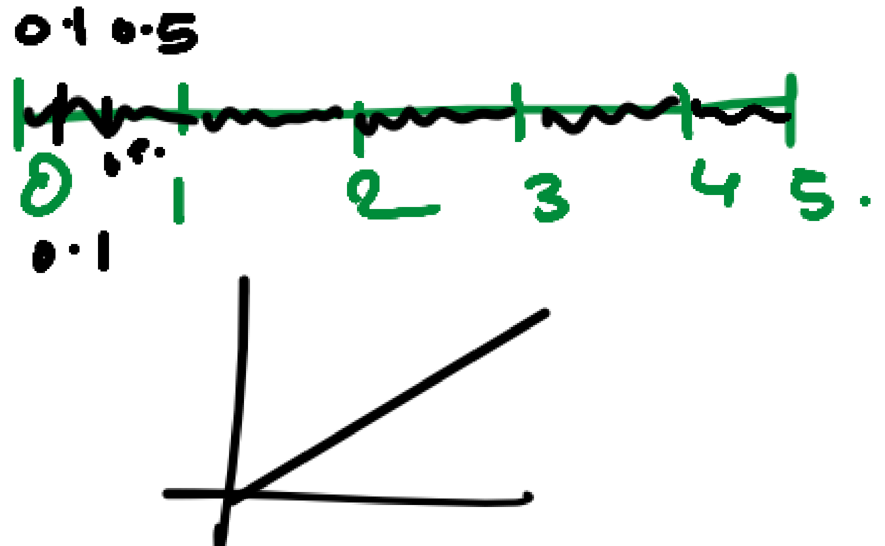
$(I\bar{a})_{\overline{n|}}$

Continuous annuity

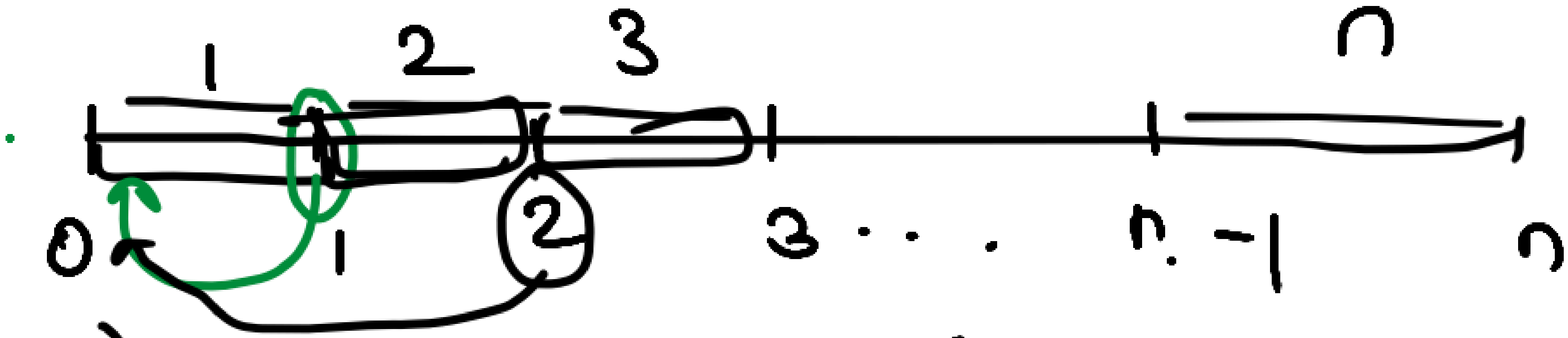
$(\bar{I}\bar{a})_{\overline{n|}}$ /

Payment cont.	**Payment cont.**
Amt increase discrete	Amt increase cont.





$(I\bar{a})_{\overline{n}|} : ?$



$$(I\bar{a})_{\overline{n}|} = 1\bar{a}_{\overline{1}|} + 2\bar{a}_{\overline{1}|}v + 3\bar{a}_{\overline{1}|}v^2 + \cdots + n\bar{a}_{\overline{1}|}v^{n-1}$$

$$= \bar{a}_{\overline{1}|}\left(1 + 2v + 3v^2 + \cdots + nv^{n-1}\right)$$

$$(I\bar{a})_{\overline{n}|} = \bar{a}_{\overline{1}|}\,(I\ddot{a})_{\overline{n}|}$$

$$= \frac{1-v^{1}}{\delta}\left(\frac{\ddot{a}_{\overline{n}|} - n v^{n}}{d}\right)$$

$$= \frac{d}{\delta}\left(\frac{\ddot{a}_{\overline{n}|} - n v^{n}}{d}\right)$$

$$(I\bar{a})_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - n v^{n}}{\delta} \leftarrow$$ cont annuity
increase di

Derivation

$$(\bar{I}\bar{a})_{\overline{n}|} = \int_0^n t v^t \, dt$$

$$= \int_0^n t e^{-\delta t} \, dt$$

$$\int_x^y uv\,ds = \left[u\int v\right]_x^y - \int_x^y \left(\frac{d}{ds}(u)\int v\right)ds.$$

$$\int_0^n te^{-\delta t}\,dt = \left[t\int e^{-\delta t}\right]_0^n - \int_0^n \left[\frac{d}{dt}(t)\int e^{-\delta t}\right]dt$$

$$= \left[t\cdot\frac{e^{-\delta t}}{-\delta}\right]_0^n - \int_0^n (1)\left[\frac{e^{-\delta t}}{-\delta}\right]dt$$

$$= \left[n\,\frac{e^{-\delta n}}{-\delta}\right] + \frac{1}{\delta}\int_0^n e^{-\delta t}\,dt$$

$$\left[n \cdot \frac{e^{-\delta n}}{-\delta} \right] + \frac{1}{\delta} \int_0^n e^{-\delta t}\, dt$$

$$= \left[n \frac{e^{-\delta n}}{-\delta} \right] + \frac{1}{\delta} \left[\frac{e^{-\delta t}}{-\delta} \right]_0^n = \left[\frac{n e^{-\delta n}}{-\delta} \right] - \frac{1}{\delta^2} \left[e^{-\delta n} - 1 \right]$$

$$= \frac{-n e^{-\delta n}}{\delta} - \frac{e^{-\delta n}}{\delta^2} + \frac{1}{\delta^2}$$

$$= \frac{-n v^n}{\delta} + \frac{1}{\delta} \left(\frac{1}{\delta} - \frac{e^{-\delta n}}{\delta} \right) = \frac{-n v^n}{\delta} + \frac{1}{\delta} \cdot \left(\frac{1 -}{\delta} \right.$$

$$-\frac{nv^n}{\delta} + \frac{1}{\delta}\bar{a}_{\overline{n}|}$$

$$\frac{\bar{a}_{\overline{n}|} - nv^n}{\delta} = \frac{(\bar{I}\bar{a})_{\overline{n}|}}{(\bar{I}a)_{\overline{n}|}}$$

9.1 Continuous Annuities

For increasing annuities that are payable continuously, it is important to distinguish between annuities which have a constant rate of payment r (per unit time) throughout the rth period and annuities which have a rate of payment t at time t.

- For the former, the rate of payment is a step function, taking the discrete values 1, 2,..
- For the latter, the rate of payment itself increases continuously.

If the annuities are payable for n time units, their present values are denoted by $(I\bar{a})_{\overline{n|}}$ and $(\overline{Ia})_{\overline{n|}}$ respectively.

The present values are:

- $(I\bar{a})_{\overline{n|}} = \dfrac{\ddot{a}_{\overline{n|}} - nv^n}{\delta}$
- $(\overline{Ia})_{\overline{n|}} = \dfrac{\bar{a}_{\overline{n|}} - nv^n}{\delta}$

9.1 Accumulated Values

The accumulated values of varying annuities are as follows:

- $(IS)_{\overline{n|}} = (1+i)^n \, (Ia)_{\overline{n|}}$
- $(I\ddot{S})_{\overline{n|}} = (1+i)^n \, (I\ddot{a})_{\overline{n|}}$
- $(I\bar{S})_{\overline{n|}} = (1+i)^n \, (I\bar{a})_{\overline{n|}}$
- $(\bar{I}\bar{S})_{\overline{n|}} = (1+i)^n \, (\bar{I}\bar{a})_{\overline{n|}}$

9.2 Decreasing Annuities

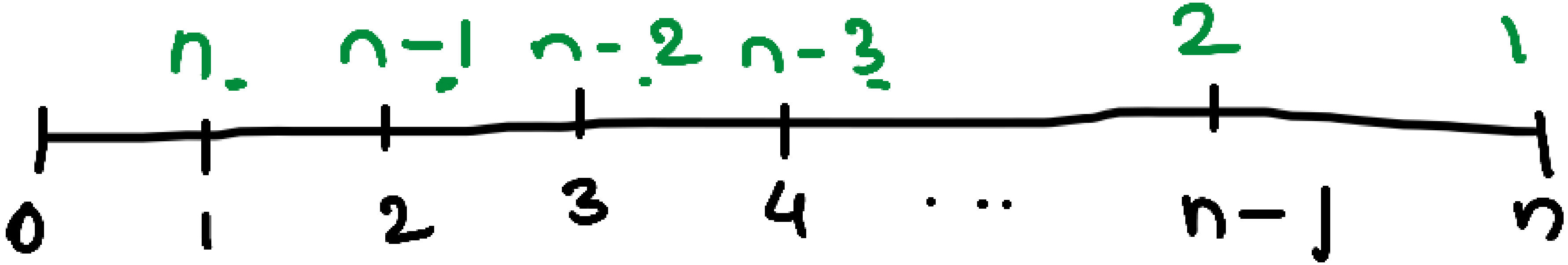
We can use the increasing annuity functions to evaluate annuities with decreasing payments.

Consider the timeline:'

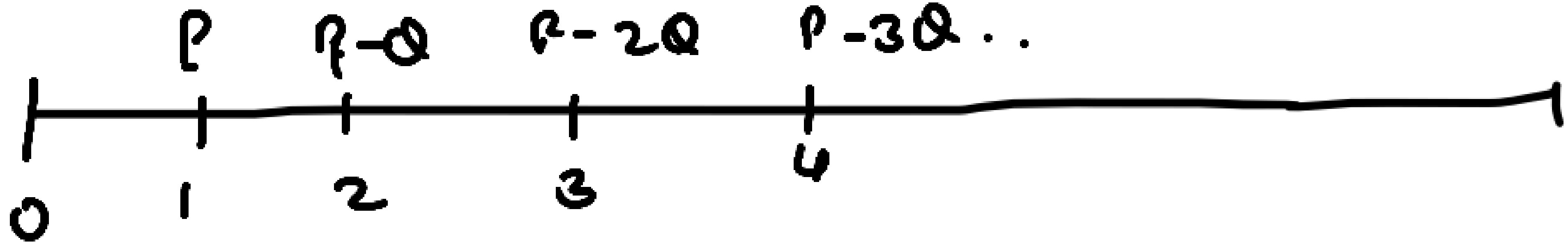
Time	0	1	2	3	……	n
Payment		P	P–Q	P-2Q	…	Q
Value						

The present value is then given as:

$$(Da)_{\overline{n}|} = \mathbf{(P+Q)}\ a_{\overline{n}|} - \mathbf{Q}\ (Ia)_{\overline{n}|}$$



n
n-1
n-2
n-3
2
1
0
1
2
3
4
...
n-1
n



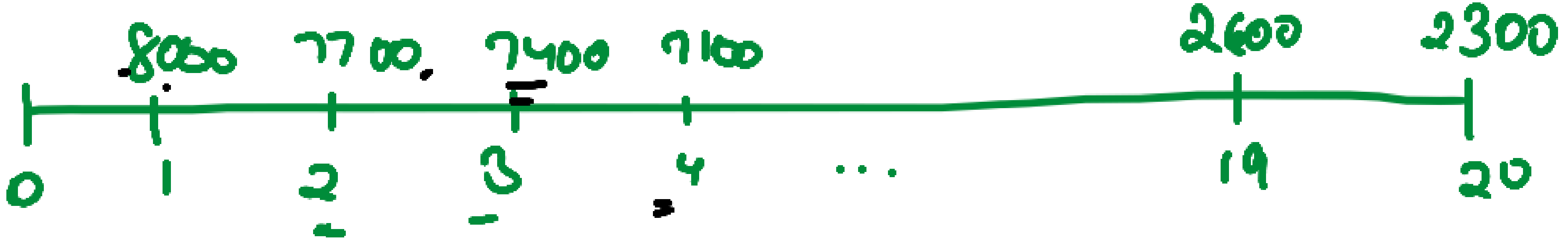
Decreasing annuity $= (P+Q)\,a_{\overline{n}|} - Q\,(Ia)_{\overline{n}|}$

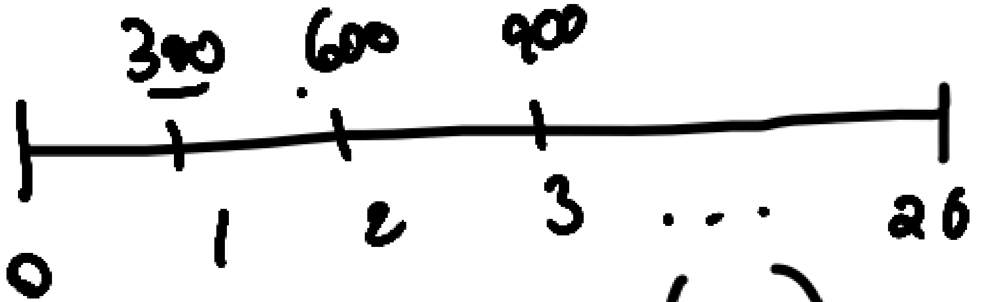
$(Da)_{\overline{n}|}$

Question

$n = 20$

An annuity is payable annually in arrears for 20 years. The first payment is of amount £8,000, and the amount of each subsequent payment decreases by £300 each year. Find the present value of the annuity on the basis of an interest rate of 5% per annum effective.





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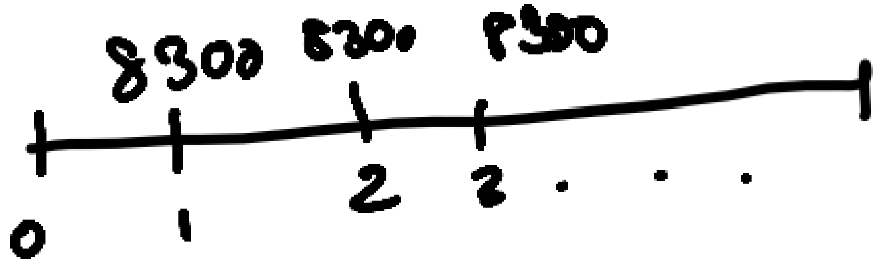


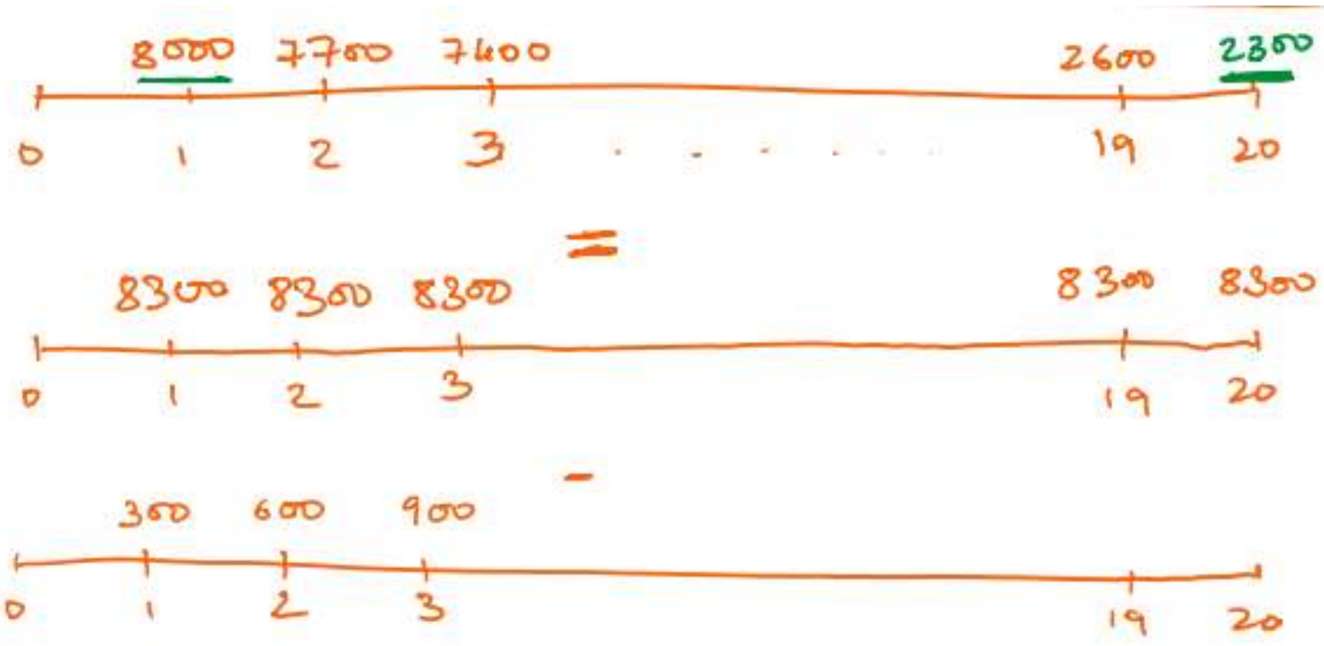
$300(Ia)_{\overline{20}|}$ $-$ $8300\, a_{\overline{20}|}$

$$8300\, a_{\overline{20}|} - 300\, (Ia)_{\overline{20}|} \qquad i = 5\%$$

$-$ $\qquad$ $-$

$=$

Solution



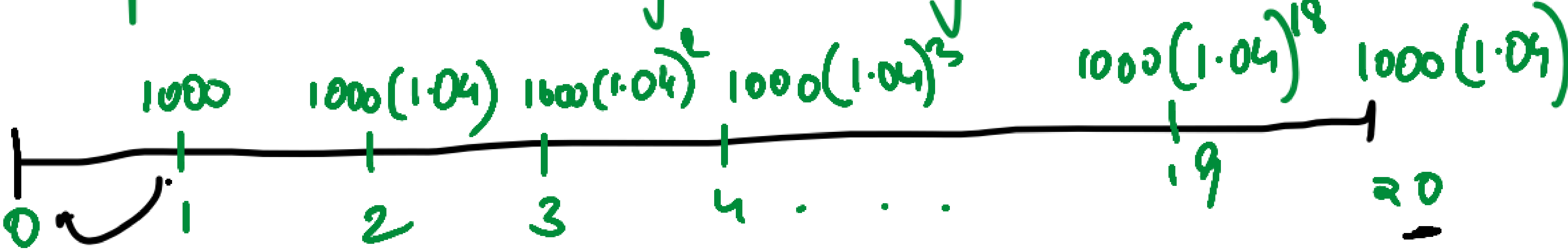
$70{,}151.08$

$= 8300\, a_{\overline{20}|}$

$-$

$300\, (Ia)_{\overline{20}|}$

Compound increasing annuity



CF = 4%

n : 20 years

arrears.

$i = 7\%$

$$PV = 1000v + 1000(1.04)v^2 + 1000(1.04)^2v^3 + \dots + 1000(1.04)^{19}v^{20}$$

$$= 1000v\left(\overset{1}{1} + \overset{v'}{\underline{1.04v}} + \overset{v'^2}{1.04^2 v^2} + \ldots + \overset{v'^{19}}{1.04^{19} v^{19}}\right)$$

let $1.04v$ be v'

$$1.04v = v'$$

$$1.04 \times \frac{1}{1.07} = v' = \frac{1}{1+i'}$$

$$\frac{1.07}{1.04} = 1 + i' \qquad \therefore \boxed{i' = 2.884615\%}$$

$$= 1000v\left(1 + v' + (v')^2 + (v')^3 + \ldots + (v')^{19}\right).$$

$$= 1000v \ \ddot{a}_{\overline{20}|} \ @ \ i' = 2.884165\%$$

$$d = \frac{i}{1+i}$$

$$= 1000 \cdot \frac{1}{1.07}\left[\frac{1-(v')^{20}}{d'}\right].$$

$$= \frac{1000}{1.07}\left[\frac{1-(1+2.884165\%)^{-20}}{\left(\frac{2.884165\%}{1+2.884165\%}\right)}\right]$$

$$\frac{i}{1+i}$$

$$= 14459.07187.$$

9.3 Compound Increasing Annuities

We will also need to be able to value compound increasing annuities where the payments increase by a constant factor each time.

We consider an example to understand better.

An annuity provides for 20 annual payments, the payment a year hence being \$1000. The payments increase in such a way that each payment is 4% greater than the preceding payment. Find the present value of this annuity at an annual effective rate of interest of 7%.

Solution

The present value of this annuity is:

$$1000v \left[1 + \frac{1.04}{1.07} + \ldots\ldots + \frac{(1.04)^{19}}{(1.07)^{19}} \right]$$

The expression in brackets is simply a geometric progression of the form:

$$1 + x + x^2 + \cdots + x^{n-1} = \frac{1 - x^n}{1 - x}$$

Where, $x = \frac{1.04}{1.07}$

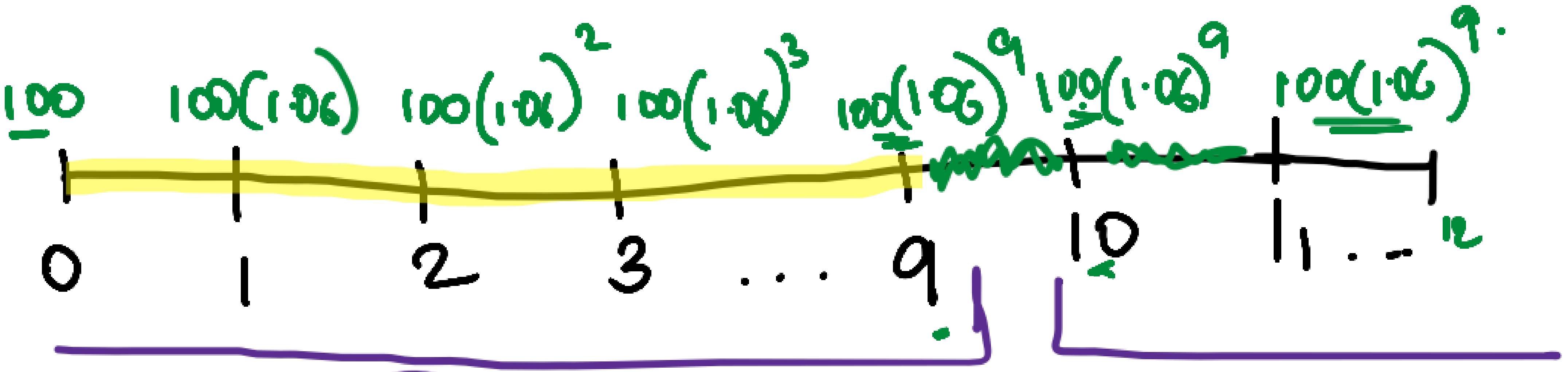
Thus:

$$1000\,v \left[\frac{1 - \left(\frac{1.04}{1.07}\right)^{20}}{1 - \frac{1.04}{1.07}} \right] = 14{,}459$$

Question

A perpetuity-due makes annual payments which begin at $100 for the first year, then increase at 6% p.a. through the 10th year, and then remain level thereafter.

Calculate the present value of this perpetuity, if the annual effective rate of interest is equal to 8%.



Imm. 10

① Comp Inc annuity

920.654

+

② perpetuity

1056.452

$$PV = 100 + 100(1.06)v + 100(1.06)^2 v^2 + \cdots + 100(1.06)^9 v^9$$

$$= 100\left(1 + 1.06v + 1.06^2 v^2 + \cdots + 1.06^9 v^9\right)$$

Let $1.06v$ be v'

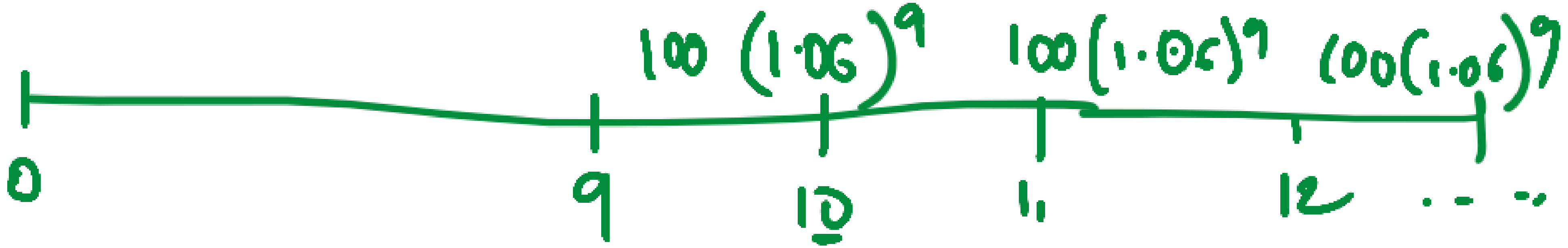
$$\frac{1.06}{1.08} = \frac{1}{1+i'} \quad \therefore \quad 1+i' = \frac{1.08}{1.06}$$

$$i' = 1.886793\%$$

$$= 100\left(1 + v' + (v')^2 + \cdots + (v')^9\right)$$

$$= 100\, \ddot{a}_{\overline{10|}} \text{ @ } i' = 1.886793\% = \boxed{920.6514}$$

$$\frac{1 - v^{n}}{d'}$$



$$PV = 100(1.06)^9 v^{10} + 100(1.06)^9 v^{11} + 100(1.06)^9 v^{12} + \dots$$

$$= 100(1.06)^9 v^{10} \left(1 + v + v^2 + \dots\right)$$

$$= \frac{100(1.06)^9}{(1.08)^{10}} \ddot{a}_{\overline{\infty}|} \ @ \ i = 8\%$$

$$\frac{100 \cdot (1.06)^9}{1.08^{10}} \left(\frac{1}{d}\right) \qquad \left[d = \frac{8\%}{1+8\%}\right]$$

$$= \underline{\underline{1056.45232}}$$

Solution

The present value of the first 10 payments is

$$100\left[1+\frac{1.06}{1.08}+\cdots+\left(\frac{1.06}{1.08}\right)^{9}\right]$$

$$= 100\left[\frac{1-\left(\frac{1.06}{1.08}\right)^{10}}{1-\frac{1.06}{1.08}}\right] = 920.65.$$

Note that the 10th annual payment is made at time $t=9$.
The present value of the rest of the payments is

$$100(1.06)^{9}\left[\frac{1}{(1.08)^{10}}+\frac{1}{(1.08)^{11}}+\cdots\right]$$

$$= 100\left(\frac{1.06}{1.08}\right)^{9}\left[\frac{1}{1.08}+\frac{1}{(1.08)^{2}}+\cdots\right]$$

$$= 100\left(\frac{1.06}{1.08}\right)^{9}\frac{1}{.08} = 1056.45.$$

Thus, the total present value equals

$$920.65+1056.45 = \$1977.10.$$