Lecture 2

INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Class: FY MSc

Subject : Financial Mathematics

Subject Code: PPSAS102

Chapter: Unit 1 – Chapter 1

Chapter Name: Measurement of Interest

Today's Agenda

1. Interest
1. Measures of Interest Rates
1. Fund Value
1. Principle of Consistency
1. Effective Rate of Interest
1. Simple Interest
1. Compound Interest
1. Present Value

9. Effective Rate of Discount
9. Equivalent Rates
9. Simple Discount
9. Compound Discount
9. Useful Insights
9. Nominal Rates of Interest
9. Force of Interest

0 Introduction

What is Interest? (In financial terms)

1 Interest

Interest – **Compensation that a borrowers pays for use of capital.**

Form of a rent that the borrower pays to the lender to compensate for the **loss of use of the capital** by the lender while it is loaned to the borrower.

1.1 Rationale

- **Investment opportunity theory/ Economic productivity of capital** – if you borrow money to run a successful business, the borrowed money allows you to produce more money and you should assign some of that gain to the lender (Good faith)

- **Time preference theory** – people prefer to have money now rather than the same amount of money at some later date. If you lend it, you no longer have the option of immediately using your money. Interest compensates a lender for this loss of choice.

- **Default risk** – lender should be compensated for the possibility that the borrower defaults and the capital is lost.

int erest

Russia

In the real world, investments have an element of risk and investors sometimes lose money.

1.2 Importance of Interest

The Reserve Bank of India sets the " repo rates" and " reverse repo rates", a target rate at which banks can borrow and invest funds with one another. This rate affects the more general cost of borrowing and also has an effect on the **stock** and **bond markets**.

Higher interest rates tend to reduce the value of other investments.

Irrational Exuberance

After the close of trading on North American financial markets on Thursday, December 5, 1996, Federal Reserve Board chairman Alan Greenspan delivered a lecture at The American Enterprise Institute for Public Policy Research.

In that speech, Mr Greenspan commented on the possible negative consequences of “irrational exuberance” in the financial markets.

The speech was widely interpreted by investment traders as indicating that stocks in the US market were overvalued and that the Federal Reserve Board might increase US interest rates, which might affect interest rates worldwide.

Although US markets had already closed, those in the Far East were just opening for trading on December 6, 1996. Japan’s main stock market index dropped 3.2%, the Hong Kong stock market dropped almost 3%. As the opening of trading in the various world markets moved westward throughout the day, market drops continued to occur. The German market fell 4% and the London market fell 2%. When the New York Stock Exchange opened at 9:30 AM EST on Friday, December 6, 1996, it dropped about 2% in the first 30 minutes of trading, although the market did recover later in the day.

Sources: www.federalreserve.gov , www.pbs.org/newshour/bb/economy/december96/greenspan_12-6.html

Key Rates & Mortgage rates (%)

	Current	1 month prior	3 months prior	6 months prior	1 year prior
RBI Target rate (Repo Rate)	4.00	4.00	4.40	5.15	5.40
3-month LIBOR	0.25	0.25	0.34	1.25	2.08
Prime Rate	3.25	3.25	3.25	4.25	5.00
15-year Mortgage	2.42	2.44	2.62	2.75	3.00
30-year Mortgage	2.93	2.88	3.18	3.50	3.60

7% - 8%.

Rates are taken as at 15 September, 2020.

2 Measures of Interest Rates

Commonly used growth patterns for investment – simple and compound interest.

Alternative measures:

- Nominal annual rate of interest
- Rate of discount – Simple and compound
- Force of interest

dt

$\boxed{P \times R \times T}$

2.1 Interest Accumulation

$A(t_1, t_2)$

$A(n)$ $A(t_1, t_1)$

$A(2, 2) =$

$A(0) =$

Common financial transaction – investment of an amount of money at interest.

- Initial amount of money (capital) invested – **Principal**
- Total amount received after a period of time – **Accumulated value**
- Difference between the accumulated value and the principal – **Amount of interest/ interest**, earned during the period of investment

The unit in which time is measured is called the measurement period, or just period.
The most common measurement period is one year, and this will be assumed unless stated otherwise.

$I = y - x$ $\frac{120}{100} = 1.2$

100 120

0 1

Principal x y Accumul value

100

0

0

$A(0) = \frac{100}{100} = 1$

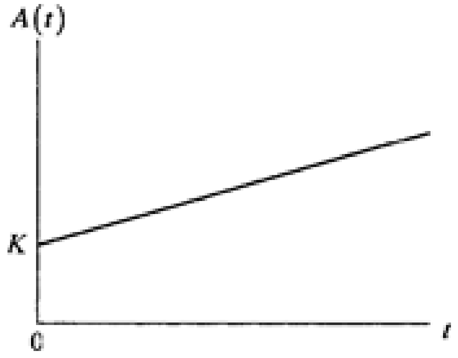
2.2 Accumulation Factor

- Consider an investment of one unit of principal.
- Accumulating factor A(t) gives the accumulated value at time t ≥ 0 of an original investment of 1.
- A(0) = 1 i.e. accumulated value at the time of investment is amount invested, which is 1.
- If C is the amount invested at time 0, accumulated value after t time is C.A(t)
- A(t) is generally an increasing function.
- A decrease in the functional values for increasing t would imply negative interest which is very rare.
- If interest accrues continuously, as is usually the case, the function will be continuous.

$A(t)$.

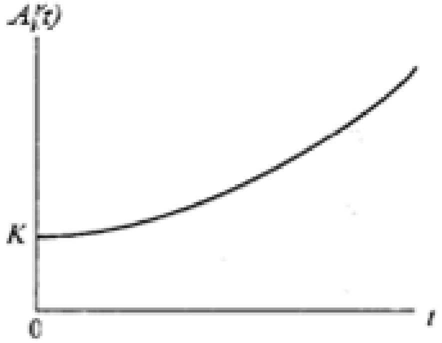
$C \times A(t)$.

2.3 Types of Accumulation Factor

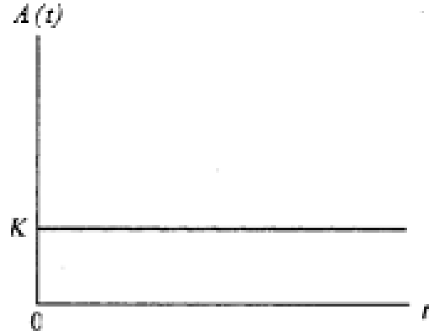


A. Linear accumulation

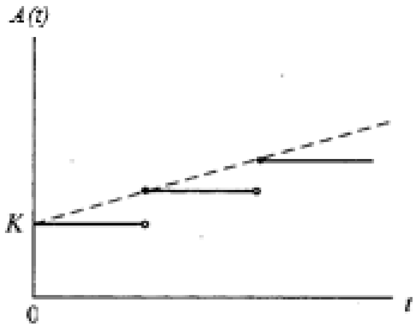
simple int



B. Non-linear accumulation, in this case an exponential curve.



C. An accumulation factor which is horizontal, i.e. the slope is zero.
This figure represents an accumulation factor in which the principal is accruing with no interest



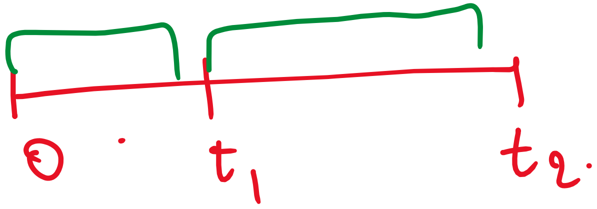
D. A step increasing accumulation factor. This is an accumulating factor in which interest is not accruing continuously but is accruing in discrete segments with no interest accruing between interest payment dates.

3 Fund Value

The **fund value is the total amount an investment currently holds**, including the capital invested and the interest it has earned to date (Accumulated value).

For instance, if 100 is invested at $t = 0$ in an investment giving 6% interest p.a., then the fund value at time $t = 1$ is 106 i.e. (100 + 6% of 100).

4 Principle of Consistency



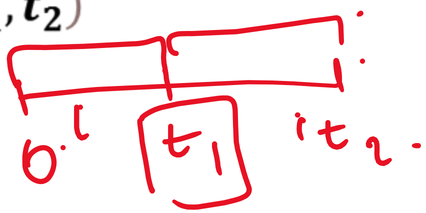
- For $t_1 \le t_2$, we define $A(t_1, t_2)$ to be the accumulation at time t_2 of a unit investment made at time t_1 for a term of (t_2 - t_1)

- The quantity $A(t_1, t_2)$ is called an accumulation factor for a term of (t_2 - t_1). For an investment of sum C at time t_1, the accumulation at time t_2 is = C. $A(t_1, t_2)$

- We define A(t, t) = 1 for all t, reflecting that the accumulation factor must be unity over zero time.

- The proceeds at time t_2 will be $A(t_0, t_2)$ if one invests at time t_0 for term (t_2 - t_0), or $A(t_0, t_1)$ x $A(t_1, t_2)$ if one invests at time t_0 for term (t_1 - t_0) and then, at time t_1, reinvests the proceeds for term (t_2 - t_1) . In a consistent market, these proceeds should not depend on the course of action taken by the investor.

- Accordingly, we say that under the principle of consistency: $\boldsymbol{A(t_0, t_2) = A(t_0, t_1) \; x \; A(t_1, t_2)}$



Question

An investment of Rs. 10,000 is made into a fund at time t = 0. The fund develops the following balances over the next 4 years:

t	A(t)
0	10000.00
1	10600.00
2	11130.00
3	11575.20
4	12153.96

If Rs. 5000 is invested at time t = 2, under the same interest environment, find the accumulated value of the Rs. 5000 at time t = 4.

Question

Arjun has saved Rs. 10,000 in the past 4 months. He plans to invest this in an investment fund. The investment provides returns at a rate of 8% per annum. The investment grows over time t, according to the following investment factor:

$A(t_1, t_2) = (1 + \mathrm{i})^{(t_2 - t_1)}$ for $t_1 < t_2$.

i) Calculate the Accumulated Value of Arjun's investment:

a) After 10 years

b) After 15 years

ii) Calculate the Accumulating factor between time 10 and 15.

5 Effective Rate of Interest

- **General notation – *i***

- **Definition:** The effective rate of interest i is the ratio of the amount of interest earned during the period to the amount of principal invested at the beginning of the period.

- **Alternative definition:** The effective rate of interest i is the amount of money that invested at the beginning of a period will earn during the period, where interest is paid at the end of the period.

- Interest is paid *once per measurement period*. This will be later contrasted with "nominal" rates of interest, in which interest is paid more frequently than once per measurement period.

- Effective rate of interest i_n, in terms of accumulation:

$$i_n = \frac{A(n) - A(n-1)}{A(n-1)} \quad for\ n = 1,2,3 \ldots$$

$\boxed{100}$ $\boxed{104}$

0 4

$$\frac{104-100}{100} = 4\%$$

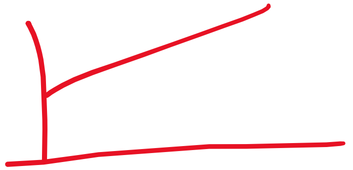
Question

An investment of Rs. 10,000 is made into a fund at time t = 0. The fund develops the following balances over the next 4 years:

t	A(t)
0	10000.00
1	10600.00
2	11130.00
3	11575.20
4	12153.96

Find the effective rate of interest for each of the 4 years.

6 Simple Interest



P×R×T

Simple interest does not compound, meaning that an investor will only gain the principal and the interest on the principal, and not interest on interest

Interest calculated on the principal portion of a loan, investment

AF

AV.

P, I, ...

Under simple interest, the **interest earned every year remains the same.** Int.

Amount under the fund grows linearly.

SI.

9%

100 × 9%

100 × 9%

100 × 9%

9% → 100

9% pa

9% × 100

9

100

109

109 + 9 = 118

109 × 9%

2. → 9

+9

6.1 Example

The current rate of interest quoted by a bank on an investment is **9% simple interest per annum**. Suraj makes an investment of Rs. 1000. Assuming that there are no other transactions, determine the accumulated value just after interest is credited at the end of 3 years.

Suraj invests 1000 at the start.

During the first year his investment will grow at the rate of 9%.
Therefore, balance at the end of first year = 1000(1 + 0.09) = 1090.

Now under the simple interest system the interest amount of 90 will not be reinvested.
Hence the principal amount remains 1000 as it is.
Balance at the end of second year = 1000(1 + 2x0.09) = 1180.

Similarly, balance at the end of third year = 1000 (1 + 3x0.09) = 1270.
Thus, Accumulated value at the end of 3 years is 1270.

C	$\boxed{1000}$	1090	1180	1270
	0	1	2	3

n . 9% SI p.a

Int amt for the 1st year $= \underset{i}{9\%} \times \underset{C}{1000} = 90$

AV at time 1 $= 1000 + 90 = 1090$ $\qquad c + ci = c(1+i)$

Int amt for the 2nd year $= \underset{i}{9\%} \times \underset{C}{1000} = 90$

AV at time 2 $= 1000 + 90 + 90 = 1180$ $\qquad C + ci + ci = c(1+2i)$

Int 3rd year $= 9\% \times 1000 = 90$

AV at 3 $= \underset{C}{1000} + \underset{ci}{90} + \underset{ci}{90} + \underset{ci}{90} = 1270 = c(1+3i)$

$$\boxed{AV = c(1+ni)}$$

$\underline{(1+ni)}$ Accumulation function.

$$\frac{(1+a\%)}{81} = (1+i)^3$$

$$(1+a\%)^{\frac{1}{3}} - 1 = i$$

6.2 Simple Interest - Generalisation

Consider:
Amount Invested – C
Simple interest rate – I

- Accumulated value at the end of the 1st period = C(1+i)
- Accumulated value at the end of the 2nd period = C(1+2i)
- And so on, the accumulated value at the end of the nth period = C(1+ni)

Simple Interest Formula = P x (1 + r x t)

R – rate of interest
T - time

The **Accumulation factor** under simple interest system for a period from t_1 to t_2, where $t_1 < t_2$ is:

$A(t_1, t_2) = (1 + (t_2 - t_1) * i)$ or
$A(n) = (1 + ni)$

$A(t_1, t_2) = (1 + (t_2 - t_1)i)$
$A(n) = (1 + ni)$

Question

1)
Amount Deposited = 10,000
Simple Interest = 7% pa

Accumulated amount after 3 years? $\boxed{12100}$

2)
Amount Deposited = 15,000 Simple Interest = 5% pa

Compare how much the investor would have after 8 years if the money was:

a. Invested for 8 years — $\boxed{21000}$ → $15000 \times (1 + 8 \times 5\%)$
b. Invested for 4 years, then immediately reinvested for further 4 years.

21,600

$C \times (1 + ni)$

10,000

$\boxed{10210}$

SI - $\boxed{7\% \text{ pa}}$

$\boxed{10,000}(1 + 3 \times 0.07)$

$10,000(1.21)$

$\boxed{12,100}$

$10,000 \times 7\% \times 3$

Int.

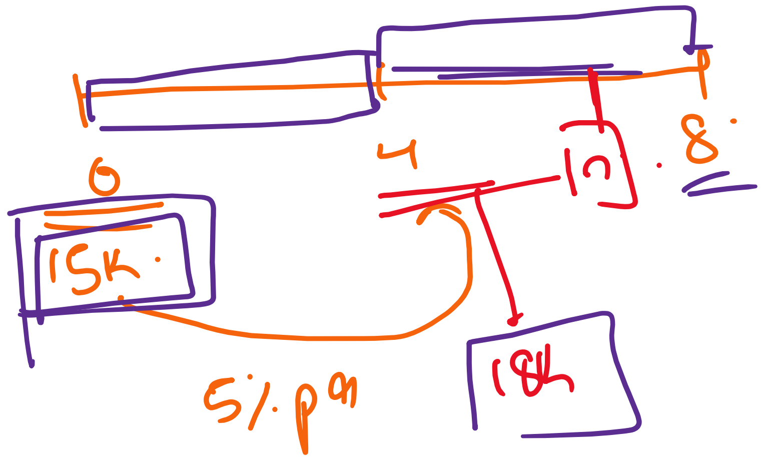
At $t=0$, Int = 15000

$C = 15000$

AV at time 4 $= C \times (1+4i)$

$= 15000\,(1+4\times 5\%)$

$= 18000$



a) $15000 = C$

b)

AV at time 8 $= C \times (1+4i)$

$= 18000\,(1+4\times 5\%)$

$= \boxed{21600}$

$C\times(1+7i) = A$, $A\cdot(1+i)$

7 Compound Interest

Int

Clearly, it is more advantageous for the investor to invest the interest earned so far, in order to earn more ahead.

1000

0 1 2 3 4

The word **“compound” refers to the process of interest being reinvested to earn additional interest**. With compound interest, the total investment of principal and interest earned to date is kept invested at all times.

9% CI pa

1000

0   1   2   3

$9\% \times 1000$

$= 90$

$AV = 1090$

$9\% \times 1090$

$= 98.1$

$AV = 1090 + 98.1$

$= 1188.1$

$9\% \times 1188.1$

$= 106.929$

$AV = 1295.029$

7.1 Example

The current rate of interest quoted by a bank on an investment is **9% compound interest per annum**. Suraj makes an investment of Rs. 1000. Assuming that there are no other transactions, determine the accumulated value just after interest is credited at the end of 3 years.

CI

1000

0 1 2 3

$1000 \times 9\% = 90$

$1000 + 90 = 1090$

Suraj invests 1000 at the start.

During the first year his investment will grow at the rate of 9%.
Therefore, balance at the end of first year = 1000 + 1000 x 0.09 = 1000(1.09) = 1090

$C + C \times i \qquad C(1+i)$

Under compound interest this balance is reinvested and earns interest in the second year, producing a balance of = 1090 + 1090 x 0.09 = 1090(1.09) = $1000(1.09)^2$ = 1188.10 at the end of the 2nd year.

$c(1+i) + c(1+i) \times i \qquad c(1+i)(1+i) = c(1+i)^2$

The balance at the end of the third year will be
= 1188.10 +1188.10 x 0.09 = (1188.10)(1.09) = $1000(1.09)^3$ = 1295.03

$C(1+i)^2 + C \times (1+i)^2 \times i \qquad c(1+i)^2(1+i) = c(1+i)^3$

$\boxed{AV = c(1+i)^n}$

$AF = (1+i)^n$

$c(1+i)^1$

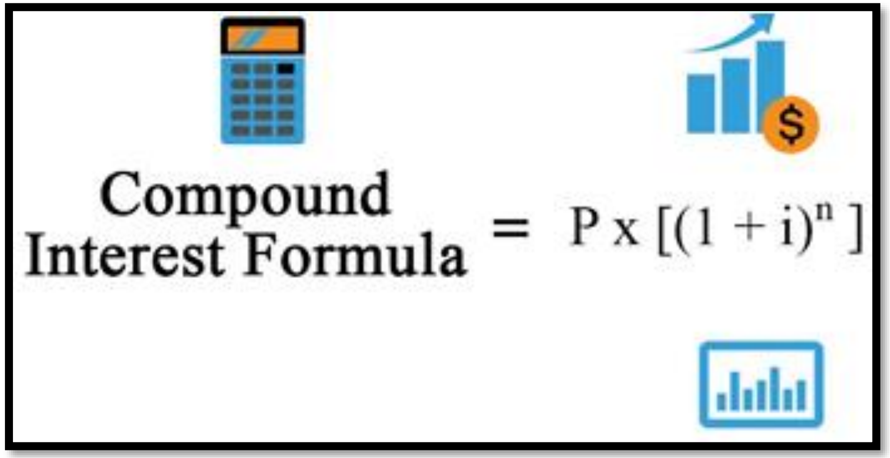
$c(1+i)^2$

$c(1+i)^3$

7.2 Compound Interest - Generalisation

Consider:
Amount Invested – C
Compound interest rate – i

- Accumulated value at the end of the 1st period = C + Ci = C(1 + i)
- Accumulated value at the end of the 2nd period = C(1 + i) + C(1 + i) x i = $C(1+i)^2$
- And so on the account will continue to grow by a factor of (1 + i) per year, resulting in a balance of $\boxed{C(1+i)^n}$ at the end of n years



The **Accumulation factor** under compound interest system for a period from t_1 to t_2, where $t_1 < t_2$ is:

$A(t_1, t_2) = (1+i)^{(t_2 - t_1)}$ or
$A(n) = (1+i)^n$

C

$A(n) = (1+ni)$

$A(t_1, t_2) = (1+i)^{(t_2-t_1)}$.

$A(n) = (1+i)^n$.

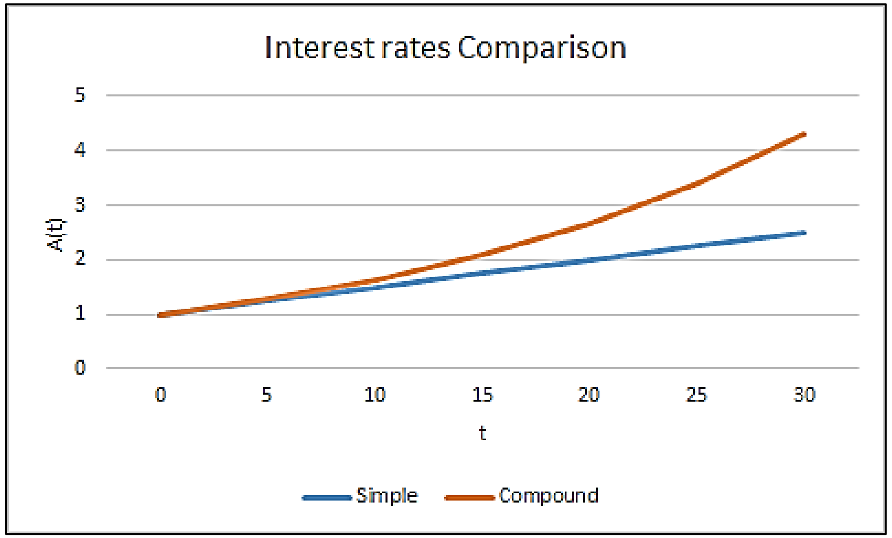
JAQS

7.3 Simple Interest vs Compound Interest

| S|P. |

1

- It is clear that simple and compound interest produces the same result over one measurement period.
- As seen above, the accumulated value at the end of first year is 1090 under both simple interest and compound interest.
- Over a longer period, compound interest produces a larger accumulated value than simple interest while the opposite is true over a shorter period.



SI
AV
AF

CI.
AV
AF

Watch – Difference between SI & CI
https://www.youtube.com/watch?v=5wpsLW5JEms

SI:

9% pa

3 years

0 — 90 — 1 — 90 — 2 — 90 — 3

1000

$1000(1+9\%)$

$= \boxed{1090}$

$1000(1+2\times 9\%)$

$= 1180$

$1000(1+3\times 9\%)$

$= 1270$

int amt

$\boxed{\text{absolute amt}}$:

1090 − 1000

1180 − 1090

1270 − 1180

contd:

C = 1000

n = 0.25

i = 9% pa SI / CI

SI :

$1000(1+0.25\times 9\%)$

$= \boxed{1022.5}$

$CI = 1000\times(1+9\%)^{0.25}$

$= \boxed{1021.778}$

CI. 9% pa.

3 years.

0 — 90 — 1 — 98.1 — 2 — 106.92. — 3

$\underline{\underline{1000.}}$

$1000(1+9\%) = \boxed{1090}$

$1000(1+9\%)^2. = 1188.1$

$1000.(1+9\%)^3. = 1295.03$

CI: $\boxed{\text{absolute amt}}$ does not stay constant.

$\boxed{\text{ratio}}$: stays constant

t = 3.

$t=1: \quad \frac{1090}{1000} = 1.09$

$t=2 = \frac{1188.1}{1090} = 1.09.$

$IA_n > IA_{n-1} > IA_{n-2} \ldots$

c = 1000 n = 0.25.

i = 9%.

i = 9%

SI =

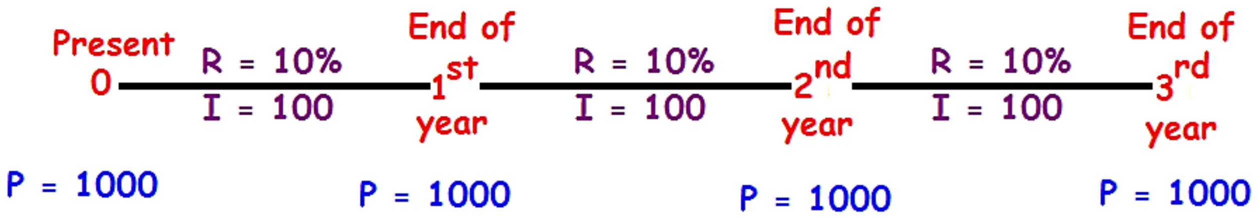
CI =

7.3 Simple Interest vs Compound Interest

2

- Under <u>simple interest</u>, it is the **absolute amount of growth** that is **constant** over equal periods of time
- Under <u>compound interest</u>, it is the **relative rate of growth** that is **constant**

SIMPLE INTEREST :



COMPOUND INTEREST :