



Class: FY MSc

Subject : Financial Mathematics

Subject Code: PPSAS102

Chapter: Unit 1 – Chapter 2

Chapter Name: Measurement of Interest

Today's Agenda

1. Interest

1. Measures of Interest Rates

1. Fund Value

1. Principle of Consistency

1. Effective Rate of Interest

1. Simple Interest

1. Compound Interest

1. Present Value

9. Effective Rate of Discount

9. Equivalent Rates

9. Simple Discount

9. Compound Discount

9. Useful Insights

9. Nominal Rates of Interest

9. Force of Interest

0 Introduction



What is Interest? (In financial terms)

1 Interest



Interest – **Compensation that a borrowers pays for use of capital.**

Form of a rent that the borrower pays to the lender to compensate for the **loss of use of the capital** by the lender while it is loaned to the borrower.

1.1 Rationale

- **Investment opportunity theory/ Economic productivity of capital** – if you borrow money to run a successful business, the borrowed money allows you to produce more money and you should assign some of that gain to the lender (Good faith)
- **Time preference theory** – people prefer to have money now rather than the same amount of money at some later date. If you lend it, you no longer have the option of immediately using your money. Interest compensates a lender for this loss of choice.
- **Default risk** – lender should be compensated for the possibility that the borrower defaults and the capital is lost.

inf cost

Russia

In the real world, investments have an element of risk and investors sometimes lose money.

1.2 Importance of Interest

The Reserve Bank of India sets the “repo rates” and “reverse repo rates”, a target rate at which banks can borrow and invest funds with one another. This rate affects the more general cost of borrowing and also has an effect on the **stock** and **bond markets**.

Higher interest rates tend to reduce the value of other investments.

Irrational Exuberance

After the close of trading on North American financial markets on Thursday, December 5, 1996, Federal Reserve Board chairman Alan Greenspan delivered a lecture at The American Enterprise Institute for Public Policy Research.

In that speech, Mr Greenspan commented on the possible negative consequences of “irrational exuberance” in the financial markets.

The speech was widely interpreted by investment traders as indicating that stocks in the US market were overvalued and that the Federal Reserve Board might increase US interest rates, which might affect interest rates worldwide.

Although US markets had already closed, those in the Far East were just opening for trading on December 6, 1996. Japan's main stock market index dropped 3.2%, the Hong Kong stock market dropped almost 3%. As the opening of trading in the various world markets moved westward throughout the day, market drops continued to occur. The German market fell 4% and the London market fell 2%. When the New York Stock Exchange opened at 9:30 AM EST on Friday, December 6, 1996, it dropped about 2% in the first 30 minutes of trading, although the market did recover later in the day.

Sources: www.federalreserve.gov , www.pbs.org/newshour/bb/economy/december96/greenspan_12-6.html

Key Rates & Mortgage rates (%)

	Current	1 month prior	3 months prior	6 months prior	1 year prior
RBI Target rate (Repo Rate)	4.00	4.00	4.40	5.15	5.40
3-month LIBOR	0.25	0.25	0.34	1.25	2.08
Prime Rate	3.25	3.25	3.25	4.25	5.00
15-year Mortgage	2.42	2.44	2.62	2.75	3.00
30-year Mortgage	2.93	2.88	3.18	3.50	3.60

7% - 8% -

Rates are taken as at 15 September, 2020.

2

Measures of Interest Rates

Commonly used growth patterns for investment – simple and compound interest.

Alternative measures:

- Nominal annual rate of interest ✓
- Rate of discount – Simple and compound ✓
- Force of interest ✓

dt .

$$P \times R \times T$$

2.1 Interest Accumulation

Common financial transaction – investment of an amount of money at interest.

- Initial amount of money (capital) invested – **Principal**
- Total amount received after a period of time – **Accumulated value**
- Difference between the accumulated value and the principal – **Amount of interest/ interest**, earned during the period of investment

The unit in which time is measured is called the measurement period, or just period.

The most common measurement period is one year, and this will be assumed unless stated otherwise.

$$A(t_1, t_2)$$

$$A(n) \quad A(t_1, t_1)$$

$$A(2, 2) =$$

$$A(0) =$$

$$2 = \frac{(4 - 1) \cdot \frac{120}{100}}{1} = \frac{1 \cdot 2}{1}$$

Principal

Accumulated value

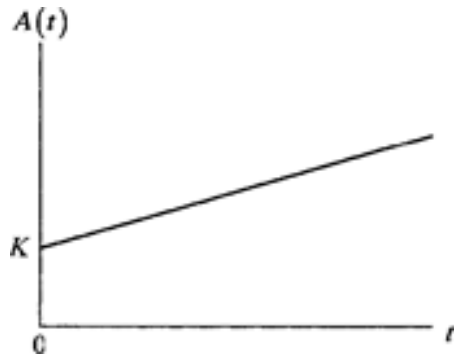
$$A(0) = \frac{100}{100} = 1$$

2.2 Accumulation Factor

- Consider an investment of one unit of principal.
- Accumulating factor $A(t)$ gives the accumulated value at time $t \geq 0$ of an original investment of 1.
- $A(0) = 1$ i.e. accumulated value at the time of investment is amount invested, which is 1.
- If C is the amount invested at time 0, accumulated value after t time is $C.A(t)$
- $A(t)$ is generally an increasing function.
- A decrease in the functional values for increasing t would imply negative interest which is very rare.
- If interest accrues continuously, as is usually the case, the function will be continuous.

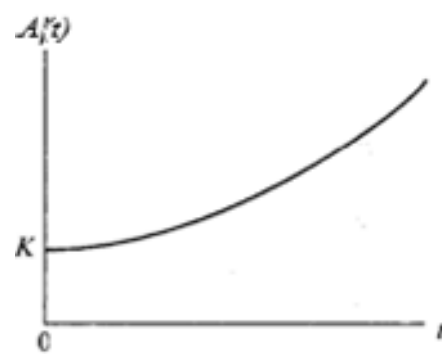
$$\underline{A(t)}$$
$$\underline{C + A(t)}$$

2.3 Types of Accumulation Factor

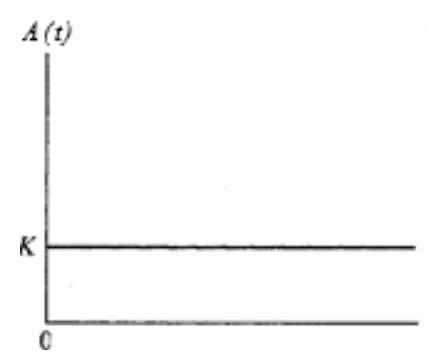


A. ~~Linear~~ accumulation

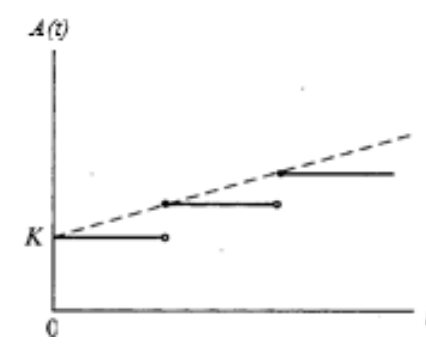
Simple int.



B. Non-linear accumulation, in this case an exponential curve.



C. An accumulation factor which is horizontal, i.e. the slope is zero. This figure represents an accumulation factor in which the principal is accruing with no interest



D. A step increasing accumulation factor. This is an accumulating factor in which interest is not accruing continuously but is accruing in discrete segments with no interest accruing between interest payment dates.

3 Fund Value

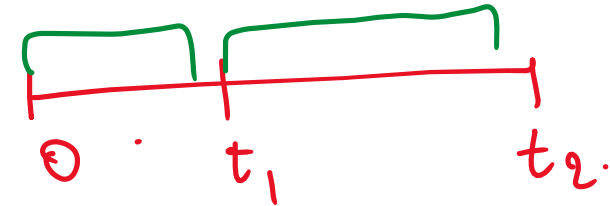


The **fund value** is the total amount an investment currently holds, including the capital invested and the interest it has earned to date (Accumulated value).

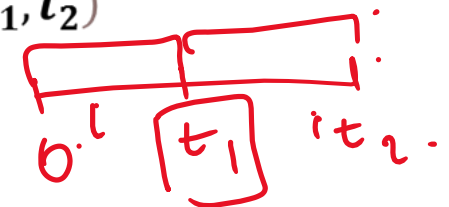
For instance, if 100 is invested at $t = 0$ in an investment giving 6% interest p.a., then the fund value at time $t = 1$ is 106 i.e. $(100 + 6\% \text{ of } 100)$.

4

Principle of Consistency



- For $t_1 \leq t_2$, we define $A(t_1, t_2)$ to be the accumulation at time t_2 of a unit investment made at time t_1 for a term of $(t_2 - t_1)$
- The quantity $A(t_1, t_2)$ is called an accumulation factor for a term of $(t_2 - t_1)$. For an investment of sum C at time t_1 , the accumulation at time t_2 is $= C \cdot A(t_1, t_2)$
- We define $A(t, t) = 1$ for all t , reflecting that the accumulation factor must be unity over zero time.
- The proceeds at time t_2 will be $A(t_0, t_2)$ if one invests at time t_0 for term $(t_2 - t_0)$, or $A(t_0, t_1) \times A(t_1, t_2)$ if one invests at time t_0 for term $(t_1 - t_0)$ and then, at time t_1 , reinvests the proceeds for term $(t_2 - t_1)$. In a consistent market, these proceeds should not depend on the course of action taken by the investor.
- Accordingly, we say that under the principle of consistency: $\underline{A(t_0, t_2)} = \underline{A(t_0, t_1)} \times A(t_1, t_2)$





Question

An investment of Rs. 10,000 is made into a fund at time $t = 0$. The fund develops the following balances over the next 4 years:

t	$A(t)$
0	10000.00
1	10600.00
2	11130.00
3	11575.20
4	12153.96

If Rs. 5000 is invested at time $t = 2$, under the same interest environment, find the accumulated value of the Rs. 5000 at time $t = 4$.



Question

Arjun has saved Rs. 10,000 in the past 4 months. He plans to invest this in an investment fund. The investment provides returns at a rate of 8% per annum. The investment grows over time t , according to the following investment factor:

$$A(t_1, t_2) = (1 + i)^{(t_2 - t_1)} \text{ for } t_1 < t_2.$$

- i) Calculate the Accumulated Value of Arjun's investment:
 - a) After 10 years
 - b) After 15 years
- ii) Calculate the Accumulating factor between time 10 and 15.

5

Effective Rate of Interest

- General notation - i

- Definition:** The effective rate of interest i is the ratio of the amount of interest earned during the period to the amount of principal invested at the beginning of the period.

- Alternative definition:** The effective rate of interest i is the amount of money that invested at the beginning of a period will earn during the period, where interest is paid at the end of the period.

- Interest is paid once per measurement period. This will be later contrasted with “nominal” rates of interest, in which interest is paid more frequently than once per measurement period.

- Effective rate of interest i_n , in terms of accumulation:

$$i_n = \frac{A(n) - A(n-1)}{A(n-1)} \text{ for } n = 1, 2, 3 \dots$$

Handwritten calculation and timeline diagram illustrating the effective rate of interest:

Timeline: 0 ————— 4

At time 0: 100 (Principal)

At time 4: 104 (Accumulated amount)

Calculation: $\frac{104 - 100}{100} = 4\%$



Question

An investment of Rs. 10,000 is made into a fund at time $t = 0$. The fund develops the following balances over the next 4 years:

t	$A(t)$
0	10000.00
1	10600.00
2	11130.00
3	11575.20
4	12153.96

Find the effective rate of interest for each of the 4 years.

6

Simple Interest



Simple interest does not compound, meaning that an investor will only gain the principal and the interest on the principal, and not interest on interest

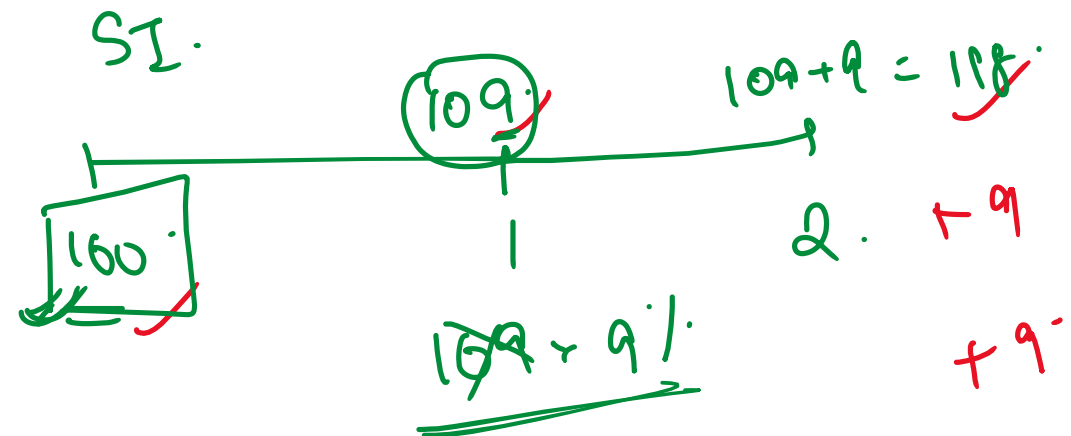
Interest calculated on the principal portion of a loan, investment

Under simple interest, the interest earned every year remains the same. *Int.*

Amount under the fund grows linearly.

$$\begin{aligned} & \underline{9\%} \\ & \underline{100 \times 9\%} \\ & \underline{100 \times 9\%} \\ & \underline{100 \times 9\%} \end{aligned}$$

$$\begin{aligned} & 9\% \text{ pa} \\ & \underline{9\% \times 100} \\ & \underline{9} \end{aligned}$$



6.1 Example

The current rate of interest quoted by a bank on an investment is 9% simple interest per annum. Suraj makes an investment of Rs. 1000. Assuming that there are no other transactions, determine the accumulated value just after interest is credited at the end of 3 years.

Suraj invests 1000 at the start.

During the first year his investment will grow at the rate of 9%.
Therefore, balance at the end of first year = $1000(1 + 0.09) = 1090$.

Now under the simple interest system the interest amount of 90 will not be reinvested.
Hence the principal amount remains 1000 as it is.
Balance at the end of second year = $1000(1 + 2 \times 0.09) = 1180$.

Similarly, balance at the end of third year = $1000(1 + 3 \times 0.09) = 1270$.
Thus, Accumulated value at the end of 3 years is 1270.



Int amt for the 1st year = $9\% \times 1000 = 90$

AV at time 1 = $\underline{1000} + 90 = 1090$ $C + Ci = C(1+i)$

Int amt for the 2nd year = $9\% \times 1000 = 90$

AV at time 2 = $\underline{1000} + 90 + 90 = 1180$

$$\boxed{AV = \underline{C(1+ni)}}$$

Int 3rd year = $9\% \times 1000 = 90$

AV at 3 = $\underline{1000} + 90 + 90 + 90 = 1270$

$$C + Ci + Ci = C(1+2i)$$

$(1+ni)$
Accumulation
Factor.

$$(1+ai)^3 = (1+i)^3.$$

$$\frac{81}{81}$$

$$(1+ai)^{\frac{1}{3}} - 1 = \sqrt[3]{1}$$

6.2 Simple Interest - Generalisation

Consider:

Amount Invested – C

Simple interest rate – i

- Accumulated value at the end of the 1st period = $C(1+i)$
- Accumulated value at the end of the 2nd period = $C(1+2i)$
- And so on, the accumulated value at the end of the n th period = $C(1+ni)$

The **Accumulation factor** under simple interest system for a period from t_1 to t_2 , where $t_1 < t_2$ is:

$$A(t_1, t_2) = (1 + (t_2 - t_1) * i) \text{ or}$$

$$A(n) = (1 + ni)$$

$$A(t_1, t_2) = (1 + (t_2 - t_1)i)$$

$$A(n) = (1 + ni)$$



Simple Interest Formula = $P \times (1 + r \times t)$

R – rate of interest

T - time





Question

1)

Amount Deposited = 10,000

Simple Interest = 7% pa

Accumulated amount after 3 years?

$$C = (1 + n\dot{r})$$

$$\underline{\underline{10,000}}$$

$$\boxed{\text{SI-}} \\ \boxed{7\% \text{ pa.}}$$

$$\boxed{12100}$$

$$\boxed{10210}$$

$$\boxed{10,000 (1 + 3 \times 0.07)} \\ 10,000 (1.21) \\ \boxed{12,100}$$

2)

Amount Deposited = 15,000 Simple Interest = 5% pa

Compare how much the investor would have after 8 years if the money was:

a. Invested for 8 years

b. Invested for 4 years, then immediately reinvested for further 4 years.

$$\boxed{21000} \rightarrow 15000 + (1 + 8 \times 5\%) \\ \underline{\underline{21,600}}$$

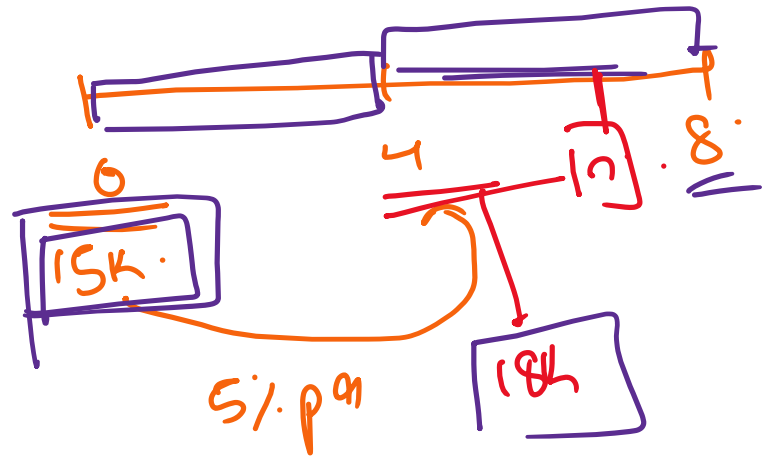
$$\underline{\underline{10,000 \times 7\% \times 8}} \\ \text{Int.}$$

At $t=0$, $\text{Int} = 15000$

$$C = 15000$$

$$\begin{aligned}\text{AV at time } 4 &= C \times (1 + 4i) \\ &= 15000 (1 + 4 \times 5\%) \\ &= 18000\end{aligned}$$

$$\begin{aligned}\text{AV at time } 8 &= C \times (1 + 4i) \\ &= 18000 (1 + 4 \times 5\%) \\ &= \boxed{21600}\end{aligned}$$



a) $15000 = C$

b).

$$\underline{C \times (1 + 4i)} = A \quad \cdot \quad A \cdot (1 + i)$$

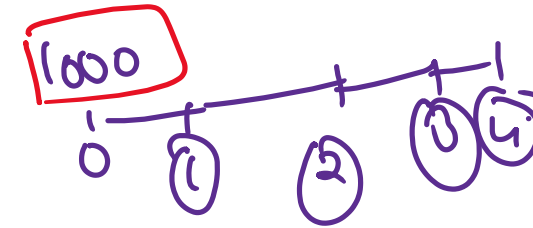
7

Compound Interest

Int =

Clearly, it is more advantageous for the investor to invest the interest earned so far, in order to earn more ahead.

The word “**compound**” refers to the process of interest being reinvested to earn additional **interest**. With compound interest, the total investment of principal and interest earned to date is kept invested at all times.



What is compound interest?

The basic principle of compound interest is earning **additional interest on interest**.

Once you earn your **first** interest payment, it is **added** to the principle.



1000

0

1

2

3

$$\frac{9\% \times 1000}{= 90}$$

$$AV = \underline{1090}$$

$$\frac{9\% \times 1090}{= 98.1}$$

$$AV = 1090 + 98.1 \\ = 1188.1$$

$$9\% \times 1188.1$$

$$= \underline{106.929}$$

$$AV = 1295.029$$

9% CI pa.

7.1 Example

The current rate of interest quoted by a bank on an investment is **9% compound interest per annum**. Suraj makes an investment of Rs. 1000. Assuming that there are no other transactions, determine the accumulated value just after interest is credited at the end of 3 years.

Suraj invests 1000 at the start.

During the first year his investment will grow at the rate of 9%.

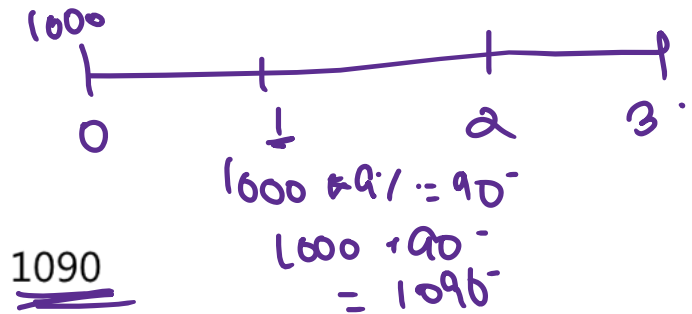
Therefore, balance at the end of first year = $\frac{1000}{C} + \frac{1000 \times 0.09}{C \times i} = \frac{1000(1.09)}{C(1+i)} = \underline{1090}$

Under compound interest this balance is reinvested and earns interest in the second year, producing a balance of = $1090 + 1090 \times 0.09 = 1090(1.09) = 1000(1.09)^2 = 1188.10$ at the end of the 2nd year.

The balance at the end of the third year will be

= $1188.10 + 1188.10 \times 0.09 = (1188.10)(1.09) = 1000(1.09)^3 = 1295.03$

CI



$$AV = \frac{C(1+i)^n}{AF}$$

$$AF = (1+i)^n$$

$$\begin{aligned} &C(1+i) \\ &C(1+i)^2 \\ &C(1+i)^3 \end{aligned}$$

7.2 Compound Interest - Generalisation

Consider:

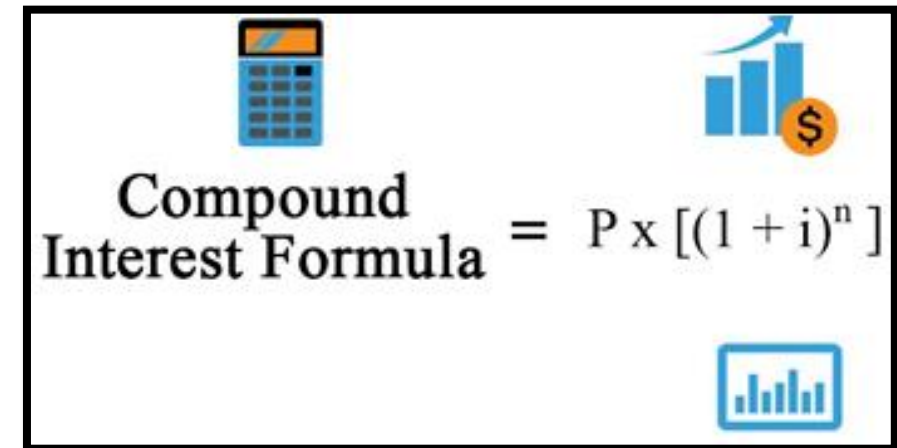
Amount Invested – C

Compound interest rate – i

- Accumulated value at the end of the 1st period = $C + Ci = C(1 + i)$
- Accumulated value at the end of the 2nd period = $C(1 + i) + C(1 + i) \times i = C(1 + i)^2$
- And so on the account will continue to grow by a factor of $(1 + i)$ per year, resulting in a balance of $C(1 + i)^n$ at the end of n years

The **Accumulation factor** under compound interest system for a period from t_1 to t_2 , where $t_1 < t_2$ is:

$$A(t_1, t_2) = (1 + i)^{(t_2 - t_1)} \text{ or } A(n) = (1 + i)^n$$



Compound Interest Formula = $P \times [(1 + i)^n]$

③

$$A(n) = (1 + ni) \quad \bigg| \quad A(t_1, t_2) = (1 + i)^{(t_2 - t_1)}$$

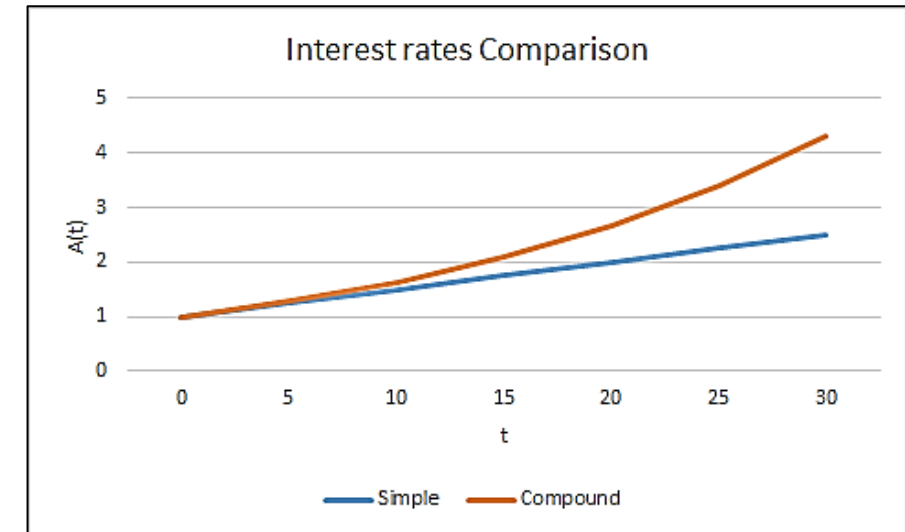
$$A(n) = (1 + i)^n$$

7.3 Simple Interest vs Compound Interest

1st yr.

1

- It is clear that simple and compound interest produces the same result over one measurement period.
- As seen above, the accumulated value at the end of first year is 1090 under both simple interest and compound interest.
- Over a longer period, compound interest produces a larger accumulated value than simple interest while the opposite is true over a shorter period.



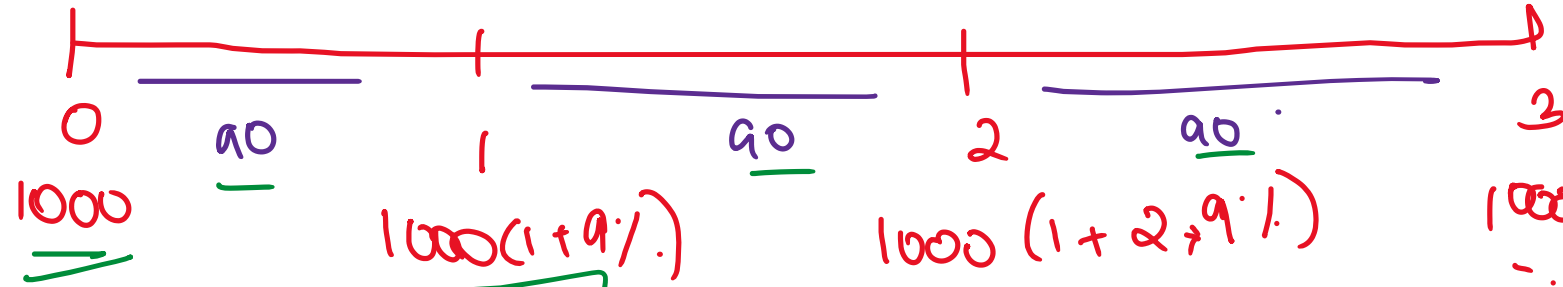
Watch – Difference between SI & CI

<https://www.youtube.com/watch?v=5wpsLW5JEms>

SI
AV
AF

CI.
AV AF

SI:



9% pa
3 years

int amt.

absolute amt :

$$\begin{aligned} 1090 - 1000 \\ 1180 - 1090 \\ 1270 - 1180 \end{aligned}$$

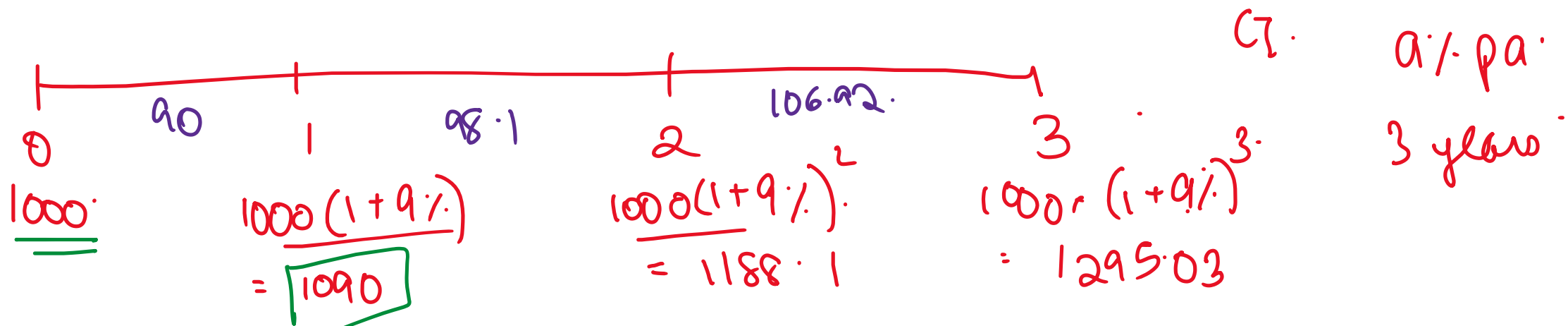
compound:-

$$\begin{aligned} CI &= 1000 \times (1 + 9\%)^{0.25} \\ &= 1021.78 \end{aligned}$$

$$\begin{aligned} C &= 1000 \\ n &= 0.25 \\ i &= 9\% \text{ pa SI / CI} \end{aligned}$$

SI :

$$\begin{aligned} 1000(1 + 0.25 \times 9\%) \\ = 1022.5 \end{aligned}$$



CI: Absolute amt does not stay constant.

ratio: stays constant

t=1:

$$\frac{1090}{1000} = 1.09$$

t=2 =

$$\frac{1188.1}{1090} = 1.09$$

t=3.

$$IA_n > IA_{n-1} > IA_{n-2}$$

SI =

$$\frac{1188.1}{1090} = 1.09 \quad \left| \begin{array}{l} i = a\% \\ i = a\% \end{array} \right.$$

C = 1000 n = 0.25

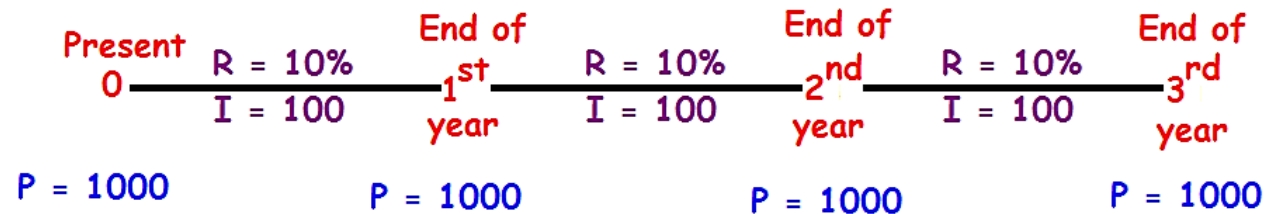
C7 =

7.3 Simple Interest vs Compound Interest

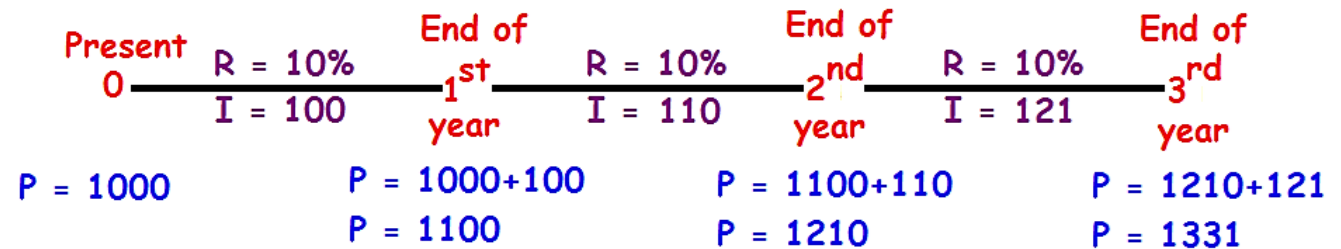
2

- Under simple interest, it is the absolute amount of growth that is constant over equal periods of time
- Under compound interest, it is the relative rate of growth that is constant

SIMPLE INTEREST :



COMPOUND INTEREST :



7.3 Simple Interest vs Compound Interest

3

- A constant rate of simple interest implies a decreasing effective rate of interest.

Let i be the constant rate of simple interest and let i_n be the effective rate of interest for the n th period. Then we have

$$i_n = \frac{A(n) - A(n-1)}{A(n-1)} = \frac{(1+n \cdot i) - (1+(n-1)i)}{(1+(n-1)i)} = \frac{i}{(1+(n-1)i)} \text{ for } n = 1, 2, 3, \dots, \text{ which is a decreasing function of } n.$$

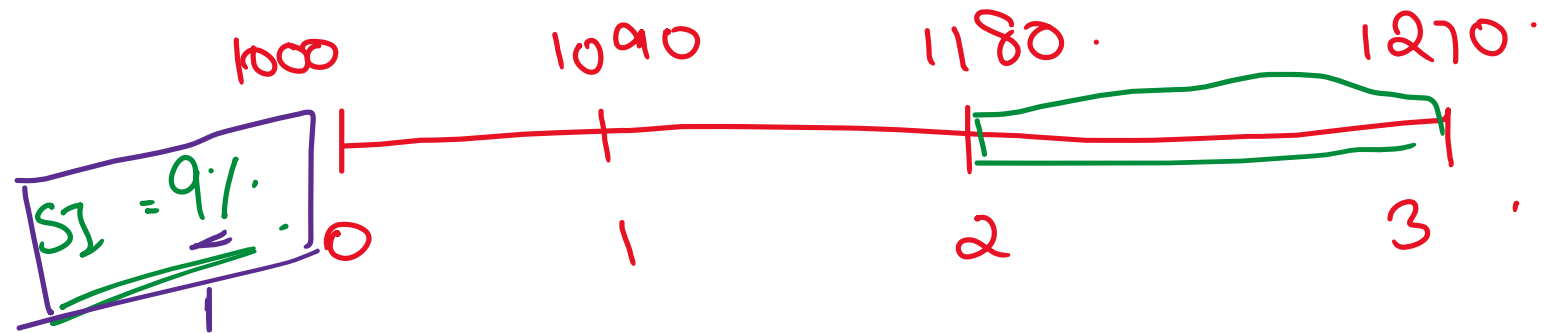
- In the case of compound interest, it can be shown that a constant rate of compound interest implies a constant effective rate of interest and, moreover, that the two are equal.

Let i be the constant rate of compound interest and let i_n be the effective rate of interest for the n th period. Then we have

$$i_n = \frac{A(n) - A(n-1)}{A(n-1)} = \frac{(1+i)^n - (1+i)^{n-1}}{(1+i)^{n-1}} = (1+i)^{n-(n-1)} - 1 = (1+i) - 1 = i$$

$$i_n = i$$

which is independent of n . Thus, although defined differently, a rate of compound interest and the corresponding effective rate of interest are same.



$$i_3 = \frac{1270 - 1180}{1180} = 7.62712\%$$

$$i_2 = \frac{1180 - 1090}{1090} = 8.2569\%$$

$$i_1 = \frac{1090 - 1000}{1000} = 9\%$$

3 years
 $SI = 9\% \text{ p.a.}$
 $i_{SI} = 9\% \text{ p.a.}$

Eff RI = $\frac{\text{Int amt}}{\text{Amt @ } n-1}$

1 → 2 → 3
 9% 8.2569% 7.62712%

Eff.
 rate of
 int

$$\begin{array}{c} A(n-1) \quad \text{-----} \quad A(n) \\ | \quad \quad \quad | \\ n-1 \quad \quad \quad n \end{array}$$

$$\underline{\underline{A(n) = (1+i)^n}}$$

SI.

$$i_n = \frac{A(n) - A(n-1)}{A(n-1)}$$

← ERI.

CI

$$i_n = \underline{\underline{\quad \quad \quad}}$$

$$i_n = \frac{(1+n i) - (1+(n-1)i)}{(1+(n-1)i)}$$

$$i_n = \frac{\cancel{1} + \cancel{n} - \cancel{1} - \cancel{n} + i}{(1+(n-1)i)}$$

$$\begin{array}{c} n \uparrow \\ i_n \downarrow \end{array}$$

simple
int

$$\begin{array}{c} i_n \\ \uparrow \\ \text{ERI} \end{array} = \frac{i}{(1+(n-1)i)}$$

CL.

$$i_n = \frac{A(n) - A(n-1)}{A(n-1)}$$

$$i_n = \frac{A(n)}{A(n-1)} - \frac{A(n-1)}{A(n-1)}$$

$$i_n = \frac{(1+i)^n}{(1+i)^{n-1}} - 1$$

$$i_n = (1+i)^1 - 1$$

$$\boxed{i_n = i}$$

$$\frac{A(n) - A(n-1)}{A(n-1)}$$

a. 1. CL. $i_n = i$

$$\begin{array}{cccc} 1000 & 1090 & 1188.1 & 1295.03 \\ | & | & | & | \\ 0 & 1 & 2 & 3 \end{array}$$

$$i_3 = \frac{1295.03 - 1188.1}{1188.1}$$

$$i_3 = 9\%$$

7.3 Simple Interest vs Compound Interest

4

- Compound interest is used almost exclusively for financial transactions covering a period of one year or more and is often used for shorter term transactions as well.
- Simple interest is occasionally used for short-term transactions and as an approximation for compound interest over fractional periods.

AV, AF, ERI, SI
CI.





Question

An investment of Rs. 10,000 in an account accumulates to Rs. 30,000 after 5 years.

1. State the accumulating factor $A(0,5)$ or $A(5)$.

2. Find the simple annual interest rate which would give the accumulation factor in part 1.

3. Find the annual compound interest rate which would give the accumulation factor in part 1.

(2)

$$A(n) = (1 + ni)$$

$$3 = (1 + 5i)$$

$$\underline{\underline{i = 40\%}}$$

(1)

$$\underline{\underline{A(5) = 3}} = 300\%$$

(3)

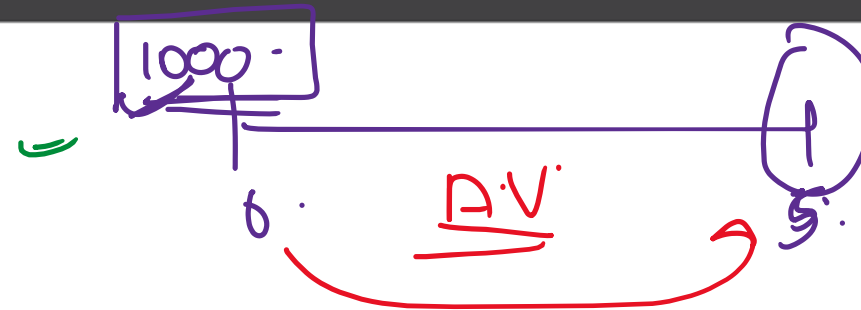
$$A(5) = (1+i)^5$$

$$3 = (1+i)^5$$

$$3^{\frac{1}{5}} = (1+i)$$

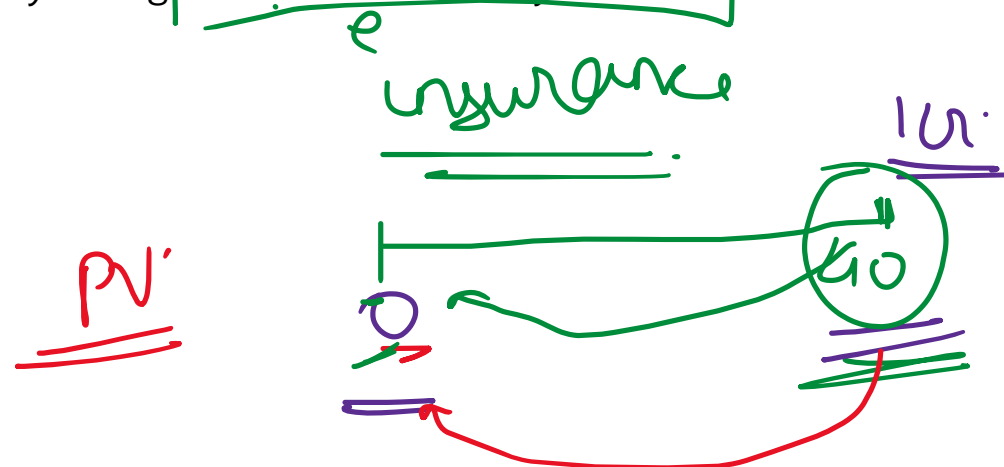
$$\underline{\underline{i = 24.5731\%}}$$

8 Present Value



- Accumulation factors were introduced to quantify the growth of an initial investment as time moves forward. However, one can consider the situation in the opposite direction.
- As an actuary we are more interested in the 2nd situation.
- If one has a future liability of known amount at a known future time, how much should one invest now (at known interest rate) to cover this liability when it falls due?

i.e. what is the current value of the future liability taking time value of money (interest) into account?



8.1 Discounting Factor



$V(t_1, t_2)$ - **Present Value factor or Discounting factor**; Gives the PV or discounted value for time t_1 to t_2 of an amount of 1 due at time t_2 .

$A(t_1, t_2)$ - **Accumulating factor**; Gives the accumulation for time t_1 to t_2 of an investment of 1 at time t_1 .

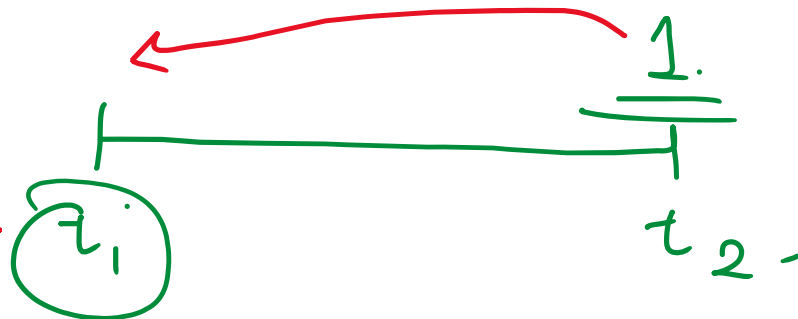


Discounting factor is the reciprocal of the accumulating factor

$$V(t_1, t_2) = \frac{1}{A(t_1, t_2)}$$

$$V(n) = \frac{1}{A(n)}$$

$$A(n) = \frac{1}{V(n)}$$



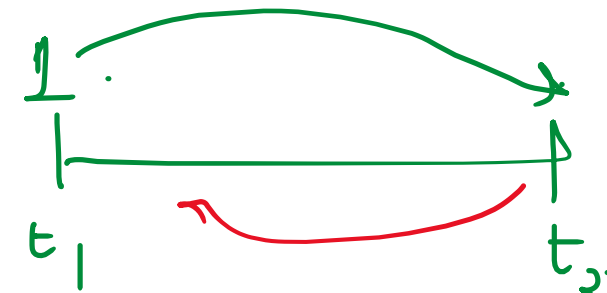
$$A(t_1, t_2)$$

$V(t_1, t_2) \Rightarrow$ PV factor / Discounting factor

$$\underline{30,000} = 10,000 \times A(5)$$

$$30,000 \times \frac{1}{A(5)} = 10,000$$

$$30,000 \times V(5) = 10,000$$

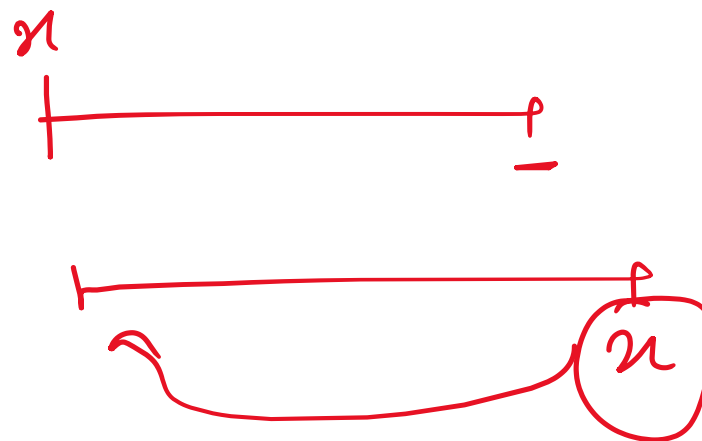


SJ: $A(n) = 1 + n^0$ ✓

$$v(n) = \left(\frac{1}{1 + n^0} \right)$$

CJ: $A(n) = (1+i)^n$ ✓

$$v(n) = \frac{1}{(1+i)^n}$$



$$(1+i)^{-n}$$

$$v = (1+i)^{-1}$$

$$v(n) = v^n$$

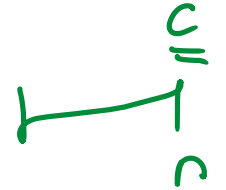
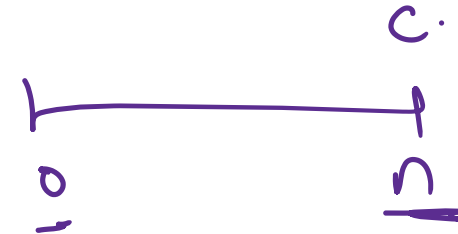
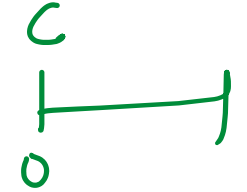
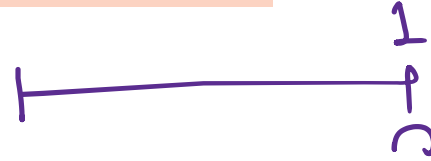
where $v = \frac{1}{(1+i)}$

8.2 Present Value – Simple Interest

- Discounting Factor: $V(t) = \frac{1}{1+it}$
- For a balance of amount C at the end of t time periods, the PV is given by

$$PV = C.V(t) = C \left(\frac{1}{1+it} \right) \text{ for } t > 0$$

$$C \times \frac{1}{(1+in)}$$



$$C \times A(n)$$

$$C \times V(n)$$



Question

1. Find the amount which must be invested at 9% p.a. in order to accumulate Rs. 5000 at the end of 5 years.

Assume Simple interest. 9%.



SI = 9% p.a.

3448.27586

Shift note.
Cin 9%
9%

2. Find the amount which must be invested at 9% p.a. in order to accumulate Rs. 1270 at the end of 3 years.

Assume Simple interest.

999.99
 ≈ 1000

$$PV = C \times \frac{1}{(1+ni)} = 5000 \times \frac{1}{(1+5 \times 9\%)} = \boxed{3448.27586}$$

8.3 Present Value – Compound Interest

- We need to determine how much a person must invest initially so that the balance will be 1 at the end of one period. The answer is $(1 + i)^{-1}$, since this amount will accumulate to 1 at the end of one period.

- Discounting Factor: $V(t) = \frac{1}{(1+i)^t} = (1 + i)^{-t}$

- Let $v = (1 + i)^{-1}$

- For a balance of amount C at the end of t time periods, the PV is given by

$$PV = C \cdot V(t) = C \cdot \frac{1}{(1+i)^t} = C v^t \text{ for } t > 0$$

- Thus, $C v^t$ is the present value at time 0 of an amount C due at time t when an investment grows according to compound interest. This means $C v^t$ is the amount that should be invested at 0 to grow to C at time t , and the present value factor v acts as 'compound present value' factor in determining present value.

Handwritten notes illustrating the present value calculation:

$$C \cdot (1+i)^{-n} = C v^n$$



Question

1. Find the amount which must be invested at 9% p.a. in order to accumulate Rs. 5000 at the end of 5 years.

Assume Compound interest.

$$\boxed{3249.6569}$$

$$PV = 5000 \cdot \frac{1}{(1+9\%)^5} =$$

2. Find the amount which must be invested at 9% p.a. in order to accumulate Rs. 1270 at the end of 3 years.

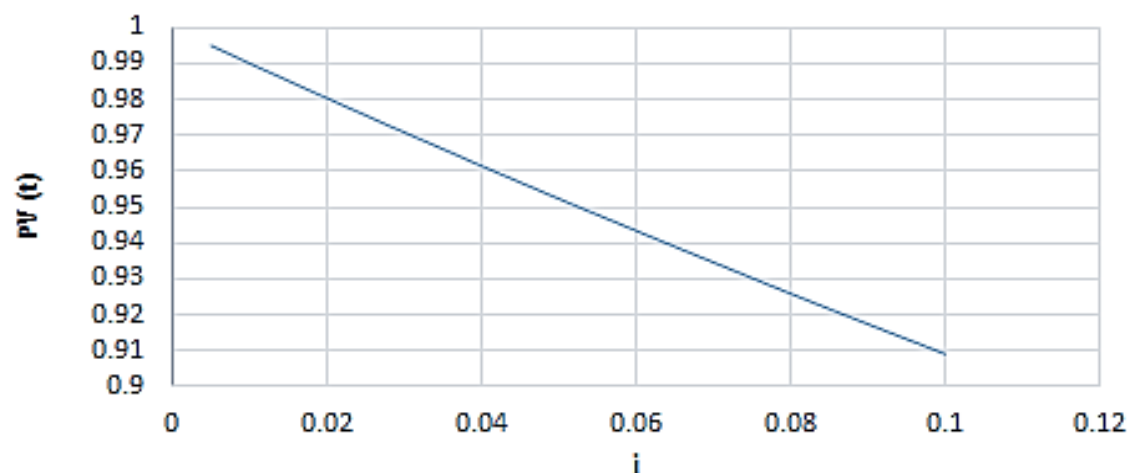
Assume Compound interest.

$$\boxed{980.6730}$$

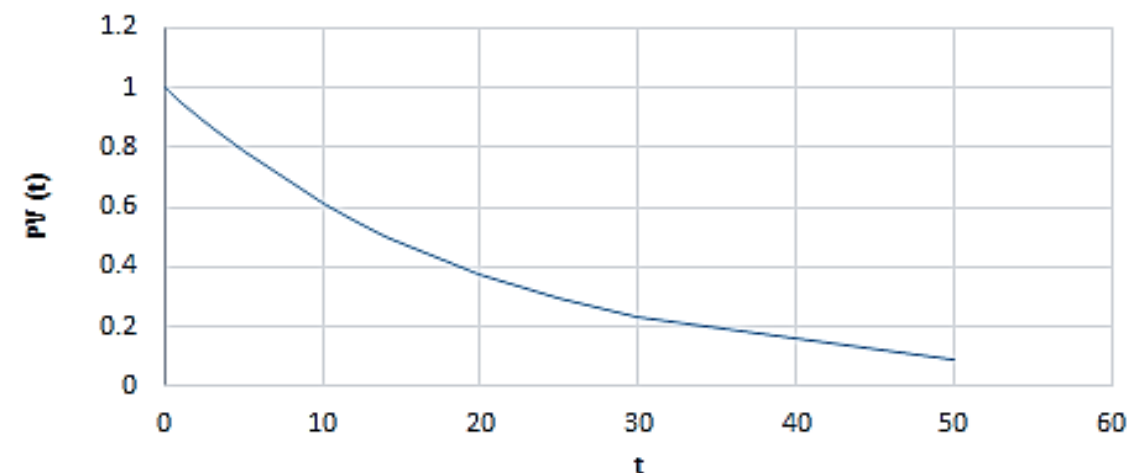
$$1270 \cdot \frac{1}{(1+9\%)^3} =$$

8.4 Simple Interest vs Compound Interest

Present value of 1 due in 1 period as function of i .



Present Value of 1 due in t time periods as a function of t





Question

Mr. Dev wants to invest a sufficient amount in a fund in order that the accumulated value will be 1 Crore on his retirement date in 25 years.

Dev considers two options. He can invest in Equity Mutual Fund (E.M. Fund), which invests in the stock market. E.M. Fund has averaged an annual compound rate of return of 19.5% since its inception 30 years ago, although its annual growth has been as low as 2% and as high as 38%. The E.M. Fund provides no guarantees as to its future performance.

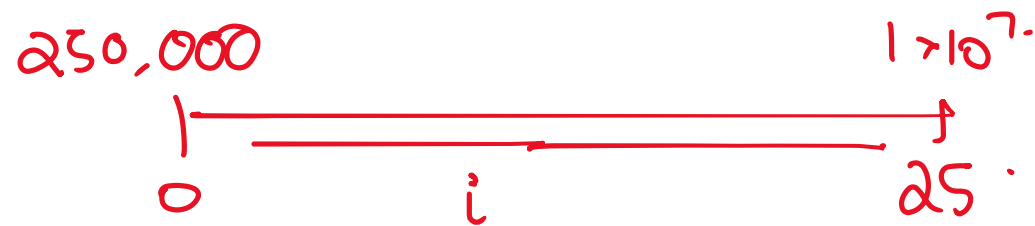
Dev's other option is to invest in a zero-coupon bond or stripped bond, with a guaranteed effective annual rate of interest of 11.5% until its maturity date in 25 years.

19.5%
25.

EMF = 19.5% pa.

ERL = 11.5% pa.

1. What amount must Mr. Dev invest if he chooses E.M. Fund and assumes that the average annual growth rate will continue for another 25 years? 116359.6397 — bond. $1 \times 10^7 \cdot \left(\frac{1}{1+19.5\%} \right)^{25}$
2. What amount must he invest if he opts for the stripped bond investment? $= 657852.1758$
3. What minimum effective annual rate is needed over the 25 years in order for an investment of Rs. 2,50,000 to accumulate to Dev's target of 1 Crore? $15.89972\% / 27.08$
4. How many years are needed for Dev to reach 1 Crore if he invests the amount found in part (a) in the stripped bond? $40.91386 \approx 41 \text{ years}$

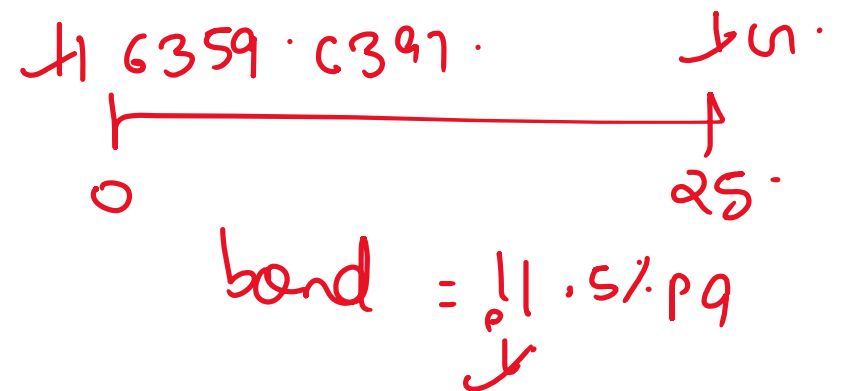


$$250,000 \quad A(25) = 1 \times 10^7$$

$$A(25) = \frac{1 \times 10^7}{250,000}$$

$$\frac{1}{(1+i)^{25}} = \frac{10000000}{250,000}$$

$$i = \underline{\underline{15.8992\%}}$$



$$n = \underline{\underline{\quad \quad \quad}}$$

$$116359.6397 = 10000000 \times \left(\frac{1}{1+11.5\%} \right)^n$$

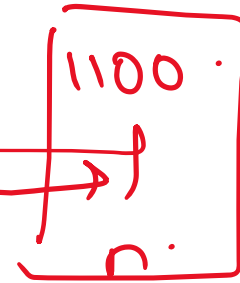
$$\frac{116359.6397}{1 \times 10^7} = \left(\frac{1}{1+11.5\%} \right)^n$$

$$85.94 = (1+11.5\%)^n$$

$$\ln 85.94 = n \ln(1+11.5\%)$$

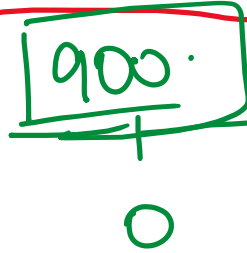
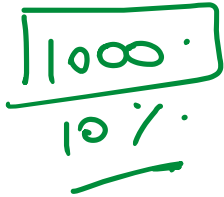
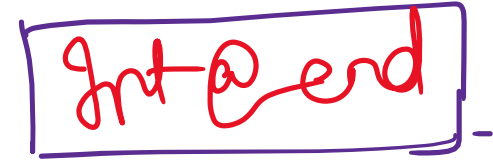
$$n = \frac{\ln 85.94}{\ln(1+11.5\%)} = \underline{\underline{40.9138}}$$

1st way

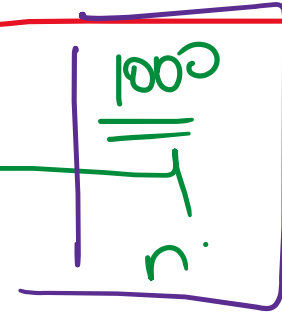


$$\frac{100}{1000}$$

$$i = \frac{\text{Amt of int @ end}}{\text{Amt @ } n-1}$$



Int @ start



- effective discount rate

$$= \frac{\text{Amt of int @ start}}{\text{Amt @ end}} = \frac{100}{1000}$$

8.4 Interest Types – Arrears & Advance

Interest type can be divided on following basis:

a) For a time period, when is the interest charged?

b) Interest amount is defined in what terms?

start/end

$$= \frac{\text{Amt @ } i}{\text{Amt @ } n}$$

Arrears = end
 Advance = start

▪ Interest paid in arrears (Interest rate)

- Interest amount paid/charged at the end of the period.
- corresponding interest rate is the ratio of the amount of interest paid for the period to the amount of principal at the start of the period

▪ Interest paid in advance (discount rate)

- Interest amount paid/charged at the start of the period.
- corresponding interest rate is the ratio of the amount of interest paid for the period to the amount due at the end of the period

9

Effective Rate of Discount

- General notation - d

- Definition:** The effective rate of discount d is the ratio of the amount of interest (sometimes called the “amount of discount” or just “discount”) earned during the period to the amount due at the end of the period.

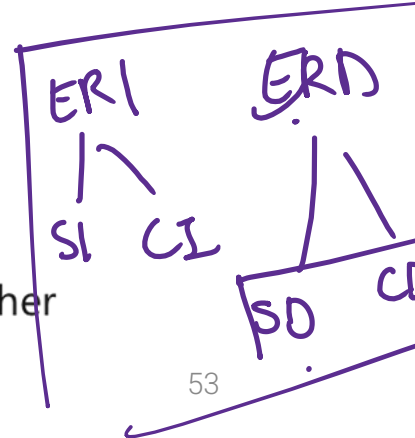
Am't of int
Am't @ n.

- Alternative definition:** The effective rate of discount d is the amount of money that invested at the beginning of a period will earn during the period, where interest is paid at the start of the period.
- Interest is paid *once per measurement period*.
- Effective rate of interest, d_n in terms of accumulation:

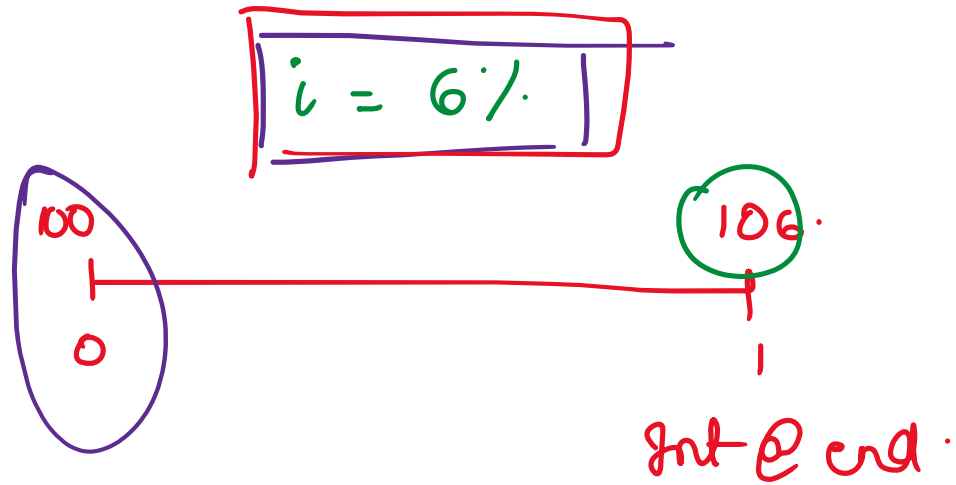
1000 ·
800 · (20%)

$$d_n = \frac{A(n) - A(n-1)}{A(n)} \text{ for } n = 1, 2, 3 \dots$$

- Effective annual interest measures growth on the basis of the initially invested amount, whereas effective annual discount measures growth on the basis of the year-end accumulated amount. Either measure can be used in the analysis of a financial transaction



ERI

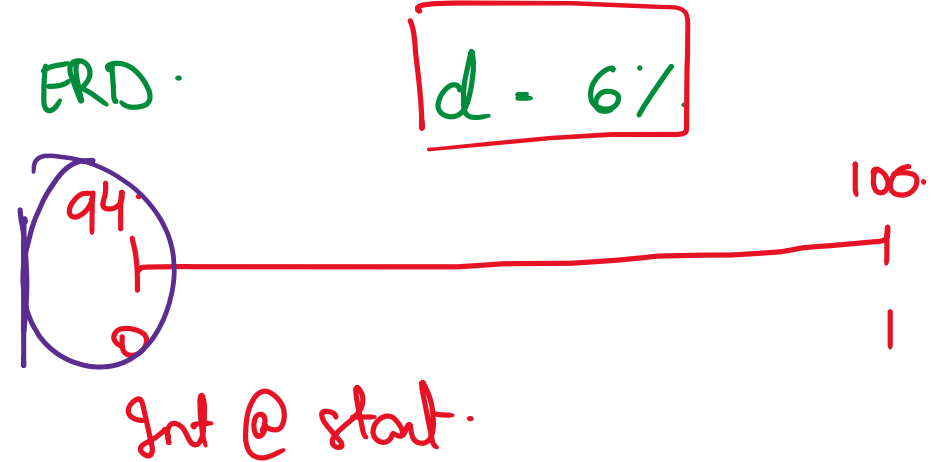


$$d = \frac{106 - 100}{106} \quad \frac{\text{Amt of Int}}{\text{Amt @ n.}}$$

$$d = 5.6604\%$$

$$\boxed{i = 6\%} \longleftrightarrow \boxed{d = 5.6604\%}$$

ERD



$$i = \frac{100 - 94}{94} = \frac{\text{Amt of Int}}{\text{Amt @ n-1}}$$

$$i = 6.3829\%$$

Illustration

100

1 year

$i = 6\%$

100 106
0 (A) 1 = 106

If A goes to a bank and borrows Rs.100 for one year at an effective rate of interest of 6%, then the bank will give A Rs.100.

At the end of the year, A will repay the bank the original loan of Rs.100, plus interest of Rs.6, or a total of Rs.106.

However, if A borrows Rs.100 for one year at an effective rate of discount of 6%, then the bank will collect its interest of 6% in advance and will give A only Rs. 94. At the end of the year, A will repay Rs.100.

Thus, it is clear that an effective rate of interest of 6% is not the same as an effective rate of discount of 6%. In the above example, A paid Rs.6 interest in both cases. However, in the case of interest paid at the end of the year, A had the use of Rs.100 for the year, while in the case of interest paid at the beginning of the year, A had the use of only Rs.94 for the year.

Note: Some readers may find the use of the word “paid” in connection with rates of discount somewhat confusing, since the borrower does not directly “pay” the interest as with rates of interest. However, the net result of deducting the interest in advance is no different than if the full amount is borrowed and then the borrower immediately pays the interest.

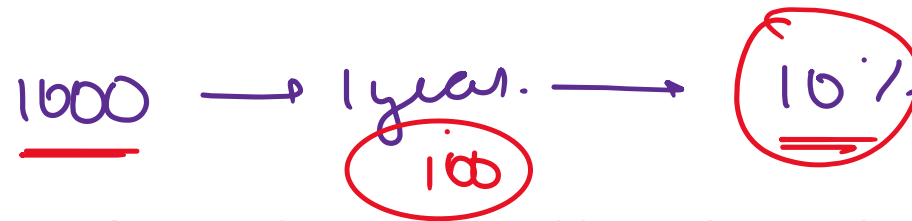
6
 $6\% \times 100$

100

(A)

6

9.1 Example



Suraj borrows 1000 for one year at a quoted rate of 10% with interest payable in advance, the 10% is applied to the loan amount of 1000, resulting in an amount of interest of 100 for the year. The interest is paid at the time the loan is made.

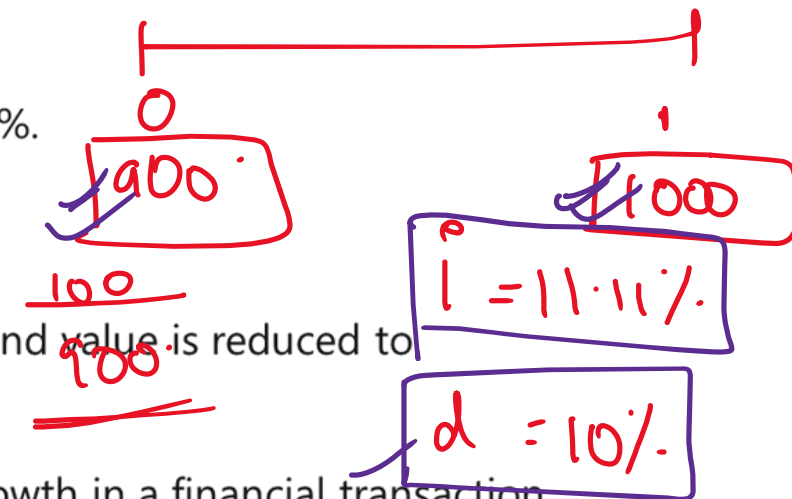
Suraj receives the loan amount of 1000 and must immediately pay the lender 100, the amount of interest on the loan. One year later he must repay the loan amount of 1000. The net effect is that Suraj receives 900 and repays 1000 one year later.

The effective annual rate of interest on this transaction is $\frac{100}{900} = 0.1111$, or 11.11%.

This 10% payable in advance is called the rate of discount for the transaction.

The rate of discount is the rate used to calculate the amount by which the year end value is reduced to determine the present value.

The effective annual rate of discount is another way of describing investment growth in a financial transaction. In the example just considered we see that an effective annual interest rate of 11.11% is equivalent to an effective annual discount rate of 10%, since both describe the same transaction.



10

Equivalent Rates



Assume that a person borrows 1 at an effective rate of discount d .

Then, in effect, the original principal is $1 - d$ and the amount of interest (discount) is d . However, from the basic definition of i as the ratio of the amount of interest to the principal, we obtain:



$$i = \frac{d}{1 - d}$$

$$d = \frac{i}{1 + i}$$

Another important relationship between i , v and d is that; **$d = i \cdot v$**

This relationship has an interesting verbal interpretation. Interest earned on an investment of 1 paid at the beginning of the period is d . Interest earned on an investment of 1 paid at the end of the period is i . Therefore, if we discount i from the end of the period to the beginning of the period with the discount factor v , we obtain d .

$i \rightarrow d$



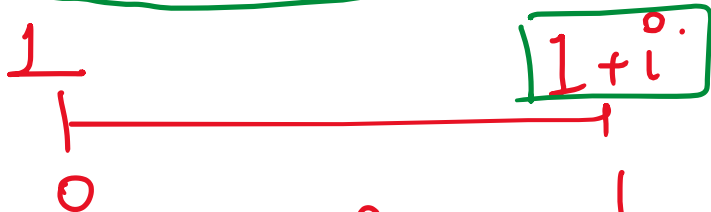
(d)

$$i = \frac{1 - (1 - d)}{1 - d} = \frac{d}{1 - d}$$

$i = \frac{\text{Int}}{\text{Amt} @ \text{st}}$ | $d = \frac{\text{Int}}{\text{Amt} @ \text{cd}}$

$d = 10\%$
 $i = 11.11\%$

$d \rightarrow i$



$d = \frac{i}{1 + i}$

~~i~~ $\downarrow$ 11.11%

$\frac{1}{1 + i}$

$$d = \frac{1 + i - 1}{1 + i}$$

$$d = \frac{i}{1 + i}$$

$$\textcircled{a} \quad \text{ERI} = \frac{\text{Int@end}}{\text{Amt@start}}.$$

$$\text{SI} : \quad AV = C(1+ni)$$

$$\text{CI} \quad AV = C \cdot (1+i)^n.$$

$$\underline{\underline{i = \frac{d}{1-d}}}$$

$$\underline{\underline{d = \frac{i}{1+i}}}$$

$$\textcircled{b} \quad \text{ERD} = \frac{\text{Int@start}}{\text{Amt@end}}.$$

SD :

CD :