INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Subject: Financial Mathematics

Chapter: Unit 2

Category: Practice Questions

IFOA PQs

1. CT1 April 2010 Q11

The force of interest δ(t) is a function of time and at any time t, measured in years, is given by the formula

$$\delta(t)=\begin{cases}0.04+0.02t & 0\le t<5\\ 0.05 & 5\le t\end{cases}$$

(i) Derive and simplify as far as possible expressions for v(t), where for v(t) is the present value of a unit sum of money due at time t.

(ii) (a)Calculate the present value of £1000 due at the end of 17 years.

(b)Calculate the rate of interest per annum convertible monthly implied by the transaction in part (ii)(a).

A continuous payment stream is received at a rate of $10e^{0.01t}$ units per annum between t = 6 and t = 10.

(iii) Calculate the present value of the payment stream.

Ans:

(i) $e^{-[0.05t+0.2]}$

(ii)(a) 349.94, (b) 6.1924%

(iii) 23.806

2. CT1 September 2010 Q8

The force of interest, δ(t), is a function of time and at any time t, measured in years, is given by the formula

$$\delta(t)=\begin{cases}0.05+0.001t & 0\le t\le 20\\ 0.05 & t>20\end{cases}$$

(i) Derive and simplify as far as possible expressions for v(t), where v(t) is the present value of a unit sum of money due at time t.

(ii) (a)Calculate the present value of £100 due at the end of 25 years.

(b)Calculate the rate of discount per annum convertible quarterly implied by the transaction in part (ii)(a).

(iii) A continuous payment stream is received at rate $30e^{-0.015t}$ units per annum between t = 20 and t = 25. Calculate the accumulated value of the payment stream at time t = 25.

Ans:

(i) $e^{-0.02-0.05t}$

(ii) (a) £23.46, (b) 0.05758

(iii) 121.82

3. CT1 April 2011 Q1

The force of interest, δ(t), is a function of time and at any time t, measured in years, is given by the formula

$$\delta(t) = \begin{cases} 0.04 + 0.003t^2 & \text{for} \quad 0 < t \le 5 \\ 0.01 + 0.03t & \text{for} \quad 5 < t \end{cases}$$

(i) Calculate the amount to which £1,000 will have accumulated at t =7 if it is invested at t = 3.

(ii) Calculate the constant rate of discount per annum, convertible monthly, which would lead to the same accumulation as that in (i) being obtained.

Ans:

(i) 1747.17

(ii) 0.138692

4. CT1 September 2011 Q6

The force of interest, δ(t), is a function of time and at any time t, measured in years, is a+ bt where a and b are constants. An amount of £45 invested at time t = 0 accumulates to £55 at time t = 5 and £120 at time t = 10.

(i) Calculate the values of a and b.

(ii) Calculate the constant force of interest per annum that would give rise to the same accumulation from time t = 0 to time t = 10.

Ans:

(i) -0.01781

(ii) 0.09808 or 9.808 %

5. CT1 April 2012 Q8

The force of interest, δ(t), at time t is given by:

$$\delta(t)=\begin{cases}0.04+0.003t^2 & \text{for} \quad 0<t\le 5\\ 0.01+0.03t & \text{for} \quad 5<t\le 8\\ 0.02 & \text{for} \quad t>8\end{cases}$$

(i) Calculate the present value (at time t = 0) of an investment of £1,000 due at time t = 10.

(ii) Calculate the constant rate of discount per annum convertible quarterly, which would lead to the same present value as that in part (i) being obtained.

(iii) Calculate the present value (at time t = 0) of a continuous payment stream payable at the rate of $100e^{0.01t}$ from time t = 10 to t = 18.

Ans:

(i) £375.31

(ii) 0.09681

(iii) £318.90

6. CT1 September 2012 Q8

The force of interest, δ(t), is a function of time and at any time t, measured in years, is given by the formula

$$\delta(t)=\begin{cases}0.03+0.01t & \text{for} \quad 0\le t\le 9\\ 0.06 & \text{for} \quad 9<t\end{cases}$$

(i) Derive, and simplify as far as possible, expressions for v(t) where v(t) is the present value of a unit sum of money due at time t.

(ii) (a) Calculate the present value of £5,000 due at the end of 15 years.

(b) Calculate the constant force of interest implied by the transaction in part (a).

A continuous payment stream is received at rate $100e^{-0.02t}$ units per annum between t = 11 and t = 15.

(iii) Calculate the present value of the payment stream.

Ans:

(i) $e^{-(0.135+0.06t)}$

(ii) (a) £1,776.13, (b) 0.0690

(iii) 124.055

7. CT1 April 2013 Q5

The force of interest per unit time at time t, $\delta(t)$, is given by:

$$\delta(t) = \begin{cases} 0.1 - 0.005t & \text{for } t < 6 \\ 0.07 & \text{for } t \geq 6 \end{cases}$$

(i) Calculate the total accumulation at time 10 of an investment of £100 made at time 0 and a further investment of £50 made at time 7.

(ii) Calculate the present value at time 0 of a continuous payment stream at the rate £$50e^{0.05t}$ per unit time received between time 12 and time 15.

Ans:

(i) £282.02

(ii) £104.67

8. CT1 September 2013 Q10

The force of interest, $\delta(t)$, is a function of time and at any time t, measured in years, is given by the formula:

$\delta(t) = 0.05 + 0.002t$

Calculate the accumulated value of a unit sum of money:

(i) (a) accumulated from time t = 0 to time t = 7.

(b) accumulated from time t = 0 to time t = 6.

(c) accumulated from time t = 6 to time t = 7.

Ans:

(i)(a) 1.490331

(b) 1.399339

(c) 1.06503

9. CT1 September 2014 Q7

The force of interest, $\delta(t)$, is a function of time and at any time t, measured in years, is given by the formula:

$$\delta(t) = \begin{cases} 0.03 & \text{for} \quad 0 < t \le 10 \\ 0.003t & \text{for} \quad 10 < t \le 20 \\ 0.0001t^2 & \text{for} \quad t > 20 \end{cases}$$

(i) Calculate the present value of a unit sum of money due at time t =28.

(ii) (a)Calculate the equivalent constant force of interest from t =0 to t =28.

(b)Calculate the equivalent annual effective rate of discount from t =0 to t =28.

A continuous payment stream is paid at the rate of $e^{-0.04t}$ per unit time between t=3 and t=7.

(iii) Calculate the present value of the payment stream.

Ans:

(i) 0.29669

(ii) (a) 4.340% per annum, (b) 4.247% per annum

(iii) 2.82797

10. CT1 September 2016 Q12

The force of interest, $\delta(t)$, is a function of time and at any time t (measured in years) is

$$\delta(t) = \begin{cases} 0.03 & \text{for } 0 \le t \le 10 \\ at & \text{for } 10 < t \le 20 \\ bt & \text{for } t > 20 \end{cases}$$

given by:

where a and b are constants.

The present value of £100 due at time 20 is 50.

(i) Calculate a.

The present value of £100 due at time 28 is 40.

(ii) Calculate b.

(iii) Calculate the equivalent annual effective rate of discount from time 0 to time 28.

A continuous payment stream is paid at the rate of $e^{-0.04t}$ per annum between t=3 and t=7.

(iv) (a)Calculate, showing all workings, the present value of the payment stream.

(b)Determine the level continuous payment stream per annum from time t=3 to time t=7 that would provide the same present value as the answer in part (iv)(a) above.

Ans:

(i) 0.0026210

(ii) 0.0011622

(iii) 3.220%

(iv) (a) 2.827969, (b) 0.82092

11. CT1 September 2017 Q9

The force of interest, $\delta(t)$, is a function of time and at any time t, measured in years, is given by the formula:

$$\delta(t) = \begin{cases} 0.09 - 0.003t & 0 \le t \le 10 \\ 0.06 & t > 10 \end{cases}$$

(i) Calculate the corresponding constant effective annual rate of interest for the period from t=0 to t =10.

(ii) Express the rate of interest in part(i) as a nominal rate of discount per annum convertible half-yearly.

(iii) Calculate the accumulation at time t =15 of £1,500 invested at time t =5.

(iv) Calculate the corresponding constant effective annual rate of discount for the period t=5 to t =15.

(v) Calculate the present value at time t=0 of a continuous payment stream payable at a rate of $10e^{0.01t}$ from time t=11 to time t = 15.

Ans:

(i) 0.077884

(ii) 0.073611

(iii) £2,837.62

(iv) 0.061760

(v) 18.003

12. CT1 April 2018 Q10

The force of interest $\delta(t)$ is a function of time, and at any time t, *measured* in years is given by the formula:

$$\delta(t) = \begin{cases} 0.24 - 0.02t & 0 < t \le 6 \\ 0.12 & 6 < t \end{cases}$$

(i) Derive, and simplify as far as possible, expressions in terms of t for the present value of a unit investment made at any time, t. You should derive separate expressions for each time interval $0 \le t \le 6$ and $6 \le t$.

(ii) Determine the discounted value at time t=4 of an investment of £1,000 due at time t=10.

(iii) Calculate the constant nominal annual interest rate convertible monthly implied by the transaction in part (ii).

(iv) Calculate the present value of a continuous payment stream invested from time t =6 to t =10 at a rate of $\rho(t) = 20e^{0.36+0.32t}$ per annum.

Ans:

(i) $e^{[-0.36-0.12t]}$

(ii) 467.67

(iii) 0.12734

(iv) 406.89

13. CT1 April 2018 Q3

An investor pays £80 at the start of each month into a 25-year savings plan. The contributions accumulate at an effective rate of interest of 3% per half-year for the first 10 years, and at a force of interest of 6% per annum for the final 15 years.

Calculate the accumulated amount in the savings plan at the end of 25 years.

Ans: £55,688.59

14. CT1 September 2018 Q7

The force of interest, $\delta(t)$, is a function of time and at any time t, measured in years, is given by the formula:

$$\delta(t) = \begin{cases} 0.03 & 0 \le t \le 10 \\ 0.003t & t > 10 \end{cases}$$

(i) Calculate the present value of a unit sum of money due at time $t = 20$.

(ii) Calculate the equivalent constant force of interest from $t=0$ to $t=20$.

(iii) Calculate the present value at time $t=0$ of a continuous payment stream payable at a rate of $e^{-0.06t}$ from time $t = 4$ to time $t = 8$.

Ans:

(i) 0.47237

(ii) 0.0375

(iii) 2.34360

15. CM1A April 2019 Q8

The force of interest, d(t), is a function of time and at any time t, measured in years, is given by the formula:

$$\delta(t) = \begin{cases} 0.03 + 0.005t & 0 \le t < 2 \\ 0.045 - 0.0025t & 2 \le t < 10 \\ 0.02 & t \ge 10 \end{cases}$$

(i) Calculate the accumulated amount at time t = 9 of an investment of £15,000 made at time t = 1.

(ii) Calculate the present value at time t = 0 of a payment stream paid continuously from time t = 10 to time t = 12, under which the rate of payment at time t is ρ(t) = 60 $e^{0.02t}$

Ans:

(i) 19,381.14

(ii) 107.50

16. CM1A September 2019 Q10

The force of interest, δ (t), is a function of time and at any time t, measured in years, is given by the formula:

$$\delta(t) = \{0.03 + 0.01t\ 0 \le t < 4\ 0.07\ 4 \le t < 6\ 0.09\ t \ge 6$$

(i) Calculate the accumulated amount at time t = 6 of a lump sum of 10 units invested at time t = 0.

(ii) Calculate the present value at time t = 0 of a deferred annuity certain of 5 units per year payable continuously from time t = 4 to t = 10.

(iii) Determine, to the nearest 0.1%, the constant annual effective rate of interest earned by an investor who invests the present value calculated in part (ii) at time t = 0 to obtain the payment stream described in part (ii).

Ans:

(i) 14.0495

(ii) 19.5947

(iii) 6.4% per annum.

17. CM1A April 2021 Q4

The force of interest, δ (t), is a function of time and at any time t, measured in years, is given by the formula:

$$\delta(t) = \begin{cases} 0.03 + 0.005t & 0 \le t \le 6 \\ 0.1 - 0.01t & t > 6 \end{cases}$$

A(0, t) , the accumulation at time t of a unit of money invested at time 0, can be written as:

$$A(0,t) = \begin{cases} e^{a+bt+ct^2} & 0 \le t \le 6 \\ e^{f+gt+ht^2} & t > 6 \end{cases}$$

(i) Calculate the values of a, b, c, f, g and h.

The sum of $5,000 is invested at t = 2 for 5 years.

(ii) Calculate the annual nominal rate of return convertible monthly on the investment.

Ans:

(i) a=0, b= 0.03, c=0.0025, f=−0.15, g=0.1, h=−0.005

(ii) 4.709%

18. CM1A September 2021 Q8

The force of interest, δ (t), is a function of time, and at any time, t, measured in years, is given by the formula:

$$\delta(t) = \begin{cases} 0.06 + 0.02t & 0 \le t \le 4 \\ 0.08 - 0.01t & t > 4 \end{cases}$$

A(0, t) the accumulation at time t of a unit of money invested at time 0, can be written as:

$$A(0,t) = \begin{cases} e^{a+bt+ct^2} & 0 \le t \le 4 \\ e^{f+gt+ht^2} & t > 4 \end{cases}$$

(i) Determine the values of a, b, c, f, g and h.

A sum of $600 is invested at t = 3 and a further sum of $900 is invested at t = 9.

(ii) Calculate, showing all working, the accumulated amount at t = 13.

(iii) Calculate, showing all working, the yield of the investment described in part (ii) expressed as an effective rate of interest per month to the nearest 0.1%

(iv) Comment on your answer to part (iii).

Ans:

(i) a = 0, b = 0.06, c = 0.01, f = 0.16, g = 0.08, h= -0.005

(ii) £1,451.46

(iii) 0.0 % per month

(iv) The force of interest is negative for all times after time 8. So the 900 decreases in value from the time of investment onwards. Whereas the 600 increases for a period and then decreases the overall effect is that the accumulated value of the investments returns roughly to the original amounts invested, hence the overall yield is approximately 0.0% per annum.

19. CM1A September 2023 Q10

The force of interest is a function of time and at any time t (measured in years) is given by the formula:

$$\delta(t)=\begin{cases}0.03+0.005t & 0\le t<8\\ 0.07 & 8\le t\end{cases}$$

(i) Derive, and simplify as far as possible, expressions for v(t), where v(t) is the present value of a unit sum of money due at time t. You should consider separately the cases $0 \le t < 8$ and $t \ge 8$

(ii) (a). Demonstrate that it will take more than 8 years for an investment to double in value.

(b)Calculate, showing all working, the exact time in years it will take for an investment to double in value.

Ans:

(i) $e^{[0.16-0.07t]}$

(ii)(a) more than 8years, (b) 12.188 years

Non IFOA PQs

Q1.

Given, $\delta_t = 0.08+0.005t$, calculate the accumulated value over five years of an investment of 1000 made at each of the following times:

(a) Time 0 and

(b) Time 2

Answer:

(a) In this case, $A(0)=1000$ and $A(5) = A(0)\cdot\exp\left[\int_0^5 \delta_t\, dt\right]$, so that the accumulated value is

$$1000\cdot\exp\left[\int_0^5 (.08+.005t)\, dt\right],$$

which is

$$1000\cdot\exp\left[(.08)(5)+(.0025)(25)\right]=1000\cdot e^{.4625} = 1588.04.$$

(b) This time we have $A(2)=1000$ and $A(7)=A(2)\cdot\exp\left[\int_2^7 \delta_t\, dt\right]$, so that the accumulated value at time 7 is

$$1000\cdot\exp\left[\int_2^7 (.08+.005t)\, dt\right],$$

leading to

$$1000\cdot\exp[(.08)(7-2)+(.0025)(49-4)] = 1669.46.$$

Note that both (a) and (b) involve 5 year periods, but the accumulations are different as a result of the non-constant force of interest. ❐

Q2.

Money accumulates at a varying force of interest

$$\delta_t = 0.05+0.01t \quad \text{for } 0 \le t \le 4$$

Find the present value at time = 0 of two payments of 100 each to be paid at times t =2 and t = 4

Answer:

Applying the reciprocal of formula (1.27), the present value of the first payment is

$$\begin{aligned} 100e^{-\int_0^2 \delta_t dt} &= 100e^{-\int_0^2 (.05+.01t)dt} \\ &= 100e^{-.12} \\ &= 88.692. \end{aligned}$$

Similarly, the present value of the second payment is

$$\begin{aligned} 100e^{-\int_0^4 \delta_t dt} &= 100e^{-\int_0^4 (.05+.01t)dt} \\ &= 100e^{-.28} \\ &= 75.578. \end{aligned}$$

Thus, the answer is $88.692 + 75.578 = \$164.27$.

Q3.

In Fund X money accumulates at force of interest $\delta_t = 0.10+0.01t$, for $0<t<20$. In Fund Y money accumulates at annual effective rate i. An amount of \$1 is invested in each fund, and the accumulated values are the same at the end of 20 years. Find the values in Fund Y at the end of 1.5 years.

Answer:

In Fund X, $AV_{20} = e^{\int_0^{20}(.01t+.10)\,dt} = e^{(.005t^2+.10t)_0^{20}} = 54.59815.$

In Fund Y, $AV_{20} = (1+i)^{20} = 54.59815$, so $i = .2214$.

Then $AV_{1.5} = (1.2214)^{1.5} = \underline{\underline{1.34985}}.$

Q4.

Fund A accumulates at nominal rate 12%, convertible monthly. Fund B accumulates at force of interest $\delta_t = \frac{t}{6}$, for all t. At time t=0, $1 is deposited in each fund. The accumulated values of the two funds are equal at time n>0, where n is measured in years. Find n.

Answer:

We are given

$$AV_n^A = (1.01)^{12n} = e^{\int_0^n t/6\,dt} = AV_n^B.$$

Then $(1.01)^{12n} = e^{n^2/12}$. Taking natural logs, we have

$$12n \cdot \ln(1.01) = (.00995)(12n) = \frac{n^2}{12},$$

or $n^2 = 1.433n$, so $n = \underline{\underline{1.433}}.$

Q5.

On January 1, 1997, $1000 is invested in a fund for which the force of interest at time t is given by $\delta_t = 0.10(t-1)^2$, where t is the number of years since January 1, 1997. Find the accumulated value of the fund on January 1, 1999.

Answer:

$$AV_2 = 1000e^{\int_0^2 .10(t-1)^2\,dt} = 1000\cdot\exp\left[\frac{.10}{3}(t-1)^3\right]_0^2$$

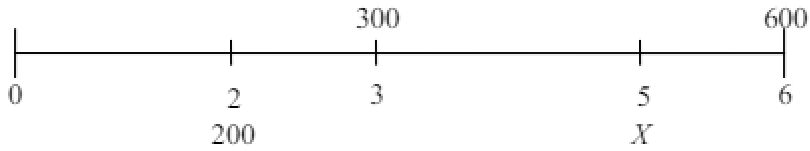
$$= 1000\cdot e^{.10/3[1-(-1)]} = 1000e^{.20/3} = \underline{\underline{1068.94}}.$$

Q6.

At $\delta_t=2(1+t)^{-1}$, payments of \$300 at t=3 and \$600 at t=6 have the same present value as payments of \$200 at t=2 and X at t=5. Find X.

Answer:

8.



$$300e^{-\int_0^3 2(1+t)^{-1}\,dt} + 600e^{-\int_0^6 2(1+t)^{-1}\,dt} = 200e^{-\int_0^2 2(1+t)^{-1}\,dt} + X\cdot e^{-\int_0^5 2(1+t)^{-1}\,dt}$$

In general,

$$-\int_0^n 2(1+t)^{-1}\,dt = -2\cdot\ln(1+t)\Big|_0^n = \ln(1+n)^{-2}.$$

Then we have

$$300(1+3)^{-2} + 600(1+6)^{-2} = 200(1+2)^{-2} + X(1+5)^{-2},$$

or

$$\frac{300}{16}+\frac{600}{49} = \frac{200}{9}+\frac{X}{36},$$

which solves for $X = \underline{\underline{315.82}}.$

Q7.

If δ_t=0.01t, 0≤t≤20, find the equivalent annual effective rate of interest over the interval 0≤t≤2.

Answer:

$$(1+i)^2 = \exp\left(\int_0^2 0.01\,t\,dt\right) = \exp\left(\left[\frac{0.01}{2}t^2\right]_0^2\right) = \exp(0.02) \quad \Rightarrow \quad i = e^{0.01} - 1$$

Q8.

Find the accumulated value of 1 at the end of 19 years if $\delta_t=0.04(1+t)^{-2}$

Answer:

$$A_{19} = \exp\left(\int_0^{19} 0.04(1+t)^{-2}dt\right) = \exp\left(\left[0.04(1+t)^{-1}\right]_0^{19}\right) = e^{0.038}$$

Q9.

At a force of interest $\delta_t=\frac{2}{k+2t}$

(i) a deposit of 75 at time t=0 will accumulate to X at time t=3 and

(ii) the present value at time t=3 of a deposit of 150 at time t=5 is also equal to X. Calculate X.

Answer:

We have that

$$a(t) = e^{\int_0^t \delta_s\,ds} = \exp\left(\int_0^t \frac{2}{k+2s}\,ds\right) = \exp\left(\ln(k+2s)\Big|_0^t\right) = e^{\ln(k+2t)-\ln k} = \frac{k+2t}{k}.$$

The information says that

$$x = 75a(3) = \frac{75(k+6)}{k}, x = 150\frac{a(3)}{a(5)} = \frac{150(k+6)}{k+10}.$$

Dividing these two equations, $1 = \frac{k+10}{2k}$ and $k = 10$. So, $x = \frac{75(k+6)}{k} = \frac{75\cdot 16}{10} = 120$.

Q10.

A customer is offered an investment where interest is calculated according to the following force of interest:

$$\delta_t = \begin{cases} 0.02t & \text{if } 0 \le t \le 3 \\ 0.045 & \text{if } 3 < t \end{cases}$$

The customer invests 1000 at time t=0. What nominal rate of interest, compounded quarterly, is earned over the first four-year period?

Answer:

First, we find $a(4) = e^{\int_0^4 \delta_s\,ds}$:

$$\int_0^4 \delta_s\,ds = \int_0^3 (0.02)s\,ds + \int_3^4 (0.045)\,ds = \frac{(0.02)s^2}{2}\Big|_0^3 + (0.045)s\Big|_3^4 = 0.135.$$

and $a(4) = e^{0.135}$. Let $i^{(4)}$ be the nominal rate of interest compounded quarterly. Then, $a(4) = \left(1+\frac{i^{(4)}}{4}\right)^{4\cdot 4} = \left(1+\frac{i^{(4)}}{4}\right)^{16}$. So, $i^{(4)} = 4(e^{0.135/16}-1) = 0.033893 = 3.3893\%$.