INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Subject: Financial Mathematics

Chapter: Unit 3 – Annuities

Category: Practice Questions

Non IFOA PQs - Easy

Q1.

A company has borrowed £500,000 from a bank. The loan is to be repaid by level instalments, payable annually in arrear for ten years from the date the loan is made. The annual instalments are calculated at an effective rate of interest of 9% per annum.

Calculate the amount of the level annual instalments.

Ans:

Let R = annual repayment

$$500{,}000 = R\, a_{\overline{10|}}^{9\%} = R \times 6.4177$$

$$\Rightarrow R = 77{,}910.04$$

Q2.

An annuity certain with payments of £150 at the end of each quarter is to be replaced by an annuity with the same term and present value, but with payments at the beginning of each month instead.

Calculate the revised payments, assuming an annual force of interest of 10%.

Ans:

We require X where:

$$600 a_{\overline{n|}}^{(4)} = 12 X \ddot{a}_{\overline{n|}}^{(12)} \Rightarrow X = 50 \frac{a_{\overline{n|}}^{(4)}}{\ddot{a}_{\overline{n|}}^{(12)}} = 50 \frac{d^{(12)}}{i^{(4)}}$$

$$d^{(12)} = 12\left(1-(1-d)^{1/12}\right) = 12\left(1-e^{-0.1/12}\right) = 0.099584$$

$$i^{(4)} = 4\left((1+i)^{1/4}-1\right) = 4\left(e^{0.1/4}-1\right) = 0.101260$$

Hence $X = 49.1724$ or £49.17

Q3.

A bank has just granted a loan of $10,000 to a business to be repaid in ten equal instalments, annually in arrears. The rate of interest is 4% per annum effective.

Calculate the amount of the annual repayment.

Ans:

Annual repayment is X where

$10,000 = Xa_{\overline{10|}}$ at 4%

$$X = \frac{10,000}{8.1109} = \$1,232.91$$

Q4.

If a college freshman invests a $10,000 gift at 8% per annum convertible quarterly, how much can be withdrawn at the end of every quarter to use up the fund exactly at the end of four years of college?

Ans:

Let R be the amount of each withdrawal. The equation of value at the date of investment is

$$R\,a_{\overline{16|}.02} = 10,000.$$

Thus, we have

$$\begin{aligned} R &= \frac{10,000}{a_{\overline{16|}.02}} \\ &= \frac{10,000}{13.57771} \\ &= \$736.50. \end{aligned}$$

The reader should verify that this answer can also be obtained using a financial calculator by setting

$$\begin{aligned} \text{N} &= 16 \\ \text{I} &= 100(.02) = 2 \\ \text{PV} &= -10,000 \end{aligned}$$

and compute PMT obtaining

$$\text{PMT} = \$736.50.$$

Q5.

A corporation sets up a sinking fund to replace some aging machinery. It deposits $100,000 into the fund at the end of each month for 10 years. The annuity earns 12% interest compounded monthly. The equipment originally cost $13 million. However, the cost of the equipment is rising 6% each year. Will the annuity be adequate to replace the equipment? If not, how much additional money is needed?

Ans:

AV of deposits = 23003868.9

AV of machine costs = 23281020.1

Extra amount required = 277151

Q6.

An investor wishes to accumulate $100,000 in a fund for retirement at the end of 12 years. To accomplish this the investor plans to make deposits at the end of each year, the final payment to be made one year prior to the end of the investment period. How large should each deposit be if the fund earns 7% effective?

Ans:

This problem should be read carefully. There will be only 11 payments, not 12. Since we are interested in the accumulated value one year after the last payment, the equation of value is

$$R\,\ddot{s}_{\overline{11}|} = 100{,}000$$

where R is the annual deposit. Solving for R we have

$$R = \frac{100{,}000}{\ddot{s}_{\overline{11}|}} = \frac{100{,}000}{16.88845} = \$5921.21.$$

This example can also be solved using a financial calculator if we set

BEG or BGN
N = 11
I = 100(.07) = 7
FV = -100,000

and compute PMT obtaining

PMT = $5921.21

Q7.

The cash price of an automobile is $10,000. The buyer is willing to finance the purchase at 18% convertible monthly and to make payments of $250 at the end of each month for four years. Find the down payment that will be necessary.

Ans: 1489.36

Q8.

A sports car enthusiast needs to finance $25,000 of the total purchase price of a new car. A loan is selected having 48 monthly level payments with a lender charging 6% convertible monthly. However, the lender informs the buyer that their policy is not to exceed a $500 monthly payment on any car loan. The buyer decides to accept the loan offer with the $500 payment and then decides to take out a second 12-month loan with a different lender at 7.5% convertible monthly to make up the shortfall not covered by the first loan. Find the amount of the monthly payment on the second loan.

Ans: 321.86

Q9.

Find $\ddot{a}_{\overline{8}|}$ if the effective rate of discount is 10%.

Ans: 5.695

Q10.

A worker aged 40 wishes to accumulate a fund for retirement by depositing $3000 at the beginning of each year for 25 years. Starting at age 65 the worker plans to make 15 annual withdrawals at the beginning of each year. Assuming that all payments are certain to be made, find the amount of each withdrawal starting at age 65 to the nearest dollar, if the effective rate of interest is 8% during the first 25 years but only 7% thereafter.

Ans: 24305

Q11.

A 48-month car loan of \$12,000 can be completely paid off with monthly payments of \$300 made at the end of each month. What is the nominal rate of interest convertible monthly on this loan?

Ans: 9.24%

Q12.

Seth wishes to borrow \$4,400 so he can pay his tuition. He qualifies for a two-year loan with a monthly effective interest rate of .25% and level payments. Find the amount of Seth's monthly payments under this loan.

Solution The loan lasts twenty-four months. Therefore, if Q is the payment amount, the time 0 equation of value stating that his payments have present value \$4,400 is $\$4{,}400 = Qa_{\overline{24}|.0025}$. So, $Q = \frac{\$4{,}400}{a_{\overline{24}|.0025}} = \frac{(\$4{,}400)(.0025)}{1-(1.0025)^{-24}} \approx \$189.1173327$. Since Seth's monthly payments must be an integral number of cents, they are \$189.12. These payments will in fact repay a loan of \$4,400.06 since $\$189.12a_{\overline{24}|.0025} \approx \$4{,}400.062057114$. (The display will just show "PV = 4,400.062057," but if you subtract 4,400.06, you obtain .002057114.) Thus the last payment may be reduced by $.062057114(1.0025)^{24} \approx .065889578 \approx .07$. The last payment should therefore be \$189.05. ■

Q13.

Calculate the present value of an annuity-immediate of amount \$100 paid annually for 5 years at the rate of interest of 9% per annum.

Table 2.1: Present value of annuity

Year	Payment (\$)	Present value (\$)
1	100	$100\,(1.09)^{-1} = 91.74$
2	100	$100\,(1.09)^{-2} = 84.17$
3	100	$100\,(1.09)^{-3} = 77.22$
4	100	$100\,(1.09)^{-4} = 70.84$
5	100	$100\,(1.09)^{-5} = 64.99$
Total		388.97

Q14.

A man borrows a loan of \$20,000 to purchase a car at annual rate of interest of 6%. He will pay back the loan through monthly installments over 5 years, with the first installment to be made one month after the release of the loan. What is the monthly installment he needs to pay?

Solution: The rate of interest per payment period is $\frac{6}{12}\% = 0.5\%$. Let P be the monthly installment. As there are $5 \times 12 = 60$ payments, from (2.1) we have

$$\begin{aligned} 20{,}000 &= P\, a_{\overline{60}|0.005} \\ &= P \times \left[\frac{1-(1.005)^{-60}}{0.005}\right] \\ &= P \times 51.7256, \end{aligned}$$

so that

$$P = \frac{20{,}000}{51.7256} = \$386.66.$$

Q15.

Find the present value of an annuity-due of \$200 per quarter for 2 years, if interest is compounded monthly at the nominal rate of 8%.

Solution: This is the situation where the payments are made less frequently than interest is converted. We first calculate the effective rate of interest per quarter, which is

$$\left[1+\frac{0.08}{12}\right]^3 - 1 = 2.01\%.$$

As there are $n = 8$ payments, the required present value is

$$200\, \ddot{a}_{\overline{8}|0.0201} = 200 \times \left[\frac{1-(1.0201)^{-8}}{1-(1.0201)^{-1}}\right] = \$1{,}493.90.$$

Q16.

Al Bundy says he paid \$25,000 down on a new house and will pay \$525 per month for 30 years. If interest is 7.8% compounded monthly, what was the selling price of the house?

First calculate the present value of the loan ($ borrowed)

$$V = \frac{P\left(1-\left(1+\frac{r}{m}\right)^{-mt}\right)}{r/m} = \frac{525\left(1-\left(1+\frac{0.078}{12}\right)^{-360}\right)}{0.078/12} = 72,929.78$$

Then add the down payment: 72929.78 + 25000 = 97929.78.

Q17.

An perpetuity–immediate provides annual payments. The first payment of 13000 is one year from now. Each subsequent payment is 3.5% more than the one preceding it. The annual effective rate of interest is i = 6%. Find the present value of this perpetuity

Ans: 520000

Q18.

Two twins Lauren & Mallory both will save $2000 at 12% compounded annually. Mallory begins at age 20 and deposits $2000 a year till age 29, for a total of 10 deposits, then does nothing till retirement at age 65 (36 years). How much will Mallory have at age 65? Lauren begins at age 29 depositing $2000 a year until retirement at age 65 (37 deposits). How much will Lauren have at retirement?

Ans:

Mallory: First determine the accumulation of the 10 deposits.

$$A = \frac{P\left(\left(1+\frac{r}{m}\right)^{mt}-1\right)}{r/m} = \frac{2000\left((1+0.12)^{10}-1\right)}{0.12} = 35097.47$$ then this is compounded annually for 36 years $\Rightarrow$ A = 35097.47(1 + 0.12)^{36} = 2,075,509.03.

Lauren: $$A = \frac{P\left(\left(1+\frac{r}{m}\right)^{mt}-1\right)}{r/m} = \frac{2000\left((1+0.12)^{37}-1\right)}{0.12} = 1,087,197.38.$$

Q19.

Chris makes annual deposits into a bank account at the beginning of each year for 10 years. Chris initial deposit is equal to 100, with each subsequent deposit k% greater than the previous year deposit. The bank credits interest at an annual effective rate of 4.5%. At the end of 10 years, the accumulated amount in Chris account is equal to 1657.22. Calculate k

Ans: 6%

Q20.

An annuity provides for 20 annuals payments, the first payment a year hence being $4500. The payments increase in such a way that each payment is 4.5% greater than the previous one. The annual effective rate of interest is 4.5%. Find the present value of this annuity.

Ans: 86124.40191

Q21.

An annuity provides for 10 annuals payments, the first payment a year hence being $2600. The payments increase in such a way that each payment is 3% greater than the previous one. The annual effective rate of interest is 4%. Find the present value of this annuity.

Ans: 23945.54454.

Q22.

A perpetuity-due pays $1000 for the first year and payments increase by 3% for each subsequent year until the 20th payment. After that the payments are the same as the 20th. Find the present value if the effective annual interest rate is 5%.

Ans: $30,641.46

Q23.

A loan is to be repaid by an increasing annuity. The first repayment will be £200 and the repayments will increase by £100 per annum. Repayments will be made annually in arrear for ten years. The repayments are calculated using a rate of interest of 6% per annum effective.

Calculate the amount of the loan.

Ans:

Amount of loan is:

$100(Ia)_{\overline{10|}} + 100a_{\overline{10|}}$ at 6% p.a.

$= 100 \times 36.9624 + 100 \times 7.3601$
$= 3696.24 + 736.01 = £4,432.25$

Q24.

Suppose a deposit of $1000 is made on the first of the months of January, February and March, $1,200 at the beginning of each month in the second quarter, $1,400 at the beginning of each month in the third quarter and $1,600 each month in the fourth quarter. If the account has a nominal 8% rate of interest compounded quarterly, what is the balance at the end of the year?

Ans: $16,226.10

Q25.

A loan is repayable by annual instalments in arrear for 20 years. The initial instalment is £5,000, with each subsequent instalment decreasing by £200. The effective rate of interest over the period of the loan is 4% per annum. Calculate the amount of the original loan.

Ans:

Original amount of loan is:

$$\begin{aligned} L &= 5,000v + 4,800v^2 + 4,600v^3 + \ldots + 1,200v^{20} \\ &= 5,200 \times \left(v + v^2 + \ldots + v^{20}\right) - 200 \times \left(v + 2v^2 + \ldots + 20v^{20}\right) \\ &= 5,200a_{\overline{20|}} - 200(Ia)_{\overline{20|}} \\ &= 5,200 \times 13.5903 - 200 \times 125.1550 \\ &= £45,638.56 \end{aligned}$$

Q26.

A 15 year annuity pays 1000 at the end of year 1 and increases by 1000 each year until the payment is 8000 at the end of year 8. Payments then decrease by 1000 each year until a payment of 1000 is paid at the end of year 15. The annual effective interest rate of 6.5%. Compute the present value of this annuity.

Ans: 39482.626.

Q27.

An annuity arrears consists of a first payment of \$100, with subsequent payments increased by 10% over the previous one until the 10th payment, after which subsequent payments decreases by 5% over the previous one. If the effective rate of interest is 10% per payment period, what is the present value of this annuity with 20 payments?

Solution: The present value of the first 10 payments is (note that $k = i$)

$$100 \times 10(1.1)^{-1} = \$909.09.$$

For the next 10 payments, $k = -0.05$ and their present value at time 10 is (note that the payment at time 11 is $100(1.10)^9(0.95)$)

$$100(1.10)^9(0.95) \times \frac{1 - \left(\frac{0.95}{1.1}\right)^{10}}{0.1 + 0.05} = 1{,}148.64.$$

Hence, the present value of the 20 payments is

$$909.09 + 1{,}148.64(1.10)^{-10} = \$1{,}351.94.$$

Q28.

A bank makes a loan to be repaid by instalments paid annually in arrear. The first instalment is £400, the second is £380 with the payments reducing by £20 per annum until the end of the 15th year, after which there are no further repayments. The rate of interest charged is 4% per annum effective.

Calculate the amount of the loan.

Ans:

PV is:

$$400v + 380v^2 + 360v^3 + L + 120v^5$$

$$420\, a_{\overline{15|}} - 20(Ia)_{\overline{15|}} \quad @\, 4\%$$

$$= 420 \times 11.1184 - 20 \times 80.8539$$

$$= 4{,}669.728 - 1{,}617.078 = £3{,}052.65.$$

Q29.

A loan of £20,000 is repayable by an annuity payable annually in arrear for 25 years. The annual repayment is calculated at an effective interest rate of 8% per annum and increases by £50 each year.

Calculate the amount of the first payment.

Ans:

Let X = initial payment

$$20000 = (X-50)a_{\overline{25|}} + 50(Ia)_{\overline{25|}}$$

$$= (X-50)\times 10.6748 + 50\times 98.4789$$

$$= 10.6748X - 533.74 + 4923.95$$

$$\Rightarrow X = \frac{15609.80}{10.6748} = £1,462.31.$$

Q30.

You have successfully started and operated a company for the past 10 years. You have decided that it is time to sell your company and spend time on the beaches in Hawaii. A potential buyer is interested in your company, but he does not have the necessary capital to pay you a lump sum. Instead, he has offered $500,000 today and annuity payments for the balance. The first payment will be for $150,000 in three months. The payments will increase at 2% per quarter and a total of 25 quarterly payments will be made. If you require an EAR of 11%, how much are you being offered for your company?

Ans:

Value of offer = $3,391,407.12 + 500,000

Value of offer = $3,891,407.12

Q31.

A loan is to be repaid by a series of instalments payable annually in arrear for 15 years. The first instalment is £1,200 and payments increase thereafter by £250 per annum.

Repayments are calculated using a rate of interest of 6% per annum effective. Determine the amount of the loan.

Ans:

$$\text{Loan} = 950\, a_{\overline{15|}} + 250(Ia)_{\overline{15|}} \text{ at } 6\%$$

$$= 950 \times 9.7122 + 250 \times 67.2668$$

$$= £26{,}043.29$$

Q32.

Show that:

$$\left(\bar{I}\bar{a}\right)_{\overline{n|}} = \frac{\bar{a}_{\overline{n|}} - nv^n}{\delta}.$$

Ans:

$$(\bar{I}\bar{a})_{\overline{n|}} = \int_0^n te^{-\delta t}\,dt = \left[t \times \frac{e^{-\delta t}}{-\delta}\right]_0^n - \int_0^n \frac{e^{-\delta t}}{-\delta}\,dt$$

$$= -\frac{n.e^{-\delta n}}{\delta} + \frac{1}{\delta}\int_0^n e^{-\delta t}\,dt$$

$$= -\frac{n.e^{-\delta n}}{\delta} + \frac{1}{\delta}\left[-\frac{e^{-\delta t}}{\delta}\right]_o^n$$

$$= -\frac{n.e^{-\delta n}}{\delta} + \frac{1}{\delta}\left[\frac{1-e^{-\delta n}}{\delta}\right] = \frac{\bar{a}_{\overline{n|}} - nv^n}{\delta}$$

Q33.

The nominal rate of interest per annum convertible quarterly is 2%. Calculate the present value of a payment stream paid at a rate of €100 per annum, monthly in advance for 12 years.

Ans:

Various approaches (e.g. effective interest period can be changed etc.).
Work in quarters. Interest rate per quarter = 0.5%. Rate of payment per quarter = 25.
Number of quarters = 48.

$$PV = 25\ddot{a}_{\overline{48|}}^{(3)} = 25\frac{i}{d^{(3)}}a_{\overline{48|}} \qquad [2]$$

$$d^{(3)} = 3\left(1-1.005^{-1/3}\right) = 0.0049834,\ a_{\overline{48|}} = 42.5803 \qquad [1]$$

Therefore, PV = 25 × (0.005/0.0049834) × 42.5803 = €1,068.05 [1]

Q34.

A bank offers two repayment alternatives for a loan that is to be repaid over sixteen year

Option 1: the borrower pays £7,800 per annum quarterly in arrear.

Option 2: the borrower makes payments at an annual rate of £8,200 every second year in arrear.

Determine which option would provide the better deal for the borrower at a rate of interest of 5% per annum effective.

Ans:

Present value for 1st option:

$$7,800\, a_{\overline{16|}}^{(4)} = 7,800 \times 1.018559 \times 10.8378$$

$$= £86,103.52$$

Present value for 2nd option:

$$16,400 \times \left(v^2 + v^4 + \ldots + v^{16}\right)$$

$$16,400\, v^2 \left(\frac{1-v^{16}}{1-v^2}\right)$$

$$16,400 \times 0.90703 \times \left(\frac{(1-0.45811)}{(1-0.90703)}\right)$$

$$= £86,702.94$$

FM Unit 3

PRACTICE QUESTION

Therefore, 1st option is better for the borrower as the total value of the repayments is less than with the 2nd option.

Q35.

An individual aged exactly 65 intends to retire in five years' time and receive an annuity-certain. The annuity will be payable monthly in advance and will cease after 20 years. The annuity will increase at each anniversary of the commencement of payment at the rate of 3% per annum.

The individual would like the initial level of annuity to be £20,000 per annum. The price of the annuity will be the present value of the payments on the date it commences using an interest rate of 7% per annum effective.

Calculate the price of the annuity

Ans:

$$\text{Value of annuity} = 20{,}000\ \ddot{a}_{\overline{1}|}^{(12)}\left(1+1.03v+1.03^2v^2+\ldots+1.03^{19}v^{19}\right)$$

$$= 20{,}000\times 1.037525\times 0.93458\times\left(\frac{1-\left(\frac{1.03}{1.07}\right)^{20}}{1-\frac{1.03}{1.07}}\right)$$

$$= 19{,}393.4173\times 14.26488$$

$$= £276{,}645.$$

Q36.

A loan is to be repaid by an increasing annuity. The first payment will be £100 and the payments will increase by £50 per annum. Payments will be made annually in arrear for ten years. The repayments are calculated using a rate of interest of 5% per annum effective.

Calculate the amount of the loan.

Ans:

Amount of loan is $50(Ia)_{\overline{10}|}+50a_{\overline{10}|}$ at 5% per annum effective

$$= 50\times 39.3738 + 50\times 7.7217$$

$$= 1968.69 + 386.09 = £2{,}354.78$$

FM Unit 3

PRACTICE QUESTION

Q37.

An n-year decreasing annuity is payable annually in arrear where the payment at the end of the first year is n, the payment at the end of the second year is (n – 1), and so on until the final payment at the end of year n is 1.

(i) Show that the present value of this annuity i

$$\frac{n - a_{\overline{n}|}}{i}$$

A loan is to be repaid over 25 years by means of annual instalments payable in arrear. The amount of the first instalment is £8,000 and each subsequent instalment reduces by £200. The effective rate of interest charged by the lender is 5.5% per annum.

(ii) Calculate the initial amount of the loan

Ans:

(i) Denote PV of annuity by:

$$(Da)_{\overline{n}|} = nv + (n-1)v^2 + (n-2)v^3 + \ldots + 2v^{n-1} + v^n$$

$$\Rightarrow (1+i) \times (Da)_{\overline{n}|} = n + (n-1)v + (n-2)v^2 + \ldots + 2v^{n-2} + v^{n-1}$$

$$\Rightarrow i \times (Da)_{\overline{n}|} = n - \left(v + v^2 + \ldots + v^n\right)$$

$$\Rightarrow (Da)_{\overline{n}|} = \frac{n - a_{\overline{n}|}}{i}$$

(ii) Initial amount of loan, L, is given by:

$$\begin{aligned} L &= 8,000v_{5.5\%} + 7,800v_{5.5\%}^2 + 7,600v_{5.5\%}^3 + \ldots + 3,200v_{5.5\%}^{25} \\ &= 3,000 \times a_{\overline{25}|}^{5.5\%} + 200 \times (Da)_{\overline{25}|}^{5.5\%} \end{aligned}$$

where $a_{\overline{25}|}^{5.5\%} = \dfrac{1 - v_{5.5\%}^{25}}{0.055} = 13.4139$, and

$$(Da)_{\overline{25}|}^{5.5\%} = \frac{25 - a_{\overline{25}|}^{5.5\%}}{0.055} = 210.6558$$

Thus, initial amount of loan is:

$$L = 3,000 \times 13.4139 + 200 \times 210.6558 = £82,372.95$$

Q38.

A loan of £80,000 was taken out on 1 January 2016. The loan was to be repaid over 10 years in level instalments payable monthly in arrears.

Calculate the level monthly instalment using an effective rate of interest of 8% per annum

Ans:

Let R denote the level monthly instalment.

Then, we have:

$$\begin{aligned} & 12R \times a_{\overline{10}|8\%}^{(12)} &= 80{,}000 \\ \Rightarrow\ & 12R \times 1.036157 \times 6.7101 &= 80{,}000 \\ \Rightarrow\ & R &= 958.86 \end{aligned}$$

Q39.

A loan is to be repaid by a series of instalments made annually in arrears. The first instalment is $200 per annum and thereafter instalments increase by $15 each year. The instalments are paid for 16 years and are calculated using an effective rate of interest of 5% per annum.

Calculate the amount of the loan.

Ans:

$\text{Loan} = 185\ a_{\overline{16}|} + 15\left(Ia\right)_{\overline{16}|}$ at 5%

$$= 185 \times 10.8378 + 15 \times 80.9975$$
$$= \$3{,}219.96$$

Q40.

Find the present value of an annuity which pays $500 at the end of each half-year for 20 years if the rate of interest is 9% convertible semiannually.

Ans:

$$500\,a_{\overline{40}|.045}.$$

This can be evaluated using formula (3.2) as

$$500\frac{1-(1.045)^{-40}}{.045} = 500(18.40158) = \$9,200.79.$$

Alternatively, using a financial calculator, we set

$$\begin{aligned} N &= 40 \\ I &= 100(.045) = 4.5 \\ PMT &= -500 \end{aligned}$$

and compute PV obtaining

$$PV = \$9,200.79.$$

Q41.

At what rate of interest, convertible quarterly, is $16,000 the present value of $1000 paid at the end of every quarter for five years?

Ans:

Let $j = i^{(4)}/4$, so that the equation of value becomes

$$1000a_{\overline{20}|j} = 16,000$$

or

$$a_{\overline{20}|j} = 16.$$

This problem is ideally set up to use a financial calculator. We set

$$\begin{aligned} N &= 20 \\ PV &= 16 \\ PMT &= -1 \end{aligned}$$

and compute I obtaining

$$I = 2.2262.$$

Thus, we have

$$j = .022262$$

so that

$$i^{(4)} = 4(.022262) = .08905.$$

Q42.

Find the accumulated value of a 10-year annuity-immediate of $100 per year if the effective rate of interest is 5% for the first 6 years and 4% for the last 4 years.

Ans:

We could apply formula (3.25); however, the following approach is simpler. The accumulated value of the first six payments after six years is

$$100\, s_{\overline{6}|.05}\,.$$

This value is accumulated to the end of the 10 years at 4%, giving

$$100\, s_{\overline{6}|.05}\,(1.04)^4.$$

The accumulated value of the last four payments is

$$100\, s_{\overline{4}|.04}\,.$$

Thus, the answer is

$$\begin{aligned} & 100\left[s_{\overline{6}|.05}(1.04)^4 + s_{\overline{4}|.04}\right] \\ &= 100[(6.8019)(1.16986)+(4.2465)] \\ &= \$1220.38. \end{aligned}$$

This is an illustration of the "portfolio rate method."

Q43.

A family wishes to accumulate \$50,000 in a college education fund at the end of 20 years. If they deposit \$1000 in the fund at the end of each of the first 10 years and \$1000+X in the fund at the end of each of the second 10 years, find X if the fund earns 7% effective.

Ans: 651.72

Q44.

An annuity provides a payment of n at the end of each year for n years. The annual effective interest rate is $1/n$. What is the present value of the annuity?

Ans:

$$n^2\left[1-\left(\frac{n}{n+1}\right)^n\right]$$

Q45.

Find the present value of payments of $200 every six months starting immediately and continuing through four years from the present, and $100 every six months thereafter through ten years from the present, if $i^{(2)}$=.06.

Ans: 2389.72

Q46.

One annuity pays 4 at the end of each year for 36 years. Another annuity pays 5 at the end of each year for 18 years. The present values of both annuities are equal at effective rate of interest *i*. If an amount of money invested at the same rate *i* will double in *n* years, find *n*.

Ans: 9

Q47.

Find the present value of an annuity-immediate which pays 1 at the end of each half-year for five years, if the rate of interest is 8% convertible semiannually for the first three years and 7% convertible semiannually for the last two years.

Ans: 8.145

Q48.

Find the present value of an annuity-immediate which pays 1 at the end of each half-year for five years, if the payments for the first three years are discounted at 8% convertible semiannually and the payments for the last two years are discounted at 7% convertible semiannually.

Ans: 8.230

Q49.

At an annual effective interest rate of *i*, both of the following annuities have a present value of *X*:

(i) a 20-year annuity-immediate with annual payments of 55.

(ii) a 30-year annuity-immediate with annual payments that pay 30 per year for the first 10 years, 60 per year for the second 10 years, and 90 per year for the final 10 years.

Calculate X.

Ans: 574.60

Q50.

Find the accumulated value at the end of four years of an investment fund in which \$100 is deposited at the beginning of each quarter for the first two years and \$200 is deposited at the beginning of each quarter for the second two years, if the fund earns 12% convertible monthly.

Ans:

We are given an interest rate of 1% per month. Let j be the equivalent rate of interest per quarter, which is the payment period. We have

$$j = (1.01)^3 - 1 = .030301.$$

The value of the annuity in symbols is

$$100\left(\ddot{s}_{\overline{16}|j} + \ddot{s}_{\overline{8}|j}\right)$$

which can be evaluated as

$$100(20.81704 + 9.17157) = \$2998.86.$$

Q51.

A loan of \$3000 is to be repaid with quarterly installments at the end of each quarter for five years. If the rate of interest charged on the loan is 10% convertible semiannually, find the amount of each quarterly payment.

Ans:

We are given an interest rate of 5% per half-year. Let j be the equivalent rate of interest per quarter, which is the payment period. We have

$$j = (1.05)^{1/2} - 1 = .024695.$$

Let the quarterly payment be denoted by R. Then the equation of value is

$$Ra_{\overline{20}|j} = 3000$$

so that

$$R = \frac{3000}{a_{\overline{20}|j}} = \frac{3000}{15.63417} = \$191.89.$$

Q52.

At what annual effective rate of interest will payments of $100 at the end of every quarter accumulate to $2500 at the end of five years?

Ans:

Let $j = i^{(4)}/4$ be the interest rate per quarter which accomplishes the above. Then the equation of value at the end of five years is

$$100\, s_{\overline{20}|j} = 2500$$

or

$$s_{\overline{20}|j} = 25.$$

We can use a financial calculator to solve for an unknown rate of interest. We set

$$\begin{aligned} N &= 20 \\ FV &= 35 \\ PMT &= -1 \end{aligned}$$

and compute I obtaining

$$I = 2.2854.$$

Thus, we have $j = .022854$. The annual effective rate of interest i is given by

$$i = (1.022854)^4 - 1 = .0946, \text{ or } 9.46\%.$$

Q53.

At six-month interval, A deposited ₹2000 in a saving account which credit interest at 10% p.a. compounded semi-annually. The first deposit was made when A' s son was six-month-old and the last deposit was made when his son was 8 years old. The money remained in the account and was presented to the son on his 10th birthday. How much did he receive?

Ans: 57511.68

Q54.

An annuity of ₹500 p.a. is flowing continuously for 10 years. Find its future value if the rate of interest is 10% compounded continuously

Soln:- 6.

F.V of an annuity when interest rate is compounding continuously @10%.

$$F.V = \int_0^n Re^{rt}\,dt$$
$$= \int_0^{10} 500\, e^{.10\times t}\,dt$$
$$= 500 \left.\frac{e^{.10t}}{.10}\right]_0^{10}$$
$$= \frac{500}{.10}\left[e^{.10\times 10} - e^{.10\times 0}\right]$$
$$= 5000\,[2.7183-1]$$
$$= 5000\times 1.7183 = ₹\ 8591.5$$

Q55.

Calculate the present value of an annuity-immediate of amount \$100 payable quarterly for 10 years at the annual rate of interest of 8% convertible quarterly. Also calculate its future value at the end of 10 years.

Solution: Note that the rate of interest per payment period (quarter) is $\frac{8}{4}\% = 2\%$, and there are $4 \times 10 = 40$ payments. Thus, from (2.1) the present value of the annuity-immediate is

$$100\, a_{\overline{40}|0.02} = 100 \times \left[\frac{1-(1.02)^{-40}}{0.02}\right] = \$2{,}735.55,$$

and the future value of the annuity-immediate is

$$2{,}735.55 \times (1.02)^{40} = \$6{,}040.20.$$

Q56.

A company wants to provide a retirement plan for an employee who is aged 55 now. The plan will provide her with an annuity-immediate of \$7,000 every year for 15 years upon her retirement at the age of 65. The company is funding this plan with an annuity-due of 10 years. If the rate of interest is 5%, what is the amount of installment the company should pay?

Solution: We first calculate the present value of the retirement annuity. This is equal to

$$7{,}000\, a_{\overline{15}|} = 7{,}000 \times \left[\frac{1-(1.05)^{-15}}{0.05}\right] = \$72{,}657.61.$$

This amount should be equal to the future value of the company's installments P, which is $P\ddot{s}_{\overline{10}|}$. Now from (2.4), we have

$$\ddot{s}_{\overline{10}|} = \frac{(1.05)^{10}-1}{1-(1.05)^{-1}} = 13.2068,$$

so that

$$P = \frac{72{,}657.61}{13.2068} = \$5{,}501.53.$$

Q57.

Find the present value of a 15–year decreasing annuity–immediate paying 150000 the first year and decreasing by 10000 each year thereafter. The effective annual interest rate of 4.5%.

Ans: 946767.616.

Q58.

Find the present value at time 0 of an annuity such that the payments start at 1, each payment thereafter increases by 1 until reaching 10, and they remain at that level until 25 payments in total are made. The effective annual rate of interest is 4%.

Solution: The cashflow is

Payments	1	2	3	···	10	10	···	10
Time	1	2	3	···	10	11	···	25

The present value at time 0 of the perpetuity is

$$(Ia)_{\overline{10}|0.04} + (1+0.04)^{-10} 10 a_{\overline{15}|0.04}$$
$$= \frac{\ddot{a}_{\overline{n}|4\%} - 10(1+0.04)^{-10}}{0.04} + (1+0.04)^{-10}(10) a_{\overline{15}|0.04}$$
$$= 41.99224806 + 75.11184164 = 117.1040897.$$

Q59.

An perpetuity provides annual payments. The first payment of 13000 is one year from now. Each subsequent payment is 3.5% more than the one preceding it. The annual effective rate of interest is i = 6%. Find the present value of this perpetuity.

Ans: 520000

Q60.

An investor receives payments half-yearly in arrears for 20 years. The first payment is £250, and each payment is 2% higher than the previous one. The interest rate is 6% pa effective for the first 10 years and 4% pa effective for the final 10 years. Calculate, showing all workings, the present value of the payments.

Ans:

We will work in years, with $v_1 = \dfrac{1}{1.06}$ denoting the one-year discount factor applicable in each of the first 10 years, and $v_2 = \dfrac{1}{1.04}$ denoting the one-year discount factor applicable in each of the final 10 years.

The total present value of the payments received is therefore:

$$250v_1^{0.5} + 250(1.02)v_1 + 250(1.02)^2 v_1^{1.5} + \cdots + 250(1.02)^{19} v_1^{10}$$
$$+ 250(1.02)^{20} v_1^{10} v_2^{0.5} + 250(1.02)^{21} v_1^{10} v_2 + \cdots + 250(1.02)^{39} v_1^{10} v_2^{10} \qquad [2]$$

The terms in the first line of the present value expression form a geometric progression of 20 terms, with first term $250v_1^{0.5}$ and common ratio $(1.02)v_1^{0.5}$, so:

$$250v_1^{0.5} + 250(1.02)v_1 + 250(1.02)^2 v_1^{1.5} + \cdots + 250(1.02)^{19} v_1^{10}$$
$$= 250v_1^{0.5} \frac{\left(1 - \left(1.02v_1^{0.5}\right)^{20}\right)}{1 - 1.02v_1^{0.5}}$$
$$= 4{,}450.86 \qquad [1½]$$

The terms in the second line of the present value expression form a geometric progression of 20 terms, with first term $250(1.02)^{20} v_1^{10} v_2^{0.5}$ and common ratio $(1.02)v_2^{0.5}$, so:

$$250(1.02)^{20} v_1^{10} v_2^{0.5} + 250(1.02)^{21} v_1^{10} v_2 + \cdots + 250(1.02)^{39} v_1^{10} v_2^{10}$$
$$= 250(1.02)^{20} v_1^{10} v_2^{0.5} \frac{\left(1 - \left(1.02v_2^{0.5}\right)^{20}\right)}{1 - 1.02v_2^{0.5}}$$
$$= 4{,}075.60 \qquad [1½]$$

So the total present value of the payments is:

$$4{,}450.86 + 4{,}075.60 = £8{,}526.46 \qquad [1]$$

[Total 6]

Non IFOA PQs - Difficult

Q1.

An actuarial student has created an interest rate model under which the annual effective rate of interest is assumed to be fixed over the whole of the next ten years. The annual effective rate is assumed to be 2%, 4% and 7% with probabilities 0.25, 0.55 and 0.2 respectively.

(a) Calculate the expected accumulated value of an annuity of £800 per annum payable annually in advance over the next ten years.

(b) Calculate the probability that the accumulated value will be greater than £10,000

Ans:

(a) Expected accumulated value

$$=800\left(0.25\ddot{s}_{\overline{10|}0.02}+0.55\ddot{s}_{\overline{10|}0.04}+0.2\ddot{s}_{\overline{10|}0.07}\right)$$

$$=800\left(0.25\left(s_{\overline{11|}0.02}-1\right)+0.55\left(s_{\overline{11|}0.04}-1\right)+0.2\left(s_{\overline{11|}0.07}-1\right)\right)$$

$$=800\left((0.25\times 11.1687)+(0.55\times 12.4864)+(0.2\times 14.7836)\right)$$

$$=(0.25\times 8934.96)+(0.55\times 9989.12)+(0.2\times 11826.88)$$

$$=£10,093.13$$

(b) Accumulation is only over £10,000 if the interest rate is 7% p.a. which has probability 0.2

Q2.

Annuities X and Y provide the following payments:

End of Year	**Annuity X**	**Annuity Y**
1-10	1	K
11-20	2	0
21-30	1	K

Annuities X and Y have equal present values at an annual effective interest rate i such that $v^{10}=1/2$. Determine K.

Ans: 1.8

Q3.

An insurance company offers a customer two payment options in respect of an invoice for £456. The first option involves 24 payments of £20 paid at the beginning of each month starting immediately. The second option involves 24 payments of £20.50 paid at the end of each month starting immediately. The customer is willing to accept a monthly payment schedule if the annual effective interest rate per annum he pays is less than 5%.

Determine which, if any, of the payment options the customer will accept.

Ans:

The annual rate of payment for the first deal is 240.

This deal is acceptable if:

$240\,\ddot{a}_{\overline{2}|}^{(12)} < 456$ at a rate of interest of 5%

$240\,\ddot{a}_{\overline{2}|}^{(12)} = 240 \times 1.8594 \times 1.026881 = 458.252$

Therefore first deal is not acceptable

The annual rate of payment on the second deal is 246.

This deal is acceptable if:

$246\,a_{\overline{2}|}^{(12)} = 246 \times 1.8594 \times 1.022715 = 467.803$

Therefore second deal is also not acceptable

Q4.

A mortgage company offers the following two deals to customers for twenty-five year mortgages.

Product A

A mortgage of £100,000 is offered with level repayments of £7,095.25 made annually in arrear. There are no arrangement or exit fees.

Product B

A mortgage of £100,000 is offered whereby a monthly payment in advance is calculated such that the customer pays an effective rate of return of 4% per annum ignoring arrangement and exit fees. In addition the customer also has to pay an arrangement fee of £6,000 at the beginning of the mortgage and an exit fee of £5,000 at the end of the twenty-five year term of the mortgage.

Compare the annual effective rates of return paid by customers on the two products.

Ans:

For Product A, the annual rate of return satisfies the equation:

$$7{,}095.25a_{\overline{25|}} = 100{,}000$$

$$\Rightarrow a_{\overline{25|}} = 14.0939$$

This equates to the value of $a_{\overline{25|}}$ at 5%. Hence the annual effective rate of return is 5%.

For Product B, the annual rate of payment is X such that:

$$X\ddot{a}_{\overline{25|}}^{(12)} = 100{,}000 \text{ at } 4\%$$

$$\ddot{a}_{\overline{25|}}^{(12)} = \frac{i}{d^{(12)}}a_{\overline{25|}} = 1.021537 \times 15.6221 = 15.95855$$

$$\Rightarrow X = \frac{100{,}000}{15.95855} = 6{,}266.23$$

The equation of value to calculate the rate of return from Product B is:

$$6{,}000 + 5{,}000v^{25} + 6{,}266.23\frac{i}{d^{(12)}}a_{\overline{25|}} = 100{,}000$$

Clearly the rate of return must be greater than 4%. Try 5%.

$$LHS = 6{,}000 + 5{,}000 \times 0.29530 + 6{,}266.2335 \times 1.026881 \times 14.0939 = 98{,}166$$

At 5% the present value of the payments is less than the amount of the loan at 5% so the rate of return must be less than 5%. Try 4%:

$$LHS = 6{,}000 + 5{,}000 \times 0.37512 + 100{,}000 = 107{,}876$$

Interpolate between 4% and 5% to get the effective rate of return, i:

$$i = 0.04 + 0.01\left(\frac{107{,}876 - 100{,}000}{107{,}876 - 98{,}166}\right) \approx 4.81\% \text{ (actual answer is 4.80\%)}$$

Therefore Product B charges a lower effective annual return than Product A.

Q5.

An individual wishes to receive an annuity which is payable monthly in arrears for 15 years. The annuity is to commence in exactly 10 years at an initial rate of £12,000 per annum. The payments increase at each anniversary by 3% per annum. The individual would like to buy the annuity with a single premium 10 years from now.

Calculate the single premium required in 10 years' time to purchase the annuity assuming an interest rate of 6% per annum effective.

Ans:

In 10 years' time the single premium P is

$$P = 12000\left(a_{\overline{1|}}^{(12)} + 1.03a_{\overline{1|}}^{(12)}v + (1.03)^2 a_{\overline{1|}}^{(12)}v^2 + \ldots + (1.03)^{14} v^{14} a_{\overline{1|}}^{(12)}\right)$$

$$= 12000a_{\overline{1|}}^{(12)}\left(1 + \frac{1.03}{1.06} + \left(\frac{1.03}{1.06}\right)^2 + \ldots + \left(\frac{1.03}{1.06}\right)^{14}\right)$$

$$= 12000a_{\overline{1|}}^{(12)}\left(\frac{1-\left(\frac{1.03}{1.06}\right)^{15}}{1-\frac{1.03}{1.06}}\right)$$

where $a_{\overline{1|}}^{(12)} = \frac{i}{i^{(12)}}v$

$$= \frac{1.027211}{1.06} = 0.969067$$

$$\Rightarrow P = 12000 \times 0.969067 \times \frac{0.3499146}{0.0283019}$$

$$= 143{,}774.45$$

Q6.

A member of a pensions savings scheme invests £1,200 per annum in monthly instalments, in advance, for 20 years from his 25th birthday. From the age of 45, the member increases his investment to £2,400 per annum. At each birthday thereafter the annual rate of investment is further increased by £100 per annum. The investments continue to be made monthly in advance for 20 years until the individual's 65th birthday.

(i) Calculate the accumulation of the investment at the age of 65 using a rate of interest of 6% per annum effective.

At the age of 65, the scheme member uses his accumulated investment to purchase an annuity with a term of 20 years to be paid half-yearly in arrear. At this time the interest rate is 5% per annum convertible half-yearly.

(ii) Calculate the annual rate of payment of the annuity.

(iii) Calculate the discounted mean term of the annuity, in years, at the time of purchase.

Ans:

(i) The accumulation is $1200\ddot{s}^{12}_{\overline{20}|}\ 1.06^{20} + 2300\ddot{s}^{12}_{\overline{20}|} + 100\ I\ddot{a}^{12}_{\overline{20}|}\ 1.06^{20}$

$$= \frac{i}{d^{12}}\ 1200 s_{\overline{20}|}\ 1.06^{20} + 2300 s_{\overline{20}|} + 100\ Ia_{\overline{20}|}\ 1.06^{20}$$

$$= 1.032211\begin{pmatrix} 1,200\times 36.7856\times 3.20714 + 2,300\times 36.7856 \\ +100\times 98.7004\times 3.20714 \end{pmatrix}$$

$$= 1.032211\ 141,571.88 + 84,606.88 + 31,654.60$$

$$= 266,138$$

(ii) Let half-yearly payment $= X$

$$Xa_{\overline{40}|} = 266,138 \text{ at } 2.5\%$$

$$\Rightarrow X = \frac{266,138}{25.1028} = 10,601.94$$

Therefore, annual rate of payment = £21,203.88

(iii) Work in half-years. Discounted mean term is:

$$10,601.94\ \ v + 2v^2 + \ldots + 40v^{40}\ \ /266,138$$

Numerator $= 10,601.94\ \ Ia_{\overline{40}|}$ at 2.5% per half year effective.

$$= 10,601.94 \times 433.3248 = 4,584,075$$

Therefore DMT = 17.26 half years or 8.63 years.

Q7.

An individual intends to retire on his 65th birthday in exactly four years' time. The government will pay a pension to the individual from age 68 of £5,000 per annum monthly in advance. The individual would like to purchase an annuity certain so that his income, including the government pension, is £8,000 per annum paid monthly in advance from age 65 until his 78th birthday. He is to purchase the annuity by a series of payments made over four years quarterly in advance starting immediately.

Calculate the quarterly payments the individual has to make if the present value of these payments is equal to the present value of the annuity he wishes to purchase at a rate of interest of 5% per annum effective. Mortality should be ignored.

Ans:

Let the annual rate of payment = X

Present value of the payments = $X\ddot{a}^{(4)}_{\overline{4}|}$

Present value of the payments needed from the annuity is:

$$8,000\ddot{a}^{(12)}_{\overline{3}|}v^4 + 3,000\ddot{a}^{(12)}_{\overline{10}|}v^7$$

$$X\ddot{a}^{(4)}_{\overline{4}|} = 8,000\ddot{a}^{(12)}_{\overline{3}|}v^4 + 3,000\ddot{a}^{(12)}_{\overline{10}|}v^7$$

$$a_{\overline{3}|} = 2.7232 \quad {}^{i}\!/_{d^{(4)}} = 1.031059$$

$$a_{\overline{4}|} = 3.5460 \quad a_{\overline{10}|} = 7.7217 \quad {}^{i}\!/_{d^{(12)}} = 1.026881 \quad v^4 = 0.82270 \quad v^7 = 0.71068$$

$$X\frac{i}{d^{(4)}}a_{\overline{4}|} = 8,000\frac{i}{d^{(12)}}a_{\overline{3}|}v^4 + 3,000\frac{i}{d^{(12)}}a_{\overline{10}|}v^7$$

$$X \times 1.031059 \times 3.5460 = 8,000 \times 1.026881 \times 2.7232 \times 0.82270 \\ +3,000 \times 1.026881 \times 7.7217 \times 0.71068$$

$$3.65614X = 18,404.80 + 16.905.51$$

$$X = £9,657.81$$

$\therefore$ Quarterly payment is: £2,414.45.

Q8.

Mrs Jones invests a sum of money for her retirement which is expected to be in 20 years' time. The money is invested in a zero coupon bond which provides a return of 5% per annum effective. At retirement, the individual requires sufficient money to purchase an annuity certain of £10,000 per annum for 25 years. The annuity will be paid monthly in arrear and the purchase price will be calculated at a rate of interest of 4% per annum convertible half-yearly.

Calculate the sum of money the individual needs to invest at the beginning of the 20-year period.

Ans:

Purchase price of the annuity (working in half-years)

$$5{,}000 a_{\overline{50}|}^{(6)} \text{ calculated at } i = 2\%$$

$$= 5{,}000 \frac{i}{i^{(6)}} a_{\overline{50}|}$$

$$i = 0.02$$

$$i^{(6)} = 0.019835$$

$$a_{\overline{50}|} = 31.4236$$

$$\text{Purchase price} = 5{,}000 \times \frac{0.02}{0.019835} \times 31.4236$$

$$= £158{,}422$$

Q9.

An investor pays £120 per annum into a savings account for 12 years. In the first four years, the payments are made annually in advance. In the second four years, the payments are made quarterly in advance. In the final four years, the payments are made monthly in advance. The investor achieves a yield of 6% per annum convertible half-yearly on the investment.

Calculate the accumulated amount in the savings account at the end of 12 years.

Ans:

We will use the ½-year as the time unit because the interest rate is convertible half yearly. The effective rate of interest is 3% per half year.

$$\text{Accumulated amount} = \frac{120}{\ddot{a}_{\overline{2}|}}\ddot{s}_{\overline{8}|}\times(1.03)^{16}+60\ddot{s}^{(2)}_{\overline{8}|}\times(1.03)^{8}+60\ddot{s}^{(6)}_{\overline{8}|} \text{ at } 3\%$$

We need $d^{(6)}$ from $\left(1-\frac{d^{(6)}}{6}\right)^{6}=1-d=\frac{1}{1+i}=\frac{1}{1.03}$

$$\Rightarrow 1-\frac{d^{(6)}}{6}=\left(\frac{1}{1.03}\right)^{1/6} \Rightarrow \quad d^{(6)}=\left(1-\left(\frac{1}{1.03}\right)^{1/6}\right)\times 6$$

$$=0.029486111$$

Thus accumulated amount =

$$\frac{120}{a_{\overline{2}|}}s_{\overline{8}|}\times(1.03)^{16}+60\frac{i}{d^{(2)}}s_{\overline{8}|}\times(1.03)^{8}+60\frac{i}{d^{(6)}}s_{\overline{8}|} \text{ at } 3\%$$

$$=\frac{120}{1.9135}*8.8923*1.60471+60\times1.022445*8.8923*1.26677+60*\frac{0.03}{0.029486111}*8.8923$$

= 894.877 + 691.040 + 542.837

= £2,128.75

(above uses factors in formulae and tables book; if book not used then exact answer is £2,128.77).

Q10.

On 1 January 2016, a student plans to take out a five-year bank loan for £30,000 that will be repayable by instalments at the end of each month. Under this repayment schedule, the instalment at the end of January 2016 will be X, the instalment at the end of February 2016 will be 2X and so on, until the final instalment at the end of December 2020 will be 60X. The bank charges a rate of interest of 15% per annum convertible monthly.

(i) Prove that

$$(Ia)_{\overline{n}|}=\frac{\ddot{a}_{\overline{n}|}-nv^{n}}{i}$$

(ii) Show that X = £26.62.

Ans:

(i) $(Ia)_{\overline{n}|} = v + 2v^2 + 3v^3 + \ldots + nv^n \quad (1)$

$(1+i)(Ia)_{\overline{n}|} = 1 + 2v + 3v^2 + \ldots + nv^{n-1} \quad (2)$

$(2)-(1) \Rightarrow i(Ia)_{\overline{n}|} = 1 + v + v^2 + \ldots + v^{n-1} - nv^n$

$$\Rightarrow (Ia)_{\overline{n}|} = \frac{\left(1 + v + v^2 + \ldots + v^{n-1}\right) - nv^n}{i} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{i}$$

(ii) Work in months i.e. use a monthly interest rate of 1.25% per month effective:

$$30{,}000 = Xv + 2Xv^2 + \ldots + 60Xv^{60} = X(Ia)_{\overline{60}|@1.25\%} = X\left(\frac{\ddot{a}_{\overline{60}|} - 60v^{60}}{i}\right)$$

$$= X\left(\frac{\frac{1-v^{60}}{d} - 60v^{60}}{i}\right) = X\left(\frac{\frac{1-1.0125^{-60}}{0.0125/1.0125} - 60\times 1.0125^{-60}}{0.0125}\right)$$

$$= 1126.8774X \Rightarrow X = £26.62$$

Q11.

An investor has a choice of two 15-year savings plans, A and B, issued by a company. In both plans, the investor pays contributions of $100 at the start of each month and the contributions accumulate at an effective rate of interest of 4% per annum before any allowance is made for expenses.

In plan A, the company charges for expenses by deducting 1% from the annual effective rate of return.

In plan B, the company charges for expenses by deducting $15 from each of the first year's monthly contributions before they are invested. In addition it deducts 0.3% from the annual effective rate of return.

Calculate the percentage by which the accumulated amount in Plan B is greater than the accumulated amount in Plan A, at the end of the 15 years.

Ans:

Accumulated amount from Fund A

$$=12\times 100\ddot{s}^{(12)}_{\overline{15}|3\%} = 1,200\frac{1.03^{15}-1}{12\left(1-1.03^{-1/12}\right)}$$

$$=\$22,679.74$$

Accumulated amount from Fund B

$$=12\times 100\ddot{s}^{(12)}_{\overline{15}|3.7\%} - 12\times 15\ddot{s}^{(12)}_{\overline{1}|3.7\%}(1.037)^{14}$$

$$=1,200\frac{1.037^{15}-1}{12\left(1-1.037^{-1/12}\right)} - 180\frac{1.037-1}{12\left(1-1.037^{-1/12}\right)}(1.037)^{14}$$

$$=23,967.992-305.313=\$23,662.68$$

The percentage by which B is greater is found from

$$\frac{23,662.68-22,679.74}{22,679.74}-1=4.33\%$$

Q12.

A university offers its students three financing options for a degree course that lasts exactly three years.

Option A

Fees are paid during the term of the course monthly in advance. The fees are £10,000 per annum in the first year and rise by 5% on the first and second anniversaries of the start of the course.

Option B

The university makes a loan to the students which is repaid in instalments after the end of the course. The instalments are determined as follows:

• No payments are made until three years after the end of the course.

• Over the following 15 years, students pay the university £1,300 per year, quarterly in advance.

• After 15 years of payments, the quarterly instalments are increased to £1,500 per year, quarterly in advance.

• After a further 15 years of payments, the quarterly instalments are increased to £1,800 per year, quarterly in advance, for a further 15-year period after which there are no more payments.

Option C

• Students pay to the university 3% of all their future earnings from work, with the payments made annually in arrear.

A particular student wishes to attend the university. He expects to leave university at the end of the three-year course and immediately obtain employment. The student expects that his earnings will rise by 3% per annum compound at the end of each year for 10 years and then he will take a five-year career break.

After the career break, he expects to restart work on the salary he was earning when the career break started. He then expects to receive salary increases of 1% per annum compound at the end of each year until retiring 45 years after graduating.

The student wishes to take the financing option with the lowest net present value at a rate of interest of 3% per annum effective.

(i) Calculate the present value of the payments due under option A.

(ii) Calculate the present value of the payments due under option B.

(iii) Calculate the initial level of salary that will lead the payments under option C to have the lowest present value of the three options.

Ans:

(i) $$PV_A = 10,000\ddot{a}_{\overline{1}|}^{(12)} + 10,000 \times 1.05v \times \ddot{a}_{\overline{1}|}^{(12)} + 10,000 \times 1.05^2 v^2 \times \ddot{a}_{\overline{1}|}^{(12)}$$

$$= 10,000\ddot{a}_{\overline{1}|}^{(12)}\left(1 + 1.05v + (1.05v)^2\right)$$

$$= 10,000\ddot{a}_{\overline{1}|}^{(12)} \frac{1-(1.05v)^3}{1-1.05v}$$

$$= 10,000 \frac{1-v}{d^{(12)}} \frac{1-(1.05v)^3}{1-1.05v}$$

$$= 10,000 \times 0.986579 \times 3.058629$$

$$= £30,176$$

[or from 2nd line in 1 above:

$$= 10,000 \frac{1-v}{d^{(12)}} \times (1 + 1.019417 + 1.039212)$$

$$= 10,000 \times 0.986579 \times 3.058629$$

$$= £30,176]$$

(ii) $PV_B(1+i)^6 = 1{,}300\ddot{a}^{(4)}_{\overline{45}|} + 200\left(\ddot{a}^{(4)}_{\overline{45}|} - \ddot{a}^{(4)}_{\overline{15}|}\right) + 300\left(\ddot{a}^{(4)}_{\overline{45}|} - \ddot{a}^{(4)}_{\overline{30}|}\right)$ [2]

$$PV_B = v^6\left[1{,}800\ddot{a}^{(4)}_{\overline{45}|} - 200\ddot{a}^{(4)}_{\overline{15}|} - 300\ddot{a}^{(4)}_{\overline{30}|}\right]$$

$$\Rightarrow PV_B = 1.03^{-6}\frac{\left[1{,}800\left(1-v^{45}\right) - 200\left(1-v^{15}\right) - 300\left(1-v^{30}\right)\right]}{4\left(1-1.03^{-1/4}\right)}$$

$$\Rightarrow PV_B = 0.837484 \times \frac{1{,}800\times 0.735561 - 200\times 0.358138 - 300\times 0.588013}{0.0294499} = £30{,}598$$

[or

$$PV_B(1+i)^6 = 1{,}300\ddot{a}^{(4)}_{\overline{15}|} + 1{,}500v^{15}\ddot{a}^{(4)}_{\overline{15}|} + 1{,}800v^{30}\ddot{a}^{(4)}_{\overline{15}|}$$

$$= \ddot{a}^{(4)}_{\overline{15}|}\left(1{,}300 + 1{,}500v^{15} + 1{,}800v^{30}\right)$$

$$= \frac{\left(1-v^{15}\right)}{4\left(1-1.03^{-1/4}\right)} \times \left(1{,}300 + 1{,}500\times 0.641862 + 1{,}800\times 0.411987\right)$$

$$= \frac{0.358138}{0.0294499} \times 3{,}004.3696$$

$$\Rightarrow PV_B = 0.837484 \times 36{,}535.91 = £30{,}598]$$

(iii) Option A has the lower present value out of A and B. Therefore, the student has to calculate the salary level so that $PV_C = 30{,}176$ [1]

Let the initial salary level in relation to option C be S_C

$$30{,}176 = 0.03S_C v^3\left(v + 1.03v^2 + \ldots + 1.03^9 v^{10}\right) + 0.03S_C 1.03^{10} v^{18}\left(v + 1.01v^2 + \ldots + 1.01^{29} v^{30}\right)$$

[3]

$$= 0.03S_C v^4\left(10 + 1.03^{10} v^{15}\left(1 + 1.01v \ + \ldots + 1.01^{29} v^{29}\right)\right)$$

$$= 0.03S_C v^4\left(10 + v^5 \frac{1 - 1.01^{30} v^{30}}{1 - 1.01v}\right)$$

$$= 0.03S_C 1.03^{-4}\left(10 + 0.862609 \times 22.90226\right) = 0.79313S_C$$

$$\Rightarrow S_C = £38{,}047$$ [3]

Therefore, the starting salary has to be less than £38,047 for option C to have the lowest net present value. [1]

Q13.

An investor pays £80 at the start of each month into a 25-year savings plan. The contributions accumulate at an effective rate of interest of 3% per half-year for the first 10 years, and at a force of interest of 6% per annum for the final 15 years.

Calculate the accumulated amount in the savings plan at the end of 25 years.

Ans:

Effective rate of interest per month for first 10 years, i_1, comes from:

$$1 + i_1 = (1.03)^{1/6} \Rightarrow i_1 = 0.49386\% \text{ per month}$$

and effective rate of interest per month for last 15 years, i_2, comes from:

$$1 + i_2 = e^{0.06/12} \Rightarrow i_2 = 0.50125\% \text{ per month}$$

$$\Rightarrow \text{Accumulation after 25 years} = 80\ \ddot{s}_{\overline{120}|}^{0.49386\%} \times (1.0050125)^{180} + 80\ \ddot{s}_{\overline{180}|}^{0.50125\%}$$

$$\text{where } \ddot{s}_{\overline{120}|}^{0.49386\%} = 1.0049386 \times \frac{(1.0049386^{120} - 1)}{0.0049386}$$

$$= 164.0318$$

$$\text{and } \ddot{s}_{\overline{180}|}^{0.50125\%} = 1.0050125 \times \frac{\left(1.0050125^{180} - 1\right)}{0.0050125} = 292.6504$$

$$\Rightarrow \text{Accumulation} = 80 \times 164.0318 \times 1.0050125^{180} + 80 \times 292.6504$$

$$= 32276.13 + 23412.03 = £55{,}688.16 \text{ (exact answer is £55,688.38)}$$

[1]

[or working in years:

$$1 + i_1 = (1.03)^2 \Rightarrow i_1 = 6.09\% \text{ per year}$$
$$1 + i_2 = e^{0.06} \Rightarrow i_2 = 6.1837\% \text{ per year}$$

$$\Rightarrow \text{Accumulation after 25 years} = 960\ \ddot{s}_{\overline{10}|}^{(12)@6.09\%} \times (1.061837)^{15} + 960\ \ddot{s}_{\overline{15}|}^{(12)@6.1837\%}$$

$$\text{where } \ddot{s}_{\overline{10}|}^{(12)@6.09\%} = \frac{(1.0609^{10} - 1)}{12 \times \left(1 - \left(1 - \frac{0.0609}{1.0609}\right)^{1/12}\right)} = 13.6693$$

$$\text{and } \ddot{s}_{\overline{15}|}^{(12)@6.1837\%} = \frac{(1.061837^{15} - 1)}{12 \times \left(1 - \left(1 - \frac{0.061837}{1.061837}\right)^{1/12}\right)} = 24.3877$$

$$\Rightarrow \text{Accumulation} = 960 \times 13.6693 \times 1.061837^{15} + 960 \times 24.3877$$

$$= 32276.42 + 23412.17 = £55{,}688.59]$$

Q14.

A company invests $50,000 now and receives the following income over the next 12 years:

During the first 4-year period: $4,000 per annum paid quarterly in arrears.

FM Unit 3

PRACTICE QUESTION

During the second 4-year period: $X per annum paid half-yearly in arrears.

During the final 4-year period: $12,000 per annum paid continuously.

There are no other payments under the investment.

Calculate X assuming the company achieves a nominal rate of return of 9% per annum convertible monthly.

Ans:

Equation of value is given by:

$$50,000 = 4,000a_{\overline{4|}}^{(4)} + Xa_{\overline{4|}}^{(2)} \times v^4 + 12,000\bar{a}_{\overline{4|}} \times v^8$$

where the effective rate of interest per annum is i such that $i^{(12)} = 9\%$.

Thus, we have: $1+i = \left(1+\frac{i^{(12)}}{12}\right)^{12} = \left(1+\frac{0.09}{12}\right)^{12} \Rightarrow i = 9.38069\%$ per annum

Thus, we have:

$$a_{\overline{4|}}^{(4)} = \frac{1-v^4}{i^{(4)}} = \frac{1-0.6986141}{0.0906767} = 3.323742 \text{, where}$$

$$v^4 = \left(\frac{1}{1+i}\right)^4 = \left(\frac{1}{1.0938069}\right)^4 = 0.6986141$$

$$\left(1+\frac{i^{(4)}}{4}\right)^4 = 1+i = 1.0938069 \Rightarrow i^{(4)} = 0.0906767$$

$$a_{\overline{4|}}^{(2)} = \frac{1-v^4}{i^{(2)}} = \frac{1-0.6986141}{0.0917045} = 3.286491 \text{, where}$$

$$\left(1+\frac{i^{(2)}}{2}\right)^2 = 1+i = 1.0938069 \Rightarrow i^{(2)} = 0.0917045$$

$$\bar{a}_{\overline{4|}} = \frac{1-v^4}{\delta} = \frac{1-0.6986141}{0.0896642} = 3.361274 \text{, where}$$

$$e^{\delta} = 1+i = 1.0938069 \Rightarrow \delta = 0.0896642$$

Thus, we have:

$$50,000 = 4,000 \times 3.323742 + 0.6986141 \times 3.286491X$$

$$+12,000 \times \left(\frac{1}{1.0938069}\right)^8 \times 3.361274$$

$$= 13,294.968 + 2.2959890X + 19,686.109$$

$$\Rightarrow X = \$7,412.46$$

FM Unit 3

PRACTICE QUESTION

Q15.

On 1 January 2022, a student plans to take out a 10-year bank loan for $15,000. Under the repayment schedule, instalments will be paid monthly in arrears until the end of the term. The first instalment, at the end of January 2022, will be X, and the second instalment, at the end of February 2022, will be 2X, and so on, until the instalment at the end of December 2026, which will be 60X. The remaining instalments from the end of January 2027 will also be 60X. The bank charges a rate of interest of 12% p.a. effective.

(i) Write down an equation of value to calculate X.

(ii) Calculate the value of X using the equation of value in part (i).

Ans:

(i)

$$\$15{,}000 = X\left(Ia\right)_{\overline{60|}@j} + 60Xv_j^{60}a_{\overline{60|}@j}$$

where $j = \frac{i^{(12)}}{12}$

Alternative

$$\$15{,}000 = X\left(Ia\right)_{\overline{60|}@j} + 12\times 60Xv_i^{5}a^{(12)}_{\overline{5|}@i}$$

(ii)

$$\left(Ia\right)_{\overline{60|}@j} = \frac{\ddot{a}_{\overline{60|}j} - 60v_j^{60}}{j} = \frac{\dfrac{1-1.0094888^{-60}}{0.0094888/1.0094888} - 60\times 1.0094888^{-60}}{0.0094888} = 1261.989$$

$$a^{(12)}_{\overline{5|}@i} = 3.6048\times\frac{0.12}{0.11387} = 3.7990 \qquad \text{OR} \qquad a_{\overline{60|}@j} = 45.58779473$$

where $i = 12\%$ and $j = \frac{i^{(12)}}{12} = 1.12^{1/12} - 1 = 0.0094888$

So $\$15{,}000 = 1261.989\times X + 60\times 45.58779473\times\left(1.0094888\right)^{-60}\times X$

$$\Rightarrow X = \frac{15{,}000}{2{,}814.053} = \$5.33$$

Q16.

The Green Investment Company has the opportunity to purchase a factory for \$400,000. The factory is to be leased and two different companies, A. The proposal made by the company is as follows:

Company A

The Green Investment Company will need to spend another \$50,000 refurbishing the factory for Company A.

Company A will pay rent annually in advance for 20 years starting immediately. The rent will increase by 3% p.a. compound each year. At the end of 20 years, Company A will purchase the factory from the Green Investment Company for \$450,000.

Calculate the initial annual rent payable by Company A, to give the Green Investment Company an internal rate of return of 9% p.a. effective on the proposal.

Ans:

$$\$450,000 = X\left[1+(1.03)v_{9\%}^{1}+(1.03)^{2}v_{9\%}^{2}+\ldots+(1.03)^{19}v_{9\%}^{19}\right]+\$450,000v_{9\%}^{20}$$

$$\text{With } \frac{1.03}{1.09}=\frac{1}{1+j}\Rightarrow j=0.058252427$$

$$\$450,000 = X\ddot{a}_{\overline{20}|j\%}+\$450,000v_{9\%}^{20}$$

$$\$450,000 = X\times 12.31216704+\$80,293.9004\Rightarrow X=\$30,027.70 \text{ per annum}$$

Q17.

Frasier is 33 years old and just received an inheritance from his parents' estate. He wants to invest an amount of money today such that he can receive \$5,000 at the end of every month for 15 years when he retires at age 65. If he can earn 9% compounded annually until age 65 and then 5% compounded annually when the fund is paying out, how much money must he invest today?

Step 2: Ordinary General Annuity (Payment Stage):

Calculate the equivalent periodic interest rate that matches the payment interval (i_{eq}, Formula 9.6), number of annuity payments (n, Formula 11.1), and present value of the ordinary general annuity (PV_{ORD}, Formula 11.3A).

$$i = \frac{I/Y}{C/Y} = \frac{5\%}{1} = 5\%$$

$$i_{eq} = (1+i)^{\frac{C/Y}{P/Y}} - 1 = (1+0.05)^{\frac{1}{12}} - 1 = 0.004074124 \text{ per month}$$

$$n = P/Y \times (\text{Number of Years}) = 12 \times 15 = 180 \text{ payments}$$

$$\begin{aligned} PV_{ORD} &= PMT\left[\frac{1-(1+i)^{-n}}{i}\right] \\ &= \$5{,}000\left[\frac{1-(1+0.004074124)^{-180}}{0.004074124}\right] \\ &= \$5{,}000\left[\frac{0.518982921}{0.004074124}\right] \\ &= \$636{,}925.79 \end{aligned}$$

Step 3: Deferral Period (Accumulation Stage):

Discount the principal of the annuity (PV_{ORD}) back to today (Age 33). Calculate the periodic interest rate (i, Formula 9.1), number of single payment compound periods (n, Formula 9.2A), and present value of a single payment (PV, Formula 9.2B), rearranged.

$$i = \frac{I/Y}{C/Y} = 9\%1 = 9\%$$

$$n = C/Y \times (\text{Number of Years}) = 1 \times 32 = 32 \text{ compounds}$$

$$\begin{aligned} PV &= \frac{FV}{(1+i)^n} \\ &= \frac{\$636{,}925.79}{(1+0.09)^{32}} \\ &= \$40{,}405.54 \end{aligned}$$

IFOA PQs

1. CT1 September 2011 Q3

An individual intends to retire on his 65th birthday in exactly four years' time. The government will pay a pension to the individual from age 68 of £5,000 per annum monthly in advance. The individual would like to purchase an annuity certain so that his income, including the government pension, is £8,000 per annum paid monthly in advance from age 65 until his 78th birthday. He is to purchase the annuity by a series of payments made over four years quarterly in advance starting immediately.

Calculate the quarterly payments the individual has to make if the present value of these payments is equal to the present value of the annuity he wishes to purchase at a rate of interest of 5% per annum effective. Mortality should be ignored.

Ans: £2,414.45

2. CT1 September 2013 Q8

Mrs Jones invests a sum of money for her retirement which is expected to be in 20 years' time. The money is invested in a zero coupon bond which provides a return of 5% per annum effective. At retirement, the individual requires sufficient money to purchase an annuity certain of £10,000 per annum for 25 years. The annuity will be paid monthly in arrear and the purchase price will be calculated at a rate of interest of 4% per annum convertible half-yearly.

(i) Calculate the sum of money the individual needs to invest at the beginning of the 20-year period.

Ans: £59,708

3. CT1 September 2016 Q2

The nominal rate of interest per annum convertible quarterly is 2%.

Calculate the present value of a payment stream paid at a rate of €100 per annum, monthly in advance for 12 years.

Ans: €1,068.05

4. CT1 April 2017 Q10

An individual aged exactly 65 intends to retire in five years' time and receive an annuity-certain. The annuity will be payable monthly in advance and will cease after 20 years. The annuity will increase at each anniversary of the commencement of payment at the rate of 3% per annum.

The individual would like the initial level of annuity to be £20,000 per annum. The price of the annuity will be the present value of the payments on the date it commences using an interest rate of 7% per annum effective.

(i) Calculate the price of the annuity.

Ans: £276,645

5. CT1 April 2018 Q3

An investor pays £80 at the start of each month into a 25-year savings plan.

The contributions accumulate at an effective rate of interest of 3% per half-year for the first 10 years, and at a force of interest of 6% per annum for the final 15 years.

Calculate the accumulated amount in the savings plan at the end of 25 years.

Ans: £55,688.16

6. CT1 September 2018 Q4

A company issues a loan stock which pays coupons at a rate of 6% per annum half-yearly in arrears. The stock is to be redeemed at 103% after 25 years.

(i)

(a) Calculate the price per £100 nominal at issue which would provide a gross redemption yield of 3% per annum convertible half yearly.
(b) Calculate the price per £100 nominal three months after issue which would provide a gross redemption yield of 3% per annum convertible half-yearly.

Ans: (i) (a) 153.925, (b) 155.075

7. CT1 September 2020 Q5

A company invests \$50,000 now and receives the following income over the next 12 years:

During the first 4-year period: \$4,000 per annum paid quarterly in arrears.

During the second 4-year period: \$X per annum paid half-yearly in arrears.

During the final 4-year period: \$12,000 per annum paid continuously.

There are no other payments under the investment.

Calculate X assuming the company achieves a nominal rate of return of 9% per annum convertible monthly.

Ans: \$7,412.46

8. CT1 April 2024 Q4

An individual pays £4,000 p.a. into a savings account for 10 years. During the first 4 years, the payments are made quarterly in advance. For the remaining years, the payments are made continuously.

The investor achieves a yield of 6% p.a. convertible quarterly on the investment.

Calculate, showing all working, the accumulated amount in the savings account at the end of 10 years.

Ans: £54,866.66

9. CT1 September 2014 Q5

Calculate, at a rate of interest of 5% per annum effective:

(i) $a_{\overline{5|}}^{(12)}$

(ii) ${}_{4|}a_{\overline{15|}}$

(iii) $(I\bar{a})_{\overline{10|}}$

(iv) $(\bar{I}\bar{a})_{\overline{10|}}$

FM Unit 3

PRACTICE QUESTION

(v) the present value of an annuity that is paid annually in advance for 10 years with a payment of 12 in the first year, 11 in the second year and thereafter reducing by 1 each year.

Ans:

(i) 4.4278

(ii) 8.5394

(iii) 40.3501

(iv) 36.3613

(v) 64.0592