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Subject: Probability & Statistics - 1

Chapter: Generating Functions

Category: Practice Questions

All the questions are taken from IAI

1. CS1 Nov'24 Q9

Select the correct expression which represents the moment generating function of the random variable X i.e. $M_X(t)$ –
A. $M_X(t)$ = exp (½ t^2)
B. $M_X(t)$ = exp (25t + 36t^2)
C. $M_X(t)$ = exp (25t + 18t^2)
D. $M_X(t)$ does not exist in closed form.
Ans: C

2. CS1 May'25 Q10

Q. 10) Let X be a random variable with probability density function:

$$f(x) = \frac{4}{5}e^{-x} + \frac{2}{5}e^{-2x} \qquad x \geq 0$$

i) Find the moment generating function of the above function. **[3]**

A) $Mx(t) = \frac{1}{(1-t)}$ $\quad for\ t < 1$
B) $Mx(t) = \frac{1}{5(1-t)} + \frac{1}{5(1-t)}$ $\quad for\ t < 1$
C) $Mx(t) = \frac{2}{5(1-t)} + \frac{4}{5(1-t)}$ $\quad for\ t < 1$
D) $Mx(t) = \frac{4}{5(1-t)} + \frac{2}{5(2-t)}$ $\quad for\ t < 1$
E) $Mx(t) = \frac{4}{5(1+t)} + \frac{2}{5(1+t)}$ $\quad for\ t > -1$

ii) Hence, find the mean and variance of X using the moment generating function in part (i). **[3]**

A) Mean = 0.8, Variance = 0.89
B) Mean = 0.9, Variance = 0.04
C) Mean = 0.8, Variance = 0.56
D) Mean = 0.8, Variance = 0.80
E) Mean = 0.9, Variance = 0.89

Ans: i) D, ii) E

3. CS1 May'24 Q5

Let the joint MGF $M_{x,y}(t, s)$ of the joint distribution of X and Y be defined as follows:

$$M_{x,y}(t, s) = E[\exp(xt + ys)].$$

Which of the following statements is TRUE in respect of the joint MGF of random variables X and Y i.e. $M_{x,y}(t, s)$?

A. $M_{x,y}(t, s) = M_x(t) + M_y(s)$
B. $M_{x,y}(t, s) = M_x(t) \times M_y(s)$
C. $M_{x,y}(t, s) = M_x(t) - M_y(s)$
D. None of the above

Ans: B

4. CS1 Nov'23 Q1

Q. 1) Let $M_x(t)$ denote the moment generating function (MGF) and $C_x(t)$ denote the cumulant generating function (CGF) of a random variable X.

i) Which of the following is TRUE for the relationship between $M_x(t)$ and $C_x(t)$?

A. $C_x(t) = e^{Mx(t)}$

B. $M_x(t) = e^{Cx(t)}$

C. $C_x(t) = \log_{10}(M_x(t))$

D. $M_x(t) = \log_{10}(C_x(t))$

The series expansion formula of moment generating function (MGF) for a random variable X is as follows:

$$M_X(t) = 1 + tE(X) + \frac{t^2}{2!}E(X^2) + \frac{t^3}{3!}E(X^3) + \frac{t^4}{4!}E(X^4) + \cdots$$

ii) Based on the above series expansion, show that the variance of random variable X is given by the following expression:

$$var(X) = M''_X(0) - [M'_X(0)]^2$$

iii) In terms of $C_x(t)$, how will you represent variance of random variable X? Choose the correct option from those given below:

A. $var(X) = C''_X(0) - [C'_X(0)]^2$

B. $var(X) = [C'_X(0)]^2$

C. $var(X) = C''_X(0)$

D. $var(X) = [C''_X(0)]^2$

Ans: i) B, ii) - , iii) C

5. CS1 May'23 Q1

i) Let M(t) be the Moment Generating Function (MGF) of a random variable Y. Given below are four MGFs written in terms of M(t) of four different random variables. Identify which one of the following is NOT a valid MGF.

A. $M(t) * M(3t)$

B. $e^{-3t} * M(0.5t)$

C. $\frac{2}{3} * M(t)$

D. $M(\frac{1}{3}t)$

Let X be a two-parameter exponential random variable such that X ~ Exp (λ, a). It has the following probability density function:

$$f(x) = \lambda e^{-\lambda(x-a)}, \quad x \geq a, \text{ where } \lambda, a > 0$$

ii) Show that the moment generating function of X is given by: $\left(1 - \frac{t}{\lambda}\right)^{-1} * e^{at}$.

iii) Using the result derived in part (ii), calculate E(X).

Ans: i) C , ii) - , iii) a+1/λ

6. CS1 Dec'22 Q3

i) For a standard normal random variable Z, derive an expression for its Moment Generating Function (MGF) using first principles.

ii) Using the results obtained in part (i), prove that a normal variable X with mean μ and variance δ^2 , is perfectly symmetrical about its mean i.e. the coefficient of skew-ness of the normal variable X is equal to 0.

Hints:
a) Standard relationship between normal variable X and standard normal variable Z i.e. "Z = (X - μ) / δ" can be directly used without proof.
b) Use the fact that E (Z^r) is the coefficient of the term t^r / r! in the Taylor expansion.
A random variable Y has probability density function

7. CS1 July'22 Q3

A Student Actuary is analysing the time taken between two consecutive claims in a health insurance policy. It is believed that the time period (denoted by random variable X) between two consecutive claims in a health insurance policy follows an exponential distribution with mean μ.

i) Identify the correct expression for moment generating function of X.

A. $E(e^{tX}) = (\frac{1}{\mu} - t)\int_0^\infty e^{-z}\, .\, dz$

B. $E(e^{tX}) = \frac{1}{\mu}(\frac{1}{\mu} - t)^{-1}\int_0^\infty e^{-z}\, .\, dz$

C. $E(e^{tX}) = \frac{1}{\mu}\int_0^\infty e^{-z}\, .\, dz$

D. $E(e^{tX}) = \frac{1}{\mu}(\frac{1}{\mu} - t)\int_0^\infty e^{-z}\, .\, dz$

E. None of the above

ii) If random variable Y denotes sum of time periods of two consecutive claims of N policies, determine the moment generating function of Y.

iii) Identify the distribution of Y.

Ans: i) B, ii) $M_Y(t) = (1 - t\mu)^{-N}$**, iii)** Gamma(N, $1/\mu$)

8. **CS1 March'22 Q3**

The claim amounts X and Y (in units of INR 1000) for two different types of insurance policy are modelled using a gamma distribution with parameters $\alpha = 5$, $\lambda = 1/8$ and $\alpha = 3$, $\lambda = 1/4$ respectively i.e. X ~ Gamma(5,1/8), Y ~ Gamma(3,1/4). Assume that X and Y are independent of each other.

i) Identify which one of the following options describes the moment generating function of X

A. $(1-8t)^{(-4)}$

B. $(1-t/8)^{(-4)}$

C. $(1-8t)^{(-5)}$

D. None of the above

ii) Use moment generating function to show $(1/4)X \sim \chi^2_{10}$

iii) Calculate the probability that the claim amount, X, exceeds INR 40,000.

iv) An analyst argues that sum of X and Y must follow Gamma(8,3/8) i.e. X + Y ~ Gamma(8,3/8)

Comment on the analyst's argument using moment generating function.

Ans: i) C, ii) , iii) 0.4405, **iv)** X + Y does not have a Gamma(8,3/8) distribution

9. **CS1 Sep'21 Q3**

The amount of claims of a motor insurance company is modelled as an exponential random variable with λ = 1.25 (in '000s).
A data analyst is interested in assessing the probability of Y exceeding 10 (INR 10,000 if represented in absolute amounts), wherein; Y is the total of 10 independent motor claim amounts.

i) Show, using moment generating functions, that:
a) Y has gamma distribution and
b) 2.5Y has a χ^2_{20} distribution

10. **CS1 Nov'20 Q7**

A person is observing time taken for a bus to appear at the bus stop. It is believed that the time period (denoted by random variable X) between the time at which two consecutive buses arrive at the stop follows an exponential distribution with mean θ.

iii) The following equation on further evaluation yields the moment generation function of X.

a) $E(e^{tX}) = \frac{1}{\theta}\left(\frac{1}{\theta} - t\right)\int_0^\infty exp(-z).dz$

b) $E(e^{tX}) = \frac{1}{\theta}\int_0^\infty exp(-z).dz$

c) $E(e^{tX}) = \left(\frac{1}{\theta} - t\right)^{-1}\int_0^\infty exp(-z).dz$

d) $E(e^{tX}) = \frac{1}{\theta}\left(\frac{1}{\theta} - t\right)^{-1}\int_0^\infty exp(-z).dz$

iv) If random variable Y denotes the total observation time to count N buses, determine the moment generating function of Y.

v) Hence identify the distribution followed by Y.

Ans: iii) D, **iv)** $M_Y(t) = (1 - t\theta)^{-N}$, **v)** *Gamma* (N, $1/\theta$)

11. **CS1 Nov'19 Q1**

A discrete random variable X has the following probability density function:
$P(X = x) = p(1 - p)^{x-1}$ where $0 < p < 1$ and $x>0$

i) Derive the Moment Generating Function (MGF) and the Cumulant Generating Function (CGF) for the above distribution.

ii) Using either the MGF or the CGF determine E(X)

Ans: ii) 1/ p

12. **CS1 Jun'19 Q1**

i) In a game, a player pulls three cards at random from a full deck of 52 cards, and earns as many points as the number of red cards among the three. Assume 2 people each play this game once with their own decks, and let X be the sum of their combined points. Derive the moment generating function of X.

ii) Hence prove that he mean of total points earned by the players is

$$\frac{12}{\binom{52}{3}}\left(\binom{26}{3} + \binom{26}{1}\binom{26}{2}\right)^2$$

13. CT3 Dec'18 Q5

The moment generating function (mgf) of a random variable X is given by:

$$Mx\ (t) = \sum_{k=0}^{\infty} \frac{e^{kt-1}}{k!}$$

Find $P(X = 3)$.

Ans: 0.06131

14. **CT3 Dec'18 Q6**

Let X_1 and X_2 be two i.i.d. random variables having the probability mass function
$P(X_1 = i) = 1/4$; i = 1, 2, 3, 4.
0 ; otherwise.
Using their mgf s or otherwise, find the probability mass function of $X1 + X2$.

Ans:

Hence$P(Y=2)=\frac{1}{16}$; $P(Y=3)=\frac{2}{16}$; $P(Y=4)=\frac{3}{16}$; $P(Y=5)=\frac{4}{16}$;
$P(Y=6)=\frac{3}{16}$; $P(Y=7)=\frac{2}{16}$; $P(Y=8)=\frac{1}{16}$

Or$P(Y=y)=\frac{4-|y-5|}{16}$ y=2,3,...,8

15. **CT3 Sep'18 Q4**

A bivariate random variable $X = (X_1, X_2)$ has the following moment generating function
$M_X(t_1, t_2) = (1/3)(1 + e^{(t1+2*t2)} + e^{(2*t1+t2)})$.

Determine the covariance between X_1 and X_2.

Ans: 0.33

16. **CT3 Sep'18 Q7**

Let $C = W_1 + W_2 + \cdots + W_N$, where C represents the aggregate claim,
N represents the random number of claims received during the time period and
W_j represents the size of the j th claim.
Assume that W_j 's are iid random variables and independent of N.

i) Derive the distribution function of C.

ii) Derive the cumulant generating function (cgf) of C

Ans: i) , ii) Hence, log $M_C(t) = \psi_C(t) = \psi_N(\psi_W(t))$ which is the cumulant generating function of C.

17. CT3 Mar'18 Q5

Let X and Y be iid random variables from an exponential distribution with mean 0.5.

i) Defining Z = Min (X ,Y), obtain the cumulative distribution function of Z.

ii) Hence, find the mean of Z.

Ans: i) $1 - e^{-4t}$ **, ii)** ¼

18. CT3 Mar'18 Q7

Let Y be a random variable such that $Y = \sum_{i=1}^{n} X_i$ i=1 ; where X_i, i = 1,2, … , n are independent random variables ~$Exp(\theta)$.
Let Z be a random variable such that the moment generating function of Z is
$M_z(t) = \sqrt{M_y(t)}$.
By comparing $M_z(t)$with the MGF of chi square distribution, determine the value of θ for which Z follows a chi square distribution with appropriate degrees of freedom

Ans: The MGF of chi-squared distribution is $(1 - 2t)^{-n/2}$. Hence $M_z(t)$ is the MGF of a chi-squared distribution with 'n' degrees of freedom when θ = 0.5.