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Subject: Numerical methods and algebra

Chapter: Unit 1 & 2

Category: Assignment 1 solutions

Solution:

1. (i) Absolute error :

Absolute error = Actual value - Approximate value

Actual length = 12.5 cm approximate value = 12.65 cm

Area of circle $A(r) = \pi r^2$

Actual change in area $= \pi(12.65)^2 - \pi(12.5)^2$

$= \pi\, [160.0225 - 156.25]$

$= \pi\, (3.7725)$

$= 3.7725\, \pi$ --(1)

Approximate change = A' (12.5) x change in radius

$= 2\, \pi\, (12.5) \times 0.15$

$= 25\, \pi \times 0.15$

$= 3.75\, \pi$ --(2)

(1) - (2)

Absolute error $= 3.7725\, \pi - 3.75\pi$

$= 0.0225\pi\ cm^2$

(ii) Relative error = (Actual value - Approximate value)/Actual value

Relative error $= 0.0225\, \pi / 3.7725\pi$

$= 0.0059\ cm^2$

(iii) Percentage error = Relative error x 100%

= 0.0059 x 100%

= 0.6 %

2.(a) Linear interpolation. $F(x) = f(x_0) + (x - x_0)\, f[x_1, x_0]$

$X_0 = 4$, $x_1 = 6$

$F[x_1, x_0] = [f(6) – f(4)] / (6 - 4) = 0.0880046$

$F(5) = f(4) + (5 - 4)\, 0.0880046 = 0.690106$

(b) $x_0 = 4.5$, $x_1 = 5.5$

$F[x_1, x_0] = [f(5.5) – f(4.5)] / (5.5 - 4.5) = 0.0871502$

$F(5) = f(4.5) + (5 – 4.5)\, 0.0871502 = 0.696788$

3. Iteration table is given as follows:

No.	a	b	c	f(a)*f(c)
1	1.5	2	1.75	15.264(+ve)
2	1.75	2	1.875	2.419(+ve)
3	1.75	1.875	1.812	2.419(+ve)
4	1.812	1.875	1.844	-0.303(-ve)
5	1.844	1.875	1.86	-0.027(-ve)

Answer: 1.86

4. Here $x^3-x-1=0$

Let $f(x)=x^3-x-1$

$\therefore f'(x)=3x^2-1$

X	0	1	2
F(x)	-1	-1	5

Here $f(1)=-1<0$ and $f(2)=5>0$

$\therefore$ Root lies between 1 and 2

$x0=1+22=1.5$

1st iteration :

$f(x0)=f(1.5)=0.875$

$f'(x0)=f'(1.5)=5.75$

$x1=x0$ - $f(x0)$ $f'(x0)$

$x1=1.5-0.8755.75$

$x1=1.34783$

2*nd* iteration :

$f(x1)= f(1.34783)=0.10068$

$f'(x1)= f'(1.34783)=4.44991$

$x2=x1- f(x1) f'(x1)$

$x2=1.34783-0.10068 * 4.44991$

$x2=1.3252$

3*rd* iteration :

$f(x2)=f(1.3252)=0.00206$

$f'(x2)=f'(1.3252)=4.26847$

$x3=x2-f(x2)f'(x2)$

$x3=1.3252-0.002064.26847$

$x3=1.32472$

4*th* iteration :

$f(x3)=f(1.32472)=0$

$f'(x3)=f'(1.32472)=4.26463$

$x4=x3-f(x3)f'(x3)$

$x4=1.32472-0 * 4.26463$

$x4=1.32472$

Approximate root of the equation $x3$-x-1=0 using Newton Raphson mehtod is 1.32472

n	x_0	$f(x_0)$	$f'(x_0)$	x_1
1	1.5	0.875	5.75	1.34783
2	1.34783	0.10068	4.44991	1.3252
3	1.3252	0.00206	4.26847	1.32472
4	1.32472	0	4.26463	1.32472

5.

We have, $A^2 = A$

Now,

$$7A - (I + A)^3 = 7A - [I^3 + A^3 + 3IA\,(I + A)]$$

$$[\because (x + y)^3 = x^3 + y^3 + 3xy\,(x + y)]$$

$$= 7A - [I + A^2 \cdot A + 3A\,(I + A)] \qquad [\because I^3 = I]$$

$$= 7A - [I + A \cdot A + 3AI + 3A^2] \; [\because A^2 = A, \text{ given}]$$

$$= 7A - [I + A + 3A + 3A] \qquad [\because AI = A]$$

$$= 7A - [I + 7A] = -I \qquad \textbf{(1)}$$

6. Let A =

$$\begin{bmatrix} 1 & -1 & 2 \\ 4 & 0 & 6 \\ 0 & 1 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{adj(A)}{|A|}$$

To find out the adj (A), first we have to find out cofactor(A).

a_{11} = -6, a_{12} = 4, a_{13} = 4

a_{21} = 1, a_{22} = -1, a_{23} = -1

a_{13} = -6, a_{32} = 2, a_{33} = 4

So, cofactor (A) =

$$\begin{bmatrix} -6 & 4 & 4 \\ 1 & -1 & -1 \\ -6 & 2 & 4 \end{bmatrix}$$

adj(A) = $[cofactor(A)]^T$

adj(A) =

$$[cofactor(A)]^T = \begin{bmatrix} -6 & 4 & 4 \\ 1 & -1 & -1 \\ -6 & 2 & 4 \end{bmatrix}^T$$

adj(A) = $\begin{bmatrix} -6 & 1 & -6 \\ 4 & -1 & 2 \\ 4 & -1 & 4 \end{bmatrix}$

Then, | A | = 1(0-6)+1(-4-0)+2(4-0) = -6-4+8 = -2

$$A^{-1} = \frac{adj(A)}{|A|} = \frac{\begin{bmatrix} -6 & 1 & -6 \\ 4 & -1 & 2 \\ 4 & -1 & 4 \end{bmatrix}}{-2}$$

A^{-1} =

$$\begin{bmatrix} 3 & -\frac{1}{2} & 3 \\ -2 & \frac{1}{2} & -1 \\ -2 & \frac{1}{2} & -2 \end{bmatrix}$$

7. Slide the image above to see the resultant vector. The resultant vector magnitude calculation is as follows.

$$\begin{aligned} R &= \sqrt{A^2 + B^2 + 2AB\cos\theta} \\ &= \sqrt{75^2 + 25^2 + 2(75)(25)\cos 30^\circ} \\ &= 97.46 \text{ meter} \end{aligned}$$

8.

$$\det\left(\begin{pmatrix}2 & -3 & 0\\ 2 & -5 & 0\\ 0 & 0 & 3\end{pmatrix} - \lambda\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\right)$$

$$\begin{pmatrix}2 & -3 & 0\\ 2 & -5 & 0\\ 0 & 0 & 3\end{pmatrix} - \lambda\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$\lambda\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}\lambda & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \lambda\end{pmatrix}$$

$$= \begin{pmatrix}2 & -3 & 0\\ 2 & -5 & 0\\ 0 & 0 & 3\end{pmatrix} - \begin{pmatrix}\lambda & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \lambda\end{pmatrix}$$

$$= \begin{pmatrix}2-\lambda & -3 & 0\\ 2 & -5-\lambda & 0\\ 0 & 0 & 3-\lambda\end{pmatrix}$$

$$= \det\begin{pmatrix}2-\lambda & -3 & 0\\ 2 & -5-\lambda & 0\\ 0 & 0 & 3-\lambda\end{pmatrix}$$

$$= (2-\lambda)\det\begin{pmatrix}-5-\lambda & 0\\ 0 & 3-\lambda\end{pmatrix} - (-3)\det\begin{pmatrix}2 & 0\\ 0 & 3-\lambda\end{pmatrix} + 0\cdot\det\begin{pmatrix}2 & -5-\lambda\\ 0 & 0\end{pmatrix}$$

$$= (2-\lambda)\left(\lambda^2+2\lambda-15\right) - (-3)\cdot 2\left(-\lambda+3\right) + 0\cdot 0$$

$$= -\lambda^3 + 13\lambda - 12$$

$$-\lambda^3 + 13\lambda - 12 = 0$$

$$-(\lambda-1)(\lambda-3)(\lambda+4) = 0$$

The eigenvalues are :

$\lambda = 1,\ \lambda = 3,\ \lambda = -4$

Eigenvectors for $\lambda = 1$

$$\begin{pmatrix}2 & -3 & 0\\ 2 & -5 & 0\\ 0 & 0 & 3\end{pmatrix} - 1\cdot\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & -3 & 0\\ 2 & -6 & 0\\ 0 & 0 & 2\end{pmatrix}$$

$$(A-1I)\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}1 & -3 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

$$\begin{Bmatrix}x-3y=0\\ z=0\end{Bmatrix}$$

Plug into $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$\eta = \begin{pmatrix} 3y \\ y \\ 0 \end{pmatrix} \quad y \neq 0$

Let $y = 1$

$\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$

Similarly

Eigenvectors for $\lambda = 3$: $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Eigenvectors for $\lambda = -4$: $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

The eigenvectors for $\begin{pmatrix} 2 & -3 & 0 \\ 2 & -5 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

$= \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

9.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Here is the derivative of the function since we'll need that.

$f'(x)=3x^2-14x+8$

The first iteration through the formula for x1 is,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 5 - \frac{f(5)}{f'(5)} = 5 - \frac{-13}{13} = 6$$

The second iteration through the formula for x2x2 is,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 6 - \frac{f(6)}{f'(6)} = 6 - \frac{9}{32} = 5.71875$$

10.

No.	a	b	c	f(a)*f(b)
1	0.5	1.5	1	0.555(+ve)
2	1	1.5	1.25	$-1.555*10^{-3}$(-ve)
3	1	1.25	1.125	0.063(+ve)
4	1.125	1.25	1.187	0.014(+ve)

Hence we stop the iterations after 4. Therefore the approximated value of x is 1.187.

11.

Cofactor matrix = $\begin{bmatrix} a^2 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & a^2 \end{bmatrix}$

adj A = (cofactor Matrix)^T = $\begin{bmatrix} a^2 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & a^2 \end{bmatrix}$

|adj A| = $\begin{vmatrix} a^2 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & a^2 \end{vmatrix}$ = a^6.

12.

We have $\begin{bmatrix}1 & x & 1\end{bmatrix}\begin{bmatrix}1 & 3 & 2\\0 & 5 & 1\\0 & 3 & 2\end{bmatrix}\begin{bmatrix}x\\1\\-2\end{bmatrix}=0$

$\Rightarrow \begin{bmatrix}1 & 5x+6 & x+4\end{bmatrix}\begin{bmatrix}x\\1\\-2\end{bmatrix}=0$

$\Rightarrow x + 5x + 6 - 2x - 8 = 0$

$\Rightarrow 4x - 2 = 0 \Rightarrow x = \frac{1}{2}$

13.

Cofactor of $1 = A_{11} = +\begin{vmatrix}5 & 0\\1 & 8\end{vmatrix} = +(40-0) = 40$

Cofactor of $2 = A_{12} = -\begin{vmatrix}3 & 0\\2 & 8\end{vmatrix} = -(24-0) = -24$

Cofactor of $3 = A_{13} = +\begin{vmatrix}3 & 5\\2 & 1\end{vmatrix} = +(3-10) = -7$

Cofactor of $4 = A_{21} = -\begin{vmatrix}0 & 0\\1 & 8\end{vmatrix} = 0$

Cofactor of $5 = A_{22} = +\begin{vmatrix}1 & 0\\2 & 8\end{vmatrix} = +(8-0) = 8$

Cofactor of $6 = A_{23} = -\begin{vmatrix}1 & 0\\2 & 1\end{vmatrix} = -(1-0) = -1$

Cofactor of $7 = A_{31} = +\begin{vmatrix}0 & 0\\5 & 0\end{vmatrix} = 0$

Cofactor of $8 = A_{32} = -\begin{vmatrix}1 & 0\\3 & 0\end{vmatrix} = -(1-0) = -1$

Cofactor of $9 = A_{33} = +\begin{vmatrix}1 & 0\\3 & 5\end{vmatrix} = +(5-0) = 5$

The Cofactor matrix of A is

$$[A_{ij}] = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} 40 & -24 & -7 \\ 0 & 8 & -1 \\ 0 & -1 & 5 \end{bmatrix}$$

Now find the transpose of A_{ij}

$$adj\,A = \begin{bmatrix} 40 & -24 & -7 \\ 0 & 8 & -1 \\ 0 & -1 & 5 \end{bmatrix}^T = \begin{bmatrix} 40 & 0 & 0 \\ -24 & 8 & -1 \\ -7 & -1 & 5 \end{bmatrix}$$

Since $|A| = 40$

So, The inverse of matrix A will be $A^{-1} = \frac{adj\,A}{|A|} = \frac{1}{40}\begin{bmatrix} 40 & 0 & 0 \\ -24 & 8 & -1 \\ -7 & -1 & 5 \end{bmatrix}$

14. From the given question, we come to know that we have to construct a matrix with 3 rows and 3 columns.

i = 1, j = 1	i = 1, j = 2	i = 1, j = 3
$a_{11} = 0$	$a_{12} = -1$	$a_{13} = -2$
i = 2, j = 1	i = 2, j = 2	i = 2, j = 3
$a_{21} = 1$	$a_{22} = 0$	$a_{23} = -1$
i = 3, j = 1	i = 3, j = 2	i = 3, j = 3
$a_{31} = 2$	$a_{32} = 1$	$a_{33} = 0$

So, the matrix A with order 3 x 3 is

$$\begin{pmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix} \quad A^T = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & 1 \\ -2 & -1 & 0 \end{pmatrix}$$

$$-A = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & 1 \\ -2 & -1 & 0 \end{pmatrix}$$

are equal

Hence it is skew symmetric matrix.