Lecture

INSTITUTE OF ACTUARIAL
& QUANTITATIVE STUDIES

Class: FY BSc

Subject : Numerical methods

Subject Code: PUSAS201

Chapter: Unit 1 Chapter 2

Chapter Name: Iterative Methods

Today's Agenda

1. Iteration

2. Bisection
    1. Steps
    2. Why to study ?

3. Newton Raphson method
    1. Understanding the Newton Raphson method
    2. Steps
    3. Why to study

1 Iteration

Iteration is the idea of repeating a process over and over with the purpose of getting closer to an answer.

- In maths, iterative methods are often used when finding an exact answer is not so simple.
- An **iterative process** should be convergent, i.e., it should come closer to the desired result as the number of **iterations** increases
- Each repetition of the process is a single iteration, and the outcome of each iteration is then the starting point of the next iteration.
- Use of iteration in mathematics is in iterative methods which are used to produce approximate numerical solutions to certain mathematical problems. Newton's method is an example of an iterative method. Manual calculation of a number's square root is a common use and a well-known example.

https://www.youtube.com/watch?v=kMSuVL6mgo4

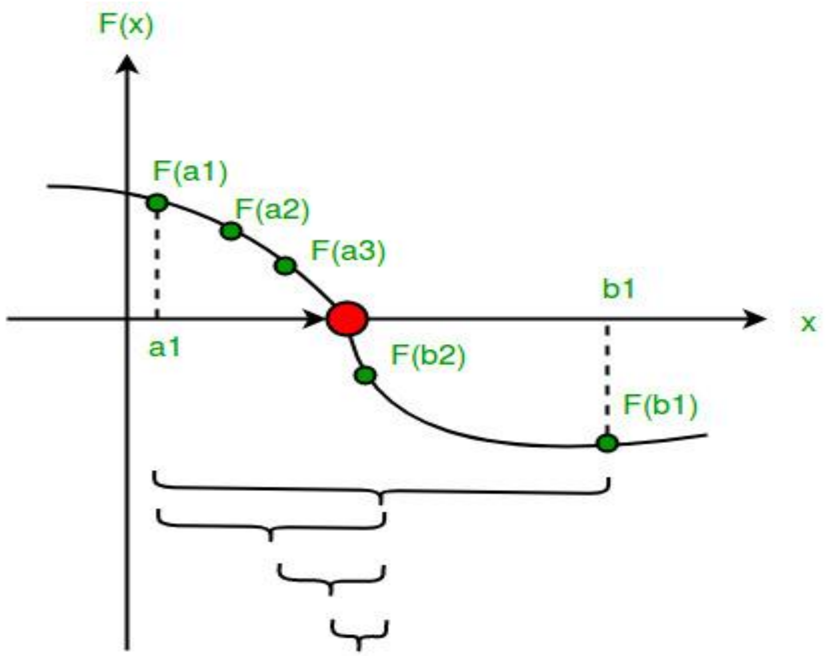
2

Bisection

- In mathematics, the **bisection method** is a root-finding method that applies to any continuous functions for which one knows two values with opposite signs. The method consists of repeatedly bisecting the interval defined by these values and then selecting the subinterval in which the function changes sign.

- The bisection method is based on finding two values between which the solution lies and then using the midpoint of the values for the next approximation. It is a very simple and robust method, but it is also relatively slow.

- The method is also called the **interval halving** or method or the **dichotomy method**.



2.1 Steps to solve equation using Bisection method

For any continuous function f(x),

- Find two points, say a and b such that a < b and f(a)* f(b) < 0
- Find the midpoint of a and b, say "t"
- t is the root of the given function if f(t) = 0; else follow the next step
- Divide the interval [a, b]
- If f(t)*f(b) <0, let a = t
- Else if f(t) *f(a)< 0, let b = t
- Repeat above three steps until f(t) = 0.

2.2 Why to study..?

Advantages

Disadvantages

2 Example

Determine the root of the given equation $x^2-3 = 0$ for $x \in [1,2]$

Solution:
$x^2-3 = 0$.
Let $f(x) = x^2-3$

Now, find the value of f(x) at a= 1 and b=2.

$f(x=1) = 1^2-3 = 1 - 3 = -2 < 0$ and $f(x=2) = 2^2-3 = 4 - 3 = 1 > 0$

The given function is continuous and the root lies in the interval [1, 2].

Let "t" be the midpoint of the interval. I.e., $t = (1+2)/2$ i.e $t = 3 / 2 = 1.5$

Therefore, the value of the function at "t" is
$f(t) = f(1.5) = (1.5)^2-3 = 2.25 - 3 = -0.75 < 0$
f(t) is negative, so b is replaced with t= 1.5 for the next iterations.

2 Working further

The iterations to the function are:

Iterations	a	b	t	f(a)	f(b)	f(t)
1	1	2	1.5	-2	1	-0.75
2	1.5	2	1.75	-0.75	1	0.062
3	1.5	1.75	1.625	-0.75	0.0625	-0.359
4	1.625	1.75	1.6875	-0.3594	0.0625	-0.1523
5	1.6875	1.75	1.7188	-01523	0.0625	-0.0457
6	1.7188	1.75	1.7344	-0.0457	0.0625	0.0081
7	1.7188	1.7344	1.7266	-0.0457	0.0081	-0.0189

So, at the seventh iteration, we get the final interval [1.7266, 1.7344]
Hence, 1.7344 is the approximated solution.

3 Newton – Raphson Method

- If you've ever tried to find a root of a complicated function algebraically, you may have had some difficulty. One of the methods that we use to solve complicated function is the Newton-Raphson method.

- This iterative process follows a set guideline to approximate one root, considering the function, its derivative, and an initial x-value.

- Newton's method is an extremely powerful technique—in general the convergence is quadratic: as the method converges on the root.

- However, there are some difficulties with the method like difficulty in calculating derivative of a function, poor initial estimate, slow convergence, stationary point etc

3.1 Understanding the Newton - Raphson

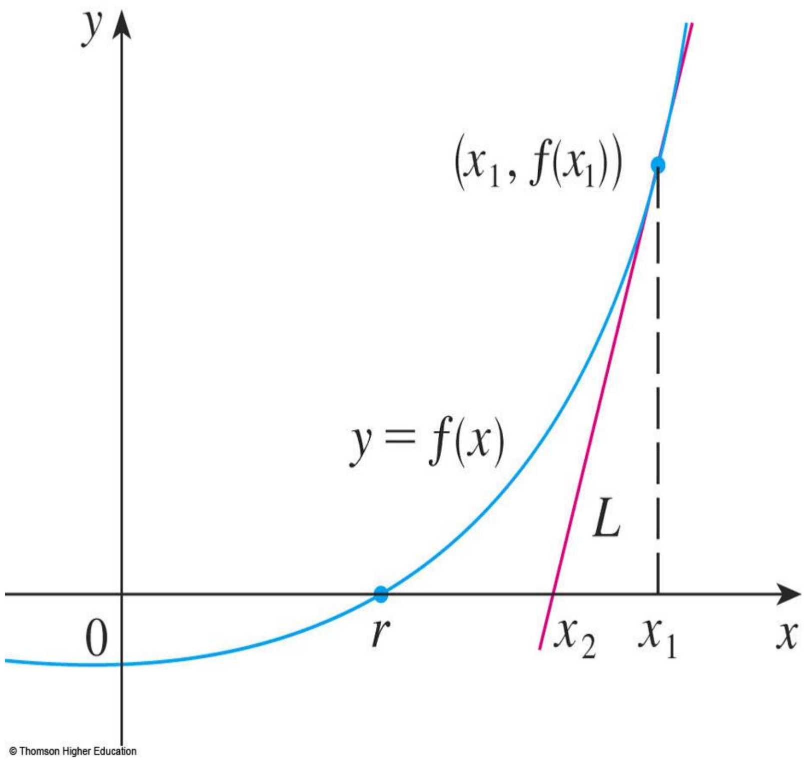
- The geometry behind Newton's method is shown here, the root we are trying to find is labeled as r.

- We start with a first approximation x_1, which is obtained by one of the following methods: i) Guessing, ii)A rough sketch of the graph of *f*, iii) A computer generated graph of f.

- Consider the tangent line *L* to the curve $y = f(x)$ at the point $(x_1, f(x_1))$ and look at the *x*-intercept of *L*, labeled x_2.

- Here's the idea behind the method.

The tangent line is close to the curve. So, its *x*-intercept, x_2 , is close to the *x*-intercept of the curve. (namely, the root *r* that we are seeking).
As the tangent is a line, we can easily find its *x*-intercept.



3.2 Steps

- To find a formula for x_2 in terms of x_1, we use the fact that the slope of L is $f'(x_1)$.

So, its equation is: $y - f(x_1) = f'(x_1)(x - x_1)$

- As the x-intercept of L is x_2, we set $y = 0$ and obtain: $0 - f(x_1) = f'(x_1)(x_2 - x_1)$
- If $f'(x_1) \neq 0$, we can solve this equation for x_2:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

We use x_2 as a second approximation to r.

- Next, we repeat this procedure with x_1 replaced by x_2, using the tangent line at $(x_2, f(x_2))$. This gives a third approximation:

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

3.2 Steps

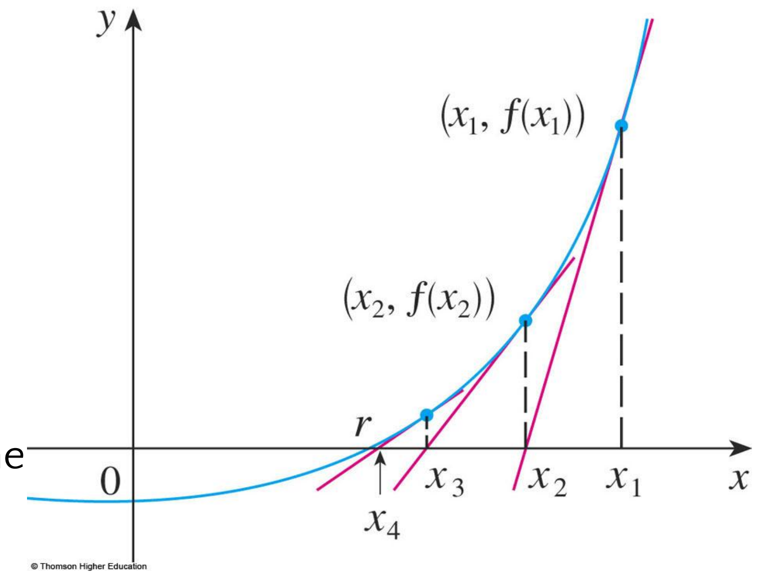
- If we keep repeating this process, we obtain a sequence of approximations $x_1, x_2, x_3, x_4, \ldots$
- In general, if the nth approximation is x_n and $f'(x_n) \neq 0$,

then the next approximation is given by:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

- If the numbers x_n become closer and closer to r as n becomes large, the we say that the sequence converges to r and we write:

$$\lim_{n\to\infty} x_n = r$$



3.3 Why to study..?

Advantages

- 1. Convergence rate is one of the fastest when it does converge.
- 2. Converges on the root quadratically
  - Near a root, the number of significant digits approximately doubles with each step.
  - This leads to the ability of the Newton-Raphson Method to "polish" a root from another convergence technique
- 3.Easy to convert to multiple dimensions
- 4. Can be used to "polish" a root found by other methods

Disadvantages

- 1. The method is very expensive - It needs the function evaluation and then the derivative evaluation.
- 2. f the tangent is parallel or nearly parallel to the x-axis, then the method does not converge.
- 3. Poor global convergence properties
- 4. Dependent on initial guess
  - May be too far from local root
  - May encounter a zero derivative
  - May loop indefinitely

3 Example

Starting with $x_1 = 2$, find the third approximation x_3 to the root of the equation $x^3 - 2x - 5 = 0$

Solution:
We apply Newton's method with $f(x) = x^3 - 2x - 5$ and $f'(x) = 3x^2 - 2$
With $n = 1$, we have and using the newton- raphson formula for nth approximation, we get

$$x_2 = x_1 - \frac{x_1^3 - 2x_1 - 5}{3x_1^2 - 2}$$

$$= 2 - \frac{2^3 - 2(2) - 5}{3(2)^2 - 2}$$

$$= 2.1$$

3 Example

With $n = 2$, we obtain:

$$x_3 = x_2 - \frac{x_2^3 - 2x_2 - 5}{3x_2^2 - 2}$$

$$= 2.1 - \frac{2.1^3 - 2(2.1) - 5}{3(2.1)^2 - 2}$$

$$\approx 2.0946$$

It turns out that this third approximation $x_3 \approx 2.0946$ is accurate to four decimal places.
Notice that the procedure in going from n to $n + 1$ is the same for all values of n. It is called an iterative process.